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Research Article

Precise Large Deviations of Aggregate Claims with Dominated Variation in Dependent Multi-Risk Models

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We consider a dependent multirisk model in insurance, where all the claims constitute a linearly extended negatively orthant dependent (LENOD) random array, and then upper and lower bounds for precise large deviations of nonrandom and random sums of random variables with dominated variation are investigated. The obtained results extend some related existing ones.

1. Introduction

In a classic insurance risk model the surplus is described as the initial surplus plus the premium income with the claims taken off. Since asymptotic behavior for precise large deviations of the loss process of insurance risk models has theoretical significance and extensive applications, it has been widely investigated and there appeared to be a great deal of research literature. Some earlier works on precise large deviations can be found in Nagaev [1, 2], Heyde [3, 4], and Nagaev [5, 6], among others. Recent works on this field can be found in Cline and Hsing [7], Klüppelberg and Mikosch [8], Tang et al. [9], Tang [10], Liu [11], Chen and Zhang [12], Shen and Lin [13], Liu [14], Chen et al. [15], and Chen and Yuen [16], among others.

In consideration of insurance reality, some researchers have begun to focus on the precise large deviations of multi-risk models in the past few years. See S. Wang and W. Wang [17], Lu [18, 19], He at al. [20], and S. J. Wang and W. S. Wang [21], among others. For convenience of representation, we adopt the notations of S. Wang and W. Wang [17]. Assume that the insurer manages k types of insurance contracts at the same time, where k is any fixed positive integer. The ith related loss amounts (claims) are denoted by random variables $\{X_{ij}, j \geq 1\}$ with common distribution $F_i(x) = P(X_i \leq x) := 1 - \overline{F}_i(x)$ satisfying $\overline{F}_i(x) > 0$ for all $x \in (-\infty, \infty)$. Let $\{N_i(t), i = 1, \ldots, k, t > 0\}$ be independent nonnegative integer-valued counting processes independent of $\{X_{ij}, i = 1, \ldots, k, j \geq 1\}$, satisfying $EN_i(t) = \lambda_i(t) \to \infty$ as $t \to \infty$. In insurance multi-risk models, $\{N_i(t), t > 0\}_{i=1}^{k}$ always denote the ith claim numbers of the related insurance contracts. Obviously, all the claims constitute the following random array:

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & \cdots \\ X_{21} & X_{22} & X_{23} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ X_{k1} & X_{k2} & X_{k3} & \cdots \end{pmatrix}. \tag{1}$$

Therefore, the loss process of the insurer during the period $[0, t]$ can be described as $S(k; t) = \sum_{i=1}^{k} \sum_{j=1}^{N_i(t)} X_{ij}$. If the random arrays (1) are independent (i.e., to say all random variables $\{X_{ij}, i = 1, \ldots, k, j \geq 1\}$ are independent) and all the claims have consistently varying tails (see the definition below), under some mild conditions, S. Wang and W. Wang [17] obtained the following result. For any fixed $\gamma > 0$, as $t \to \infty$, the relation

$$P\left(S(k; t) - \sum_{i=1}^{k} \mu_i \lambda_i(t) > x\right) \sim \sum_{i=1}^{k} \lambda_i(t) \overline{F}_i(x) \tag{2}$$

holds uniformly for $x \geq \max\{\gamma \lambda_i(t), i = 1, \ldots, k\} := \Gamma(k)$.

The relation (2) describes the so-called precise large deviations for random sums in multi-risk models. Subsequently, Lu [18, 19] investigated precise large deviations with subexponential and long-tailed claims in multi-risk models under independent structures, respectively. He et al. [20] obtained the lower bounds of precise large deviations of multi-risk models with nonnegative random variables (regardless of heavy or light tails) under a specific dependence structure. However, on the one hand, for practical reasons, the independence assumptions in the above-mentioned papers are quite unrealistic. On the other hand, up to now, most works on precise large deviations with heavy tails concentrated on the consistent variation or some other heavy-tailed subclasses although some specific dependence structures have been considered. Therefore, it is more interesting to study the estimation of precise large deviation probabilities of aggregate claims in the presence of dominated variation and some dependence structures, where dominated variation strictly includes consistent variation. More recently, some researchers have begun to focus on this issue and obtained some interesting results. See Wang et al. [22], Wang et al. [23], Yang et al. [24], Yang and Wang [25], and Chen and Qu [26], among others. Motivated by the two reasons mentioned above, in this paper, upper and lower bounds for precise large deviations of aggregate claims with dominated variation in dependent multi-risk models (see the dependent structures in Definitions 1 and 2 below) are investigated. The obtained results extend some related existing ones.

To close this section, we introduce two new dependence structures called ENOD and LENOD, respectively, which are the basic assumptions in this paper. The idea of Definition 1 comes from Liu [14].

