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Research Article

Eigenvalue Problem for Nonlinear Fractional Differential Equations with Integral Boundary Conditions

Guotao Wang,[1] Sanyang Liu,[1] and Lihong Zhang[2]

[1] *Department of Applied Mathematics, Xidian University, Xi'an, Shaanxi 710071, China*
[2] *School of Mathematics and Computer Science, Shanxi Normal University, Linfen, Shanxi 041004, China*

Correspondence should be addressed to Sanyang Liu; liusanyang@126.com

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By employing known Guo-Krasnoselskii fixed point theorem, we investigate the eigenvalue interval for the existence and nonexistence of at least one positive solution of nonlinear fractional differential equation with integral boundary conditions.

1. Introduction

Fractional calculus has been receiving more and more attention in view of its extensive applications in the mathematical modelling coming from physical and other applied sciences; see books [1–5]. Recently, the existence of solutions (or positive solutions) of nonlinear fractional differential equation has been investigated in many papers (see [6–28] and references cited therein). However, in terms of the eigenvalue problem of fractional differential equation, there are only a few results [29–33].

To the best of author's knowledge, no paper has considered the eigenvalue problem of the following nonlinear fractional differential equation with integral boundary conditions:

$$\begin{aligned} &{}^{C}D^{\alpha}u(t) + \lambda f(t, u(t)) = 0, \\ &0 < t < 1, \quad n < \alpha \le n+1, \quad n \ge 2, \quad n \in N. \\ &u(0) = u''(0) = u'''(0) = \cdots = u^{(n)}(0) = 0, \\ &u(1) = \xi \int_0^1 u(s)\,ds, \end{aligned} \tag{1}$$

where $0 < \xi < 2$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

Our proof is based upon the properties of the Green function and Guo-Krasnoselskii's fixed point theorem given in [34]. Our purpose here is to give the eigenvalue interval for nonlinear fractional differential equation with integral boundary conditions. Moreover, according to the range of the eigenvalue λ, we establish some sufficient conditions for the existence and nonexistence of at least one positive solution of the problem (1).

2. Preliminaries

For the convenience of the readers, we first present some background materials.

Definition 1. For a function $f : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$$\begin{aligned} {}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds, \\ n = [\alpha] + 1, \end{aligned} \tag{2}$$

where $[\alpha]$ denotes the integer part of the real number α.

Definition 2. The Riemann-Liouville fractional integral of order α for a function f is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \quad \alpha > 0, \tag{3}$$

provided that such integral exists.

Lemma 3. *Let $\alpha > 0$; then*

$$I^{\alpha}\,{}^{C}D^{\alpha}u(t) = u(t) + C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1}t^{n-1}, \quad (4)$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $n = [\alpha] + 1$.

Lemma 4 (see [34]). *Let E be a Banach space, and let $P \subset E$ be a cone. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that*

(i) *$\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$, or*

(ii) *$\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$.*

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 5. *Let $n < \alpha \leq n+1$, $n \geq 2$, $n \in N$, and $\xi \neq 2$. Assume $y \in C[0,1]$; then the unique solution of the problem*

$$\begin{gathered} {}^{C}D^{\alpha}u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = u'''(0) = \cdots = u^{(n)}(0) = 0, \\ u(1) = \xi\int_0^1 u(s)\,ds, \end{gathered} \quad (5)$$

is given by the expression

$$u(t) = \int_0^1 G(t,s)\,y(s)\,ds, \quad (6)$$

where

$$G(t,s) = \begin{cases} \dfrac{2t(1-s)^{\alpha-1}(\alpha-\xi+\xi s) - (2-\xi)\alpha(t-s)^{\alpha-1}}{(2-\xi)\Gamma(\alpha+1)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{2t(1-s)^{\alpha-1}(\alpha-\xi+\xi s)}{(2-\xi)\Gamma(\alpha+1)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (7)$$

Proof. It is well known that the equation ${}^{C}D^{\alpha}u(t) + y(t) = 0$ can be reduced to an equivalent integral equation:

$$u(t) = -I^{\alpha}y(t) - \sum_{i=0}^{n} b_i t^i = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds - \sum_{i=0}^{n} b_i t^i, \quad (8)$$

for some $b_i \in \mathbb{R}$ $(i = 0, 1, 2, \ldots, n)$.