Definition 1. We call random arrays $\{X_{ij},\ i = 1, \ldots, k,\ j \geq 1\}$

(1) Extended Negatively Lower Orthant Dependent (ENLOD) if there exists some constant $M > 0$ such that for each $n = 1, 2, \ldots$, $\{i_1, \ldots, i_n\} \subset \{1, \ldots, k\}$, $\{j_1, \ldots, j_n\} \subset \{1, 2, \ldots\}$, and all $x_1, \ldots, x_n$,

$$P\left(X_{i_1 j_1} \leq x_1, \ldots, X_{i_n j_n} \leq x_n\right) \leq M \prod_{k=1}^{n} P\left(X_{i_k j_k} \leq x_k\right); \quad (3)$$

(2) Extended Negatively Upper Orthant Dependent (ENUOD) if there exists some constant $M > 0$ such that for each $n = 1, 2, \ldots$, $\{i_1, \ldots, i_n\} \subset \{1, \ldots, k\}$, $\{j_1, \ldots, j_n\} \subset \{1, 2, \ldots\}$, and all $x_1, \ldots, x_n$,

$$P\left(X_{i_1 j_1} > x_1, \ldots, X_{i_n j_n} > x_n\right) \leq M \prod_{k=1}^{n} P\left(X_{i_k j_k} > x_k\right); \quad (4)$$

(3) Extended Negatively Orthant Dependent (ENOD) if both (3) and (4) hold for each $n = 1, 2, \ldots$, $\{i_1, \ldots, i_n\} \subset \{1, \ldots, k\}$, $\{j_1, \ldots, j_n\} \subset \{1, 2, \ldots\}$, and all $x_1, \ldots, x_n$.

Definition 2. We call random arrays $\{X_{ij},\ i = 1, \ldots, k,\ j \geq 1\}$

(1) Linearly Extended Negatively Lower Orthant Dependent (LENLOD) if for each $m = 1, 2, \ldots$, any finite disjoint subsets $A_1, A_2, \ldots, A_m$ of $\{1, \ldots, k\} \times \{1, 2, \ldots\}$, and positive r_{ij}'s,

$$\sum_{(i,j) \in A_1} r_{ij} X_{ij}, \qquad \sum_{(i,j) \in A_2} r_{ij} X_{ij}, \ldots, \qquad \sum_{(i,j) \in A_m} r_{ij} X_{ij} \ \text{are ENLOD}; \quad (5)$$

(2) Linearly Extended Negatively Upper Orthant Dependent (LENUOD) if for each $m = 1, 2, \ldots$, any finite disjoint subsets $A_1, A_2, \ldots, A_m$ of $\{1, \ldots, k\} \times \{1, 2, \ldots\}$, and positive r_{ij}'s,

$$\sum_{(i,j) \in A_1} r_{ij} X_{ij}, \qquad \sum_{(i,j) \in A_2} r_{ij} X_{ij}, \ldots, \qquad \sum_{(i,j) \in A_m} r_{ij} X_{ij} \ \text{are ENUOD}; \quad (6)$$

(3) Linearly Extended Negatively Orthant Dependent (LENOD) if both (5) and (6) hold for each $m = 1, 2, \ldots$, any finite disjoint subsets $A_1, A_2, \ldots, A_m$ of $\{1, \ldots, k\} \times \{1, 2, \ldots\}$ and positive r_{ij}'s.

Remark 3. The idea of Definition 2 is due to Newman [27] who first put forward the concept of LNQD (linearly negative quadrant dependent) when studying the central limit theorem. Other related dependence structures are called NQD and NA. See Joag-Dev and Proschan [28] for more details. It is well known that NA implies LNQD and LNQD is slightly stronger than NQD. In the consideration of the need of studying multivariate random variables, we introduce the concept of LENOD random arrays in this paper. By definitions, one can also easily check that NA random arrays must be LENOD ones and LENOD implies ENQD. Moreover, LENOD is more comprehensive than NA in that it can reflect not only a negative association structure but also a positive one to some extent.

Remark 4. It is worth to mention that there maybe exist some flaws in the definition of extended negatively associated (ENA) structure of S. J. Wang and W. S. Wang [21]. In fact, S. J. Wang and W. S. Wang [21] stated that the relationship

$$\operatorname{Cov}\left(f\left(X_{ij}; (i,j) \in A\right), g\left(X_{ij}; (i,j) \in B\right)\right) \leq M \quad (7)$$

holds for fixed $M > 0$ and any pair of coordinate-wise increasing functions f, g. Considering the case that $\operatorname{Cov}(f(X_{ij}; (i,j) \in A), g(X_{ij}; (i,j) \in B)) > 0$, inequality (7) can not hold because

$$\lim_{c \to \infty} \operatorname{Cov}\left(cf\left(X_{ij}; (i,j) \in A\right), cg\left(X_{ij}; (i,j) \in B\right)\right) = \infty. \quad (8)$$

It might be questionable. Therefore, in this paper, we redefine a new dependence structure called LENOD for random arrays to avoid this problem. Fortunately, under this new dependence structure, the main results still hold and extend some related existing ones.

The rest of this paper is organized as follows. Section 2 gives some preliminaries. Precise large deviations for nonrandom sums and random sums with dominated variation in dependent multi-risk models are presented in Sections 3 and 4.

2. Preliminaries

For convenience, hereafter, for two positive infinitesimals $f(\cdot)$ and $g(\cdot)$ satisfying

$$a \leq \liminf \frac{f(\cdot)}{g(\cdot)} \leq \limsup \frac{f(\cdot)}{g(\cdot)} \leq b, \tag{9}$$

we write $f(\cdot) = O(g(\cdot))$ if $b < \infty$; $f(\cdot) = o(g(\cdot))$ if $b = 0$. For two positive bivariate functions $f(t,x)$ and $g(t,x)$, we say $f(t,x) \sim g(t,x)$ holds as $t \to \infty$ uniformly for all $x \in \mathscr{D}(t) \neq \phi$ in the sense that $\lim_{t\to\infty}\sup_{x\in\mathscr{D}(t)}|(f(t,x)/g(t,x))-1| = 0$.

For simplicity, for any positive integer sequence $\{n_i,\ i = 1,\ldots,k\}$, we use the following notations:

$$S_n = \sum_{k=1}^{n} X_k, \quad S_{n_i} = \sum_{j=1}^{n_i} X_{ij}, \quad S(k; n_1,\ldots,n_k) = \sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{ij},$$
$$S_{N(t)} = \sum_{k=1}^{N(t)} X_k, \quad S_{N_i(t)} = \sum_{k=1}^{N_i(t)} X_{ij}, \quad i = 1,\ldots,k. \tag{10}$$

2.1. Heavy-Tailed Distributions. In risk theory, heavy-tailed distribution functions are often used to model large claims. They play a key role in some fields such as insurance, financial mathematics, and queueing theory. We say that a nonnegative random variable X (or its distribution function F) is heavy-tailed if it has no finite exponential moments. For details, we refer the reader to Embrechts, Klüppelberg, and Mikosch [8], among others. For convenience of use, we recall some important subclasses of heavy-tailed distributions. A quite large subclass is called long-tailed distribution class denoted by $\mathscr{L}$. A distribution function F is said to belong to $\mathscr{L}$ if for all $y \in (-\infty,+\infty)$

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1. \tag{11}$$

Another important subclass is called the class of random variables with dominatedly varying tails (or the class of distribution functions with dominated variation) denoted by $\mathscr{D}$. We say that a distribution function F is in $\mathscr{D}$ if for any $0 < y < 1$ (or equivalently for some $0 < y < 1$)

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty. \tag{12}$$

Furthermore, for any $y > 1$, set $\overline{F}_*(y) = \liminf_{x\to\infty}(\overline{F}(xy)/\overline{F}(x))$ and then define

$$L_F = \lim_{y\downarrow 1} \overline{F}_*(y), \qquad J_F^+ := \inf\left\{-\frac{\log \overline{F}_*(y)}{\log y},\ y > 1\right\}. \tag{13}$$

In the terminology of Tang and Tsitsiashvili [29], J_F^+ is called the (upper) Matuszewska index of F. The following proposition is well known.

Proposition 5. *Consider $F \in \mathscr{D} \Leftrightarrow \overline{F}_*(y) > 0$ for all $y > 1 \Leftrightarrow \overline{F}_*(y) > 0$ for some $y > 1 \Leftrightarrow L_F > 0 \Leftrightarrow J_F^+ < \infty$.*

Some other subclasses are as follows. Denote by $\mathscr{C} = \{F : L_F = 1\}$ the class of random variables with consistently varying tails (or distribution functions with consistent variation). For some $0 < \alpha \leq \beta < \infty$, denote by $\mathrm{ERV}(-\alpha,-\beta) = \{F : y^{-\beta} \leq \overline{F}_*(y) \leq \overline{F}^*(y) \leq y^{-\alpha}$ for all $y > 1\}$ the extended regularly varying class. Particularly, if $\alpha = \beta$, it reduces to the regularly varying class, denoted by $\mathscr{R}_{-\alpha}$. For the heavy-tailed distribution subclasses mentioned above, it is well known that the following inclusions hold:

$$\mathscr{R}_{-\alpha} \subset \mathrm{ERV}\left(-\alpha,-\beta\right) \subset \mathscr{C} \subset \mathscr{D}\cap\mathscr{L} \subset \mathscr{D}. \tag{14}$$

Finally, we define an important quantity of heavy-tailed distribution function needed in the main results.

Definition 6. For any heavy-tailed distribution F and $y \geq 0$, define

$$\rho_F(y) = \liminf_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)}. \tag{15}$$

Remark 7. Obviously, $\rho_F(0) = 1$. Furthermore, if $F \in \mathscr{L}$, then $\rho_F(y) \equiv 1$ for any $y > 0$ by the definition of $\mathscr{L}$.

2.2. Some Lemmas. In this sequel, we will give some lemmas needed in the proofs of the main results. Lemmas 1 and 2 are due to S. J. Wang and W. S. Wang [21] (Lemmas 2.3 and 2.5). To state the results, we should introduce two assumptions added on the process $\{N(t),\ t \geq 0\}$, which also appeared in Chen et al. [15] and S. J. Wang and W. S. Wang [21].

Assumption A. For any $\delta > 0$ and some $p > J_F^+$, as $t \to \infty$,

$$EN^p(t)\,\mathbf{1}_{(N(t)>(1+\delta)\lambda(t))} = O(\lambda(t)). \tag{16}$$

Assumption B. For all $0 < \delta < 1$, as $t \to \infty$,

$$P(N(t) \leq (1-\delta)\lambda(t)) = o\left(\lambda(t)\overline{F}(\lambda(t))\right). \tag{17}$$

Lemma 1. *Let $\{X_k,\ k = 1,2,\ldots\}$ be ENOD random variables with common distribution $F \in \mathscr{D}$ and finite mean μ, satisfying $E(X_1^-)^r < \infty$ for some $r > 1$ and*

$$F(-x) = o\left(\overline{F}(x)\right), \quad x \longrightarrow \infty; \tag{18}$$

then, for any $\gamma > 0$, as $n \to \infty$, the relation

$$P(S_n - n\mu \leq -x) = o\left(n\overline{F}(x)\right) \tag{19}$$

holds uniformly for $x \geq \gamma n$.