By the conditions $u(0) = u''(0) = u'''(0) = \cdots = u^{(n)}(0) = 0$ and $u(1) = \xi\int_0^1 u(s)ds$, we can get that $b_0 = b_2 = b_3 = \cdots = b_n = 0$ and

$$b_1 = -\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds - \xi\int_0^1 u(s)\,ds. \quad (9)$$

Hence, we have

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds + t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds + \xi t\int_0^1 u(s)\,ds. \quad (10)$$

Put $\int_0^1 u(s)ds = A$; then, from (10), we deduce that

$$\begin{aligned} A &= \int_0^1 u(t)\,dt \\ &= -\int_0^1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\,dt + \iint_0^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\,dt + \int_0^1 \xi At\,dt \\ &= -\int_0^1 \frac{(1-s)^{\alpha}}{\alpha\Gamma(\alpha)}y(s)\,ds + \frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds + \frac{1}{2}\xi A, \end{aligned} \quad (11)$$

which implies that

$$A = -\frac{2}{2-\xi}\int_0^1 \frac{(1-s)^{\alpha}}{\alpha\Gamma(\alpha)}y(s)\,ds + \frac{1}{2-\xi}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds. \quad (12)$$

Replacing this value in (10), we obtain the following expression for function $u(t)$:

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds + t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds - \frac{2\xi}{2-\xi}\int_0^1 \frac{t(1-s)^{\alpha}}{\alpha\Gamma(\alpha)}y(s)\,ds \\ &\quad + \frac{\xi}{2-\xi}\int_0^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds \\ &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds + \int_0^1 \frac{2t(1-s)^{\alpha-1}(\alpha-\xi+\xi s)}{(2-\xi)\alpha\Gamma(\alpha)}y(s)\,ds \\ &= \int_0^t \Big(\big(2t(1-s)^{\alpha-1}(\alpha-\xi+\xi s) - (2-\xi)\alpha(t-s)^{\alpha-1}\big) \\ &\quad \times ((2-\xi)\Gamma(\alpha+1))^{-1}y(s)\,ds\Big) + \int_t^1 \frac{2t(1-s)^{\alpha-1}(\alpha-\xi+\xi s)}{(2-\xi)\Gamma(\alpha+1)}y(s)\,ds \\ &= \int_0^1 G(t,s)\,y(s)\,ds. \end{aligned} \quad (13)$$

This completes the proof. □

Lemma 6. *Let G be the Green function, which is given by the expression (7). For $0 < \lambda < 2$, the following property holds:*

$$t\,G(1,s) \le G(t,s) \le \frac{2\alpha}{\xi(\alpha-2)}\,G(1,s), \quad \forall t,s \in (0,1). \tag{14}$$

The proof is similar to that of Lemma 2.4 in [7], so we omit it.

Consider the Banach space $X = C[0,1]$ with general norm

$$\|u\| = \sup_{t\in[0,1]} |u(t)|. \tag{15}$$

Define the cone $P = \{u \in X : u(t) \ge (\xi(\alpha-1)/2\alpha)t\|u\|\}$.

Suppose u is a solution of (1). It is clear from Lemma 5 that

$$u(t) = \lambda \int_0^1 G(t,s)\,f(s,u(s))\,ds, \quad \forall t \in [0,1]. \tag{16}$$

Define the operator $S_\lambda : P \to X$ as follows:

$$(S_\lambda u)(t) = \lambda \int_0^1 G(t,s)\,f(s,u(s))\,ds, \quad \forall t \in [0,1]. \tag{17}$$

Lemma 7. *$S_\lambda : P \to P$ is completely continuous.*

Proof. Since $0 < \xi < 2$, it is obvious that $G(t,s) \ge 0$. So we have

$$\begin{aligned}\|S_\lambda u\| &= \sup_{t\in[0,1]} \lambda \int_0^1 G(t,s)\,f(s,u(s))\,ds\\ &\le \lambda \int_0^1 \frac{2\alpha}{\xi(\alpha-2)} G(1,s)\,f(s,u(s))\,ds,\\ (S_\lambda u)(t) &= \lambda \int_0^1 G(t,s)\,f(s,u(s))\,ds\\ &\ge \frac{\xi(\alpha-2)}{2\alpha} t\lambda \int_0^1 \frac{2\alpha}{\xi(\alpha-2)} G(1,s)\,f(s,u(s))\,ds\\ &\ge \frac{\xi(\alpha-2)}{2\alpha} t\,\|S_\lambda u\|.\end{aligned} \tag{18}$$

Therefore, $S_\lambda(P) \subset P$. The other proof is similar to that in [7], so we omit it. □

3. Main Result

For convenience, we list the denotation:

$$\begin{aligned}F_0 &= \lim_{u\to 0^+} \sup_{t\in[0,1]} \frac{f(t,u(t))}{u},\\ F_\infty &= \lim_{u\to +\infty} \sup_{t\in[0,1]} \frac{f(t,u(t))}{u},\\ f_0 &= \lim_{u\to 0^+} \inf_{t\in[0,1]} \frac{f(t,u(t))}{u},\\ f_\infty &= \lim_{u\to +\infty} \inf_{t\in[0,1]} \frac{f(t,u(t))}{u}.\end{aligned} \tag{19}$$

Next, we will establish some sufficient conditions for the existence and nonexistence of positive solution for problem (1).