Lemma 2. *Let $\{X_k,\ k = 1,2,\ldots\}$ be a sequence of ENOD random variables with common distribution function $F \in \mathscr{D}$*

and finite mean μ, satisfying (18)*, and let $\{N(t),\ t \geq 0\}$ be a nonnegative integer-valued counting process independent of $\{X_k,\ k = 1, 2, \ldots\}$, satisfying Assumptions A and B; then, for any $\gamma > 0$, as $t \to \infty$, the relation*

$$P\left(S_{N(t)} - \mu\lambda(t) \leq -x\right) = o\left(\lambda(t)\overline{F}(x)\right) \tag{20}$$

holds uniformly for $x \geq \gamma\lambda(t)$.

Lemmas 3–5 play important roles in the proofs of main results and have their own interests. Lemma 3 describes an important property of the quantity ρ_F defined in Definition 6. Lemmas 4 and 5 present precise large deviations for nonrandom and random sums of random variables with dominated variation in single-risk models, respectively.

Lemma 3. *Let any distribution $F \in \mathscr{D}$, and, for any fixed real number z, denote the distribution $F(\cdot + z)$ by $F_{+z}(\cdot)$. Then,*

(i) $L_F \leq L_{F_{+z}} \leq \rho_F^{-1}(z) L_F$ when $z \geq 0$;

(ii) $\rho_F(-z) L_F \leq L_{F_{+z}} \leq L_F$ when $z < 0$.

Proof. (i) By the definition of $\mathscr{D}$, one can easily check that $F_{+z} \in \mathscr{D}$.

When $z \geq 0$, we get

$$\begin{aligned} L_{F_{+z}} &= \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy+z)}{\overline{F}(x+z)} \\ &\geq \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}((x+z)y)}{\overline{F}(x+z)} = L_F. \end{aligned} \tag{21}$$

Conversely, for any $0 < \varepsilon < 1$ and large enough x, noticing that $F \in \mathscr{D}$, we have

$$\begin{aligned} L_{F_{+z}} &\leq \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x+z)} \\ &\leq \frac{\lim_{y\downarrow 1} \liminf_{x\to\infty} \left(\overline{F}(xy)/\overline{F}(x)\right)}{\liminf_{x\to\infty} \left(\overline{F}(x+z)/\overline{F}(x)\right)} = \rho_F^{-1}(z) L_F. \end{aligned} \tag{22}$$

(ii) For any $y \geq 0$ and fixed real number z, by Definition 6, one can easily obtain $\rho_{F_{+z}}(y) = \rho_F(y)$. Thus, when $z < 0$, (21) and (22) imply $L_F \geq L_{F_{+z}}$ and $L_F \leq \rho_{F_{+z}}^{-1}(-z) L_{F_{+z}} = \rho_F^{-1}(-z) L_{F_{+z}}$. It ends the proof of Lemma 3. □

Remark 8. It is interesting that Lemma 3 indicates that the quantity $L_{F_{+z}}$ maybe will not be equal to L_F when we let $F_{+z}(\cdot) = F(\cdot + z)$ for any fixed real number z. In essence, it is due to the fact that $\mathscr{D}$ is not a subset of $\mathscr{L}$. Thus, we should be very cautious when dealing with the random variables from the subclass $\mathscr{D}$.

Lemma 4. *Let $\{X_k,\ k \geq 1\}$ be a sequence of ENOD random variables with common distribution $F \in \mathscr{D}$ and finite mean μ, satisfying*

$$\begin{gathered} F(-x) = o\left(\overline{F}(x)\right) \quad \text{as } x \longrightarrow \infty, \\ E|X_1|^r \mathbf{1}_{\{X_1 \leq 0\}} < \infty \quad \text{for some } r > 1. \end{gathered} \tag{23}$$

Then, for any fixed $\gamma > 0$,

$$\begin{aligned} \rho_F(|\mu|) L_F &\leq \liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_n - n\mu > x)}{n\overline{F}(x)} \\ &\leq \limsup_{n\to\infty} \sup_{x\geq\gamma n} \frac{P(S_n - n\mu > x)}{n\overline{F}(x)} \leq M_{F,\mu} L_F^{-1}, \end{aligned} \tag{24}$$

where $M_{F,\mu} := \mathbf{1}_{\{\mu\geq 0\}} + \rho_F^{-2}(|\mu|)\mathbf{1}_{\{\mu<0\}}$.

Proof. If $\mu = 0$, one can mimic the proof of Theorem 1.1 of Liu [14] to prove (24). Also one can get the relation (24) directly from Corollaries 1 and 2 of Wang et al. [23]. Therefore, we omit the process here.