Theorem 8. *Let $l \in (0,1)$ be a constant. Then for each*

$$\begin{aligned}\lambda \in \Bigg(&\left(\frac{\xi(\alpha-2)\,l f_\infty}{2\alpha} \int_0^1 sG(1,s)\,ds\right)^{-1},\\ &\left(\frac{2\alpha F_0}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds\right)^{-1}\Bigg),\end{aligned} \tag{20}$$

problem (1) *has at least one positive solution.*

Proof. First, for any $\varepsilon > 0$, from (20) we have

$$\begin{aligned}&\left(\frac{\xi(\alpha-2)\,l\,(f_\infty-\varepsilon)}{2\alpha} \int_0^1 sG(1,s)\,ds\right)^{-1}\\ &\quad\le \lambda \le \left(\frac{2\alpha(F_0+\varepsilon)}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds\right)^{-1}.\end{aligned} \tag{21}$$

On the one hand, by the definition of F_0, there exists $r_1 > 0$ such that, for any $u \in [0,r_1]$, we have

$$f(t,u) \le (F_0+\varepsilon)\,u. \tag{22}$$

Choose $\Omega_1 = \{u \in X : \|u\| \le r_1\}$. For $u \in P \cap \partial\Omega_1$, we have

$$\begin{aligned}\|S_\lambda u\| &= \sup_{t\in[0,1]} \lambda \int_0^1 G(t,s)\,f(s,u(s))\,ds\\ &\le \lambda \int_0^1 \frac{2\alpha}{\xi(\alpha-2)} G(1,s)\,(F_0+\varepsilon)\,u(s)\,ds\\ &\le \lambda \frac{2\alpha(F_0+\varepsilon)}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds\,\|u\| \le \|u\|.\end{aligned} \tag{23}$$

On the other hand, by the definition of F_∞, there exists $r_2 > r_1$ such that, for any $u \in [r_2,+\infty)$, we have

$$f(t,u) \ge (f_\infty-\varepsilon)\,u. \tag{24}$$

Take $\Omega_2 = \{u \in X : \|u\| \le r_2\}$. For $u \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}\|S_\lambda u\| &\ge (S_\lambda u)(l) \ge \lambda \int_0^1 lG(1,s)\,(f_\infty-\varepsilon)\,u(s)\,ds\\ &\ge \lambda l \frac{\xi(\alpha-2)\,f_\infty}{2\alpha} \int_0^1 sG(1,s)\,ds\,\|u\| \ge \|u\|.\end{aligned} \tag{25}$$

According to (23), (25), and Lemma 4, S_λ has at least one fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \le \|u\| \le r_2$, which is a positive solution of (1). □

Remark 9. If $F_0 = 0$ and $f_\infty = \infty$, then we can get

$$\begin{aligned}\frac{2\alpha F_0}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds &= 0,\\ \frac{\xi(\alpha-2)\,l f_\infty}{2\alpha} \int_0^1 sG(1,s)\,ds &= +\infty.\end{aligned} \tag{26}$$

Theorem 8 implies that, for $\lambda \in (0, +\infty)$, problem (1) has at least one positive solution.

Theorem 10. *Let $l \in (0, 1)$ be a constant. Then for each*

$$\lambda \in \left(\left(\frac{\xi(\alpha-2) l f_0}{2\alpha} \int_0^1 sG(1,s)\,ds \right)^{-1}, \left(\frac{2\alpha F_\infty}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds \right)^{-1} \right), \tag{27}$$

problem (1) *has at least one positive solution.*

Proof. First, it follows from (27) that, for any $\varepsilon > 0$,

$$\left(\frac{\xi(\alpha-2) l (f_0-\varepsilon)}{2\alpha} \int_0^1 sG(1,s)\,ds \right)^{-1} \le \lambda \le \left(\frac{2\alpha (F_\infty+\varepsilon)}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds \right)^{-1}. \tag{28}$$

By the definition of f_0, there exists $r_1 > 0$ such that, for any $u \in [0, r_1]$, we have

$$f(t,u) \ge (f_0+\varepsilon)\,u. \tag{29}$$

Choose $\Omega_1 = \{u \in X : \|u\| \le r_1\}$. For $u \in P \cap \partial\Omega_1$, we have $\|u\| = r_1$. Similar to the proof in Theorem 8, it holds from (28) and (29) that

$$\begin{aligned} \|S_\lambda u\| &\ge (S_\lambda u)(l) \\ &\ge \lambda l \frac{\xi(\alpha-2) f_0}{2\alpha} \int_0^1 sG(1,s)\,ds\,\|u\| \ge \|u\|. \end{aligned} \tag{30}$$

Note $F_\infty = \lim_{u\to+\infty} \sup_{t\in[0,1]} f(t,u(t))/u$. There exists $r_3 > r_1$, such that

$$f(t,u) \le (F_\infty+\varepsilon)\,u, \quad u \in (r_3, +\infty). \tag{31}$$

We consider the problem on two cases. (I) Suppose f is bounded. There exists $M > 0$, such that $f(t,u(t)) \le M$, $\forall u \in (r_3, +\infty)$. Choose $r_4 = \max\{r_3, M\lambda(2\alpha/\xi(\alpha-2)) \int_0^1 G(1,s)ds\}$. Let $\Omega_2' = \{u \in X : \|u\| \le r_4\}$. For $u \in P \cap \partial\Omega_2'$, we have

$$\begin{aligned} \|S_\lambda u\| &= \sup_{t\in[0,1]} \lambda \int_0^1 G(t,s) f(s,u(s))\,ds \\ &\le \lambda \int_0^1 \frac{2\alpha}{\xi(\alpha-2)} G(1,s) f(s,u(s))\,ds \\ &\le \lambda M \frac{2\alpha}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds\,\|u\| \le r_4 \\ &= \|u\|. \end{aligned} \tag{32}$$

(II) Suppose f is unbounded. There exists $r_5 > r_3$ such that

$$f(t,u(t)) \le u, \quad u \in (r_5, +\infty). \tag{33}$$

Let $\Omega_2'' = \{u \in X : \|u\| \le r_5\}$. For $u \in P \cap \partial\Omega_2''$, we have

$$\begin{aligned} \|S_\lambda u\| &\le \lambda \int_0^1 \frac{2\alpha}{\xi(\alpha-2)} G(1,s) f(s,u(s))\,ds \\ &\le \lambda \frac{2\alpha(F_\infty+\varepsilon)}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds\,\|u\| \le \|u\|. \end{aligned} \tag{34}$$

Combining (I) and (II), take $\Omega_2 = \{u \in X : \|u\| \le r_2\}$; here, $r_2 \ge \max\{r_4, r_5\}$. Then for $u \in P \cap \partial\Omega_2$, we have

$$\|S_\lambda u\| \le \|u\|. \tag{35}$$

Hence, (30) and (42) together with Lemma 4 imply that S_λ has at least one fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \le \|u\| \le r_2$, which is a positive solution of (1). □

Theorem 11. *Assume $F_0 < +\infty$ and $F_\infty < +\infty$. Problem* (1) *has no positive solution provided*

$$\lambda < \left(\frac{2\alpha k}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds \right)^{-1}, \tag{36}$$

where k is a constant defined in (38).

Proof. Since $F_0 < +\infty$ and $F_\infty < +\infty$, together with the definitions of F_0 and F_∞, there exist positive constants k_1, k_2, r_1, and r_2 satisfying $r_1 < r_2$ such that

$$\begin{aligned} f(t,u) &\le k_1 u, \qquad u \in [0, r_1], \\ f(t,u) &\le k_2 u, \qquad u \in [r_2, +\infty]. \end{aligned} \tag{37}$$

Take

$$k = \max\left\{ k_1, k_2, \sup_{(t,u)\in(0,1)\times(k_1,k_2)} \frac{f(t,u)}{u} \right\}. \tag{38}$$

It follows that $f(t,u) \le ku$ for any $u \in (0, +\infty)$. Suppose that $v(t)$ is a positive solution of (1). That is,

$$(S_\lambda v)(t) = v(t), \quad \forall t \in J. \tag{39}$$

In sequence,

$$\begin{aligned} \|v\| = \|S_\lambda v\| &= \sup_{t\in[0,1]} \lambda \int_0^1 G(t,s) f(s,v(s))\,ds \\ &\le \lambda \int_0^1 \frac{2\alpha}{\xi(\alpha-2)} G(1,s) f(s,v(s))\,ds \\ &\le \lambda k \frac{2\alpha}{\xi(\alpha-2)} \int_0^1 G(1,s)\,ds\,\|v\| < \|v\|, \end{aligned} \tag{40}$$

which is a contradiction. Hence, (1) has no positive solution. □

Theorem 12. *Assume $f_0 > 0$ and $f_\infty > 0$. Problem* (1) *has no positive solution provided*

$$\lambda > \left(\frac{\xi k(\alpha-2)}{2\alpha} \int_0^1 s^2 G(1,s)\,ds \right)^{-1}, \tag{41}$$

where k is a constant defined in (43).