For the case of $\mu \neq 0$, let $X_i' = X_i - \mu$, $i = 1, 2, \ldots$, and note that X_i' is distributed by $F_{+\mu}$ (see the notation in Lemma 3). Therefore, if $\mu < 0$, by the fact that $F \in \mathscr{D}$ and Lemma 3, it yields that

$$\begin{aligned} &\liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_n - n\mu > x)}{n\overline{F}(x)} \\ &\quad\geq \liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_n - n\mu > x)}{n\overline{F}_{+\mu}(x)} \liminf_{x\to\infty} \frac{\overline{F}(x+\mu)}{\overline{F}(x)} \\ &\quad\geq L_{F_{+\mu}} \geq \rho_F(|\mu|) L_F, \end{aligned} \tag{25}$$

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{x\geq\gamma n} \frac{P(S_n - n\mu > x)}{n\overline{F}(x)} \\ &\quad\leq \limsup_{n\to\infty} \sup_{x\geq\gamma n} \frac{P(S_n - n\mu > x)}{n\overline{F}_{+\mu}(x)} \limsup_{x\to\infty} \frac{\overline{F}(x+\mu)}{\overline{F}(x)} \\ &\quad\leq L_{F_{+\mu}}^{-1} \left(\liminf_{x\to\infty} \frac{\overline{F}(x)}{\overline{F}(x+\mu)}\right)^{-1} \leq \rho_F^{-2}(|\mu|) L_F^{-1}. \end{aligned} \tag{26}$$

Combining (25) and (26), we get (24). Similarly as above, one can also check that the relation (24) holds if $\mu > 0$ and this ends the proof of Lemma 4. □

Lemma 5. *Let $\{X_k,\ k \geq 1\}$ be a sequence of ENOD random variables with common distribution $F \in \mathscr{D}$ and finite mean μ, satisfying* (23)*, and let $\{N(t), t \geq 0\}$ be a nonnegative integer-valued counting process independent of $\{X_k,\ k \geq 1\}$. Then, for any fixed $\gamma > 0$, the relation*

$$\begin{aligned} \rho_F(|\mu|) L_F^2 &\leq \liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{P\left(S_{N(t)} - \mu\lambda(t) > x\right)}{\lambda(t)\overline{F}(x)} \\ &\leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{P\left(S_{N(t)} - \mu\lambda(t) > x\right)}{\lambda(t)\overline{F}(x)} \leq M_{F,\mu} L_F^{-2} \end{aligned} \tag{27}$$

holds under one of the following two conditions (where $M_{F,\mu}$ is defined in Lemma 4):

(i) *when $\mu \geq 0$, Assumption A holds;*

(ii) *when $\mu < 0$, Assumption B holds.*

To prove Lemma 5, one can mimic the proof of Theorem 3.1 of S. Wang and X. Wang, [30] by replacing $\mathscr{C}$ by $\mathscr{D}$ and using Lemma 3. For simplicity, we also omit it here.

3. Large Deviations for Nonrandom Sums

In this section, we will give precise large deviations for nonrandom sums with dominated variation in multi-risk models under LENOD structures. Hereafter, M always represents a finite and positive constant whose value may vary in different places.

Theorem 9. *Let $\{X_{ij}, j \geq 1\}_{i=1}^k$ be LENOD random arrays. For all $i = 1, \ldots, k$, $\{X_{ij}, j \geq 1\}$ have common distribution function $F_i(x)$ and finite expectation μ_i, satisfying*

$$F_i(-x) = o\left(\overline{F}_i(x)\right), \quad x \longrightarrow \infty. \tag{28}$$

If $E|X_{ij}|^r 1_{\{X_{ij} \leq 0\}} < \infty$ for some $r > 1$ and $F_i \in \mathscr{D}$ for all $i = 1, \ldots, k$, then, for any fixed $\gamma > 0$, we have, for the lower bound,

$$\liminf_{n_1,\ldots,n_k \to \infty} \inf_{x \geq \Delta(k)} \frac{P\left(S(k; n_1, \ldots, n_k) - \sum_{i=1}^k n_i \mu_i > x\right)}{\sum_{i=1}^k \rho_{F_i}\left(|\mu_i|\right) L_{F_i}^k n_i \overline{F}_i(x)} \geq 1, \tag{29}$$

and, for the upper bound,

$$\limsup_{n_1,\ldots,n_k \to \infty} \sup_{x \geq \Delta(k)} \frac{P\left(S(k; n_1, \ldots, n_k) - \sum_{i=1}^k n_i \mu_i > x\right)}{\sum_{i=1}^k M_{F_i,\mu_i} L_{F_i}^{-k} n_i \overline{F}_i(x)} \leq 1, \tag{30}$$

where $\Delta(k) := \max\{\gamma n_i, i = 1, \ldots, k\}$ and $M_{F_i,\mu_i} := \mathbf{1}_{\{\mu_i \geq 0\}} + \rho_{F_i}^{-2}(|\mu_i|)\mathbf{1}_{\{\mu_i < 0\}}, i = 1, 2, \ldots, k$.

Remark 10. In Theorem 9, if we assume $F_i \in \mathscr{D} \cap \mathscr{L}$, then (29) and (30) hold with $\rho_{F_i}(|\mu_i|) = M_{F_i,\mu_i} \equiv 1$ for all $i = 1, 2, \ldots, k$. Particularly, if $F_i \in \mathscr{C}$ for all $i = 1, \ldots, k$, then $L_{F_i} = 1$. Hence, Theorem 9 reduces to Theorem 3.1 of S. J. Wang and W. S. Wang [21]. Moreover, if we also assume $\{X_{ij}, j \geq 1\}_{i=1}^k$ are nonnegative independent random arrays, one can easily see the condition (28) naturally holds. Therefore, (29) and (30) constitute the results of Theorem 3.1 of S. Wang and W. Wang [17].

Remark 11. If we suppose $k = 1$ or all $F_i(x)$ $(i = 1, \ldots, k)$ are the same distribution functions, then Theorem 9 implies Lemma 4. It means that Theorem 9 extends the result of Lemma 4 to multi-risk models. Particularly, if we also assume $F_1 \in \mathscr{C}$, then Theorem 9 reduces to Theorem 2.1 of Liu [14].