Proof. Since $f_0 > 0$ and $f_\infty > 0$, together with the definitions of f_0 and f_∞, there exist positive constants k_1, k_2, r_1, and r_2 satisfying $r_1 < r_2$ such that

$$\begin{aligned} f(t,u) &\geq k_1 u, \qquad u \in [0, r_1], \\ f(t,u) &\geq k_2 u, \qquad u \in [r_2, +\infty]. \end{aligned} \tag{42}$$

Take

$$k = \min\left\{k_1, k_2, \inf_{(t,u)\in(0,1)\times(k_1,k_2)} \frac{f(t,u)}{u}\right\}. \tag{43}$$

It follows that $f(t,u) \geq ku$ for any $u \in (0, +\infty)$. Suppose that $v(t)$ is a positive solution of (1). That is,

$$(S_\lambda v)(t) = v(t), \quad \forall t \in J. \tag{44}$$

In sequence,

$$\begin{aligned} \|v\| &\geq \lambda \int_0^1 sG(1,s) f(s, v(s))\, ds \\ &\geq \lambda k \frac{\xi(\alpha-2)}{2\alpha} \int_0^1 s^2 G(1,s)\, ds \|v\| > \|v\|, \end{aligned} \tag{45}$$

which is a contradiction. Hence, (1) has no positive solution. □

Example 13. Consider the fractional differential equation

$$\begin{aligned} &{}^{C}D^{5/2} u(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \\ &u(0) = u''(0) = 0, \qquad u(1) = \int_0^1 u(s)\, ds. \end{aligned} \tag{46}$$

In this example, take

$$f(t, u(t)) = \frac{(500u^2 + u)(7 - t^2)}{u + 7}. \tag{47}$$

Obviously, we have

$$\begin{aligned} F_0 &= \lim_{u\to 0^+} \sup_{t\in[0,1]} \frac{(500u^2+u)(7-t^2)}{u(u+7)} = 1, \\ f_\infty &= \lim_{u\to+\infty} \inf_{t\in[0,1]} \frac{(500u^2+u)(7-t^2)}{u(u+7)} = 3000. \end{aligned} \tag{48}$$

Since $\alpha = 5/2$ and $\xi = 1$, through a computation, we can get

$$\begin{aligned} &\int_0^1 G(1,s)\, ds \\ &\quad = \int_0^1 \frac{2t(1-s)^{\alpha-1}(\alpha - \xi + \xi s) - (2-\xi)\alpha(t-s)^{\alpha-1}}{(2-\xi)\Gamma(\alpha+1)} ds \\ &\quad = \int_0^1 \frac{2(1-s)^{3/2}(3/2+s) - (5/2)(1-s)^{3/2}}{\Gamma(7/2)} ds \\ &\quad \leq \frac{1}{\Gamma(7/2)}, \\ &\int_0^1 sG(1,s)\, ds \\ &\quad = \int_0^1 \frac{2s(1-s)^{3/2}(3/2+s) - (5/2)s(1-s)^{3/2}}{\Gamma(7/2)} ds \\ &\quad = \int_0^1 \frac{s(1-s)^{3/2}}{2\Gamma(7/2)} ds \geq \frac{2}{35\Gamma(7/2)}. \end{aligned} \tag{49}$$

Choose $l = 2/3$; we have

$$\begin{aligned} &\left(\frac{\xi(\alpha-2) l f_\infty}{2\alpha} \int_0^1 sG(1,s)\, ds\right)^{-1} \\ &\quad \leq \frac{7\Gamma(7/2)}{80} < \frac{\Gamma(7/2)}{10} \leq \left(\frac{2\alpha F_0}{\xi(\alpha-2)} \int_0^1 G(1,s)\, ds\right)^{-1}. \end{aligned} \tag{50}$$

Theorem 8 implies that, for $\lambda \in (7\Gamma(7/2)/80, \Gamma(7/2)/10)$, the problem (46) has at least one positive solution.

Remark 14. In particular, if we take $f(t, u(t)) = u^2(1+t)$ in Example 13, then $F_0 = 0$ and $f_\infty = \infty$. Remark 9 implies that problem (46) has at least one positive solution for $\lambda \in (0, +\infty)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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