Proof. As usual, we use induction to prove Theorem 9. For the case of $k = 2$, we first show that

$$\liminf_{n_1,n_2 \to \infty} \inf_{x \geq \Delta(2)} \frac{P\left(S(2; n_1, n_2) - \sum_{i=1}^2 n_i \mu_i > x\right)}{\rho_{F_1}\left(|\mu_1|\right) L_{F_1}^2 n_1 \overline{F}_1(x) + \rho_{F_2}\left(|\mu_2|\right) L_{F_2}^2 n_2 \overline{F}_2(x)} \geq 1. \tag{31}$$

By using the same decomposition method of S. J. Wang and W. S. Wang [21], for any $0 < \varepsilon < 1$ and any $x > 0$, it holds that

$$\begin{aligned}
&P\left(S(2; n_1, n_2) - n_1\mu_1 - n_2\mu_2 > x\right) \\
&\quad \geq P\left(S_{n_1} - n_1\mu_1 > (1+\varepsilon)x, S_{n_2} - n_2\mu_2 > -\varepsilon x\right) \\
&\quad\quad + P\left(S_{n_2} - n_2\mu_2 > (1+\varepsilon)x, S_{n_1} - n_1\mu_1 > -\varepsilon x\right) \\
&\quad\quad - P\left(S_{n_1} - n_1\mu_1 > (1+\varepsilon)x, S_{n_2} - n_2\mu_2 > (1+\varepsilon)x\right) \\
&\quad := K_1 + K_2 - K_3.
\end{aligned} \tag{32}$$

For K_1 and K_2, by the same argument of relation (3.7) of S. J. Wang and W. S. Wang [21], and Lemmas 1 and 4, one can easily obtain, for any sufficiently small $\delta > 0$, that there exist sufficiently large n_1, n_2 such that, uniformly for $x \geq \Delta(2)$,

$$\begin{aligned}
K_1 &\geq (1-\delta)\rho_{F_1}\left(|\mu_1|\right) L_{F_1} n_1 \overline{F}_1((1+\varepsilon)x) \\
&\quad + o\left(\rho_{F_2}\left(|\mu_2|\right) L_{F_2}^2 n_2 \overline{F}_2(x)\right),
\end{aligned} \tag{33}$$

$$\begin{aligned}
K_2 &\geq (1-\delta)\rho_{F_2}\left(|\mu_2|\right) L_{F_2} n_2 \overline{F}_2((1+\varepsilon)x) \\
&\quad + o\left(\rho_{F_1}\left(|\mu_1|\right) L_{F_1}^2 n_1 \overline{F}_1(x)\right).
\end{aligned} \tag{34}$$

Next we turn to estimate K_3. Noticing that random arrays $\{X_{ij}, i = 1, 2, \ldots, k, j \geq 1\}$ are LENOD, again by Lemma 4, for sufficiently large n_1, n_2 and uniformly for $x \geq \Delta(2)$, we arrive at

$$\begin{aligned}
K_3 &\leq M(1+\delta)^2 M_{F_1,\mu_1} L_{F_1}^{-1} n_1 \overline{F}_1((1+\varepsilon)x) \\
&\quad \times M_{F_2,\mu_2} L_{F_2}^{-1} n_2 \overline{F}_2((1+\varepsilon)x) \\
&\leq M(1+\delta)^2 M_{F_1,\mu_1} L_{F_1}^{-1} M_{F_2,\mu_2} L_{F_2}^{-1} n_1 \overline{F}_1(x) n_2 \overline{F}_2(x) \\
&= o\left(\rho_{F_1}\left(|\mu_1|\right) L_{F_1}^2 n_1 \overline{F}_1(x) + \rho_{F_2}\left(|\mu_2|\right) L_{F_2}^2 n_2 \overline{F}_2(x)\right),
\end{aligned} \tag{35}$$

where in the last step we use the fact that $\lim_{n_i \to \infty} \sup_{x \geq \gamma n_i} n_i \overline{F}_i(x) = 0, i = 1, 2$. Furthermore, note that $\lim_{\varepsilon \downarrow 0} \liminf_{x \to \infty} (\overline{F}_i((1+\varepsilon)x)/\overline{F}_i(x)) = L_{F_i}$ for $i = 1, 2$; hence, for sufficiently small $\varepsilon > 0$ and sufficiently large x,

$$\overline{F}_i((1+\varepsilon)x) \geq \left(L_{F_i} - \delta\right) \overline{F}_i(x), \quad i = 1, 2. \tag{36}$$

Combining (32)–(36), it holds that the left-hand side of (32) is bounded from below by

$$\begin{aligned}
&(1-\delta)\left[\rho_{F_1}\left(|\mu_1|\right) L_{F_1}\left(L_{F_1} - \delta\right) n_1 \overline{F}_1(x)\right. \\
&\quad \left. + \rho_{F_2}\left(|\mu_2|\right) L_{F_2}\left(L_{F_2} - \delta\right) n_2 \overline{F}_2(x)\right] \\
&\quad + o\left(\rho_{F_1}\left(|\mu_1|\right) L_{F_1}^2 n_1 \overline{F}_1(x) + \rho_{F_2}\left(|\mu_2|\right) L_{F_2}^2 n_2 \overline{F}_2(x)\right).
\end{aligned} \tag{37}$$

Therefore, (31) can be derived directly from above by the arbitrariness of δ.

Next we show that

$$\limsup_{n_1,n_2\to\infty}\sup_{x\geq\Delta(2)}\frac{P\left(S\left(2;n_1,n_2\right)-n_1\mu_1-n_2\mu_2>x\right)}{M_{F_1,\mu_1}L_{F_1}^{-2}n_1\overline{F}_1(x)+M_{F_2,\mu_2}L_{F_2}^{-2}n_2\overline{F}_2(x)}\leq 1. \tag{38}$$

For any $\varepsilon\in(0,1/2)$ and $x>0$, Lemma 4 yields that, for any $0<\delta<1$, sufficiently large n_1,n_2, and uniformly for $x\geq\Delta(2)$,

$$\begin{aligned}
&P\left(S\left(2;n_1,n_2\right)-n_1\mu_1-n_2\mu_2>x\right)\\
&\quad\leq P\left(S_{n_1}-n_1\mu_1>(1-\varepsilon)x\right)\\
&\qquad+P\left(S_{n_2}-n_2\mu_2>(1-\varepsilon)x\right)\\
&\qquad+MP\left(S_{n_1}-n_1\mu_1>\varepsilon x\right)P\left(S_{n_2}-n_2\mu_2>\varepsilon x\right)\\
&\quad\leq(1+\delta)\left[M_{F_1,\mu_1}L_{F_1}^{-1}n_1\overline{F}_1((1-\varepsilon)x)\right.\\
&\qquad\qquad\left.+M_{F_2,\mu_2}L_{F_2}^{-1}n_2\overline{F}_2((1-\varepsilon)x)\right]\\
&\qquad+M(1+\delta)^2M_{F_1,\mu_1}L_{F_1}^{-1}n_1\overline{F}_1(\varepsilon x)M_{F_2,\mu_2}L_{F_2}^{-1}n_2\overline{F}_2(\varepsilon x).
\end{aligned}\tag{39}$$

Note that

$$\begin{aligned}
&\frac{n_1\overline{F}_1(\varepsilon x)n_2\overline{F}_2(\varepsilon x)}{M_{F_1,\mu_1}L_{F_1}^{-2}n_1\overline{F}_1(x)+M_{F_2,\mu_2}L_{F_2}^{-2}n_2\overline{F}_2(x)}\\
&\quad=\left(M_{F_1,\mu_1}L_{F_1}^{-2}\frac{\overline{F}_1(x)}{\overline{F}_1(\varepsilon x)}\frac{1}{n_2\overline{F}_2(\varepsilon x)}\right.\\
&\qquad\left.+M_{F_2,\mu_2}L_{F_2}^{-2}\frac{\overline{F}_2(x)}{\overline{F}_2(\varepsilon x)}\frac{1}{n_1\overline{F}_1(\varepsilon x)}\right)^{-1}.
\end{aligned}\tag{40}$$

It is easy to check that $\lim_{n_i\to\infty}\sup_{x\geq\gamma n_i}n_i\overline{F}_i(\varepsilon x)=0$, $i=1,2$. Moreover, since $F_1,F_2\in\mathscr{D}$, we have, for any $0<\varepsilon<1$, $\liminf_{n_1\to\infty}\inf_{x\geq\gamma n_1}(\overline{F}_1(x)/\overline{F}_1(\varepsilon x))>0$ and $\liminf_{n_2\to\infty}\inf_{x\geq\gamma n_2}(\overline{F}_2(x)/\overline{F}_2(\varepsilon x))>0$. Therefore, we have, as $n_1,n_2\to\infty$, uniformly for $x\geq\Delta(2)$,

$$\begin{aligned}
&n_1\overline{F}_1(\varepsilon x)n_2\overline{F}_2(\varepsilon x)\\
&\quad=o\left(M_{F_1,\mu_1}L_{F_1}^{-2}n_1\overline{F}_1(x)+M_{F_2,\mu_2}L_{F_2}^{-2}n_2\overline{F}_2(x)\right).
\end{aligned}\tag{41}$$

Furthermore, by the definition of L_{F_i}, for sufficiently small $\varepsilon>0$ and sufficiently large x, similarly as (36), it yields that

$$\overline{F}_i((1-\varepsilon)x)\leq\left(L_{F_i}^{-1}+\delta\right)\overline{F}_i(x),\quad i=1,2. \tag{42}$$

Substituting (41) and (42) into (39) and letting $\delta\downarrow 0$, (38) holds and Theorem 9 is proved in the case of $k=2$.

Now suppose (29) holds for $k-1$; for the case of k, using the similar argument as (32), it holds that

$$\begin{aligned}
&P\left(S\left(k;n_1,\ldots,n_k\right)-\sum_{i=1}^{k}n_i\mu_i>x\right)\\
&\quad\geq P\left(\sum_{i=1}^{k-1}S_{n_i}-\sum_{i=1}^{k-1}n_i\mu_i>(1+\varepsilon)x,S_{n_k}-n_k\mu_k>-\varepsilon x\right)\\
&\qquad+P\left(\sum_{i=1}^{k-1}S_{n_i}-\sum_{i=1}^{k-1}n_i\mu_i>-\varepsilon x,S_{n_k}-n_k\mu_k>(1+\varepsilon)x\right)\\
&\qquad-MP\left(\sum_{i=1}^{k-1}S_{n_i}-\sum_{i=1}^{k-1}n_i\mu_i>(1+\varepsilon)x\right)\\
&\qquad\times P\left(S_{n_k}-n_k\mu_k>(1+\varepsilon)x\right)\\
&\quad:=I_1+I_2-MI_3.
\end{aligned}\tag{43}$$

To estimate I_1 and I_2, similarly as (33) and (34), by Lemma 1, Lemma 4, and induction hypothesis, we have, for any $0<\delta<1$, all $i=1,\ldots,k$, sufficiently small $\varepsilon>0$, as $n_i\to\infty$,

$$\begin{aligned}
I_1&\geq(1-\delta)\sum_{i=1}^{k-1}\rho_{F_i}\left(\left|\mu_i\right|\right)L_{F_i}^{k-1}\left(L_{F_i}-\delta\right)n_i\overline{F}_i(x)\\
&\quad+o\left(\rho_{F_k}\left(\left|\mu_k\right|\right)L_{F_k}^{k}n_k\overline{F}_k(x)\right),\\
I_2&\geq(1-\delta)\rho_{F_k}\left(\left|\mu_k\right|\right)L_{F_k}^{k-1}\left(L_{F_k}-\delta\right)n_k\overline{F}_k(x)\\
&\quad+o\left(\sum_{i=1}^{k-1}\rho_{F_i}\left(\left|\mu_i\right|\right)L_{F_i}^{k}n_i\overline{F}_i(x)\right).
\end{aligned}\tag{44}$$

Finally for I_3, similarly as (35), we arrive at

$$I_3=o\left(\sum_{i=1}^{k}\rho_{F_i}\left(\left|\mu_i\right|\right)L_{F_i}^{k}n_i\overline{F}_i(x)\right). \tag{45}$$

Combining (43)–(45) and letting $\delta\downarrow 0$, we have

$$\liminf_{n_1,\ldots,n_k\to\infty}\inf_{x\geq\Delta(k)}\frac{P\left(S\left(k;n_1,\ldots,n_k\right)-\sum_{i=1}^{k}n_i\mu_i>x\right)}{\sum_{i=1}^{k}\rho_{F_i}\left(\left|\mu_i\right|\right)L_{F_i}^{k}n_i\overline{F}_i(x)}\geq 1. \tag{46}$$

To obtain the reverse inequality, for any $\varepsilon \in (0, 1/2)$ and $x > 0$, by Lemma 4 and induction hypothesis, it holds that, for any $0 < \delta < 1$, sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
&P\left(S(k; n_1, \ldots, n_k) - \sum_{i=1}^{k} n_i\mu_i > x\right) \\
&\le P\left(\sum_{i=1}^{k-1} S_{n_i} - \sum_{i=1}^{k-1} n_i\mu_i > (1-\varepsilon)x\right) \\
&\quad + P\left(S_{n_k} - n_k\mu_k > (1-\varepsilon)x\right) \\
&\quad + MP\left(\sum_{i=1}^{k-1} S_{n_i} - \sum_{i=1}^{k-1} n_i\mu_i > \varepsilon x\right) P\left(S_{n_k} - n_k\mu_k > \varepsilon x\right) \\
&\le (1+\delta)\left[\sum_{i=1}^{k-1} M_{F_i,\mu_i} L_{F_i}^{-(k-1)} n_i \overline{F}_i((1-\varepsilon)x)\right. \\
&\qquad \left. + M_{F_k,\mu_k} L_{F_k}^{-1} n_k \overline{F}_k((1-\varepsilon)x)\right] \\
&\quad + (1+\delta)^2 M \sum_{i=1}^{k-1} M_{F_i,\mu_i} L_{F_i}^{-(k-1)} n_i \overline{F}_i(\varepsilon x) \\
&\qquad \times M_{F_k,\mu_k} L_{F_k}^{-1} n_k \overline{F}_k(\varepsilon x) \\
&\le (1+\delta)\left[\sum_{i=1}^{k-1} M_{F_i,\mu_i} L_{F_i}^{-(k-1)} \left(L_{F_i}^{-1} + \delta\right) n_i \overline{F}_i(x)\right. \\
&\qquad \left. + M_{F_k,\mu_k} L_{F_k}^{-(k-1)} \left(L_{F_k}^{-1} + \delta\right) n_k \overline{F}_k(x)\right] \\
&\quad + o\left(\sum_{i=1}^{k} M_{F_i,\mu_i} L_{F_i}^{-k} n_i \overline{F}_i(x)\right).
\end{aligned} \tag{47}$$

Letting $\delta \downarrow 0$, we get

$$\limsup_{n_1,\ldots,n_k \to \infty} \sup_{x \ge \Delta(k)} \frac{P\left(S(k; n_1, \ldots, n_k) - \sum_{i=1}^{k} n_i\mu_i > x\right)}{\sum_{i=1}^{k} M_{F_i,\mu_i} L_{F_i}^{-k} n_i \overline{F}_i(x)} \le 1. \tag{48}$$

Therefore, Theorem 9 follows from (46) and (48) immediately. □

4. Large Deviations for Random Sums

In this section, we will give precise large deviations for random sums with dominated variation in multi-risk models under LENOD structures.

Remark 12. It is worth to mention that Assumption B was firstly introduced by Chen et al. [15]. Furthermore, one can easily see that both Assumptions A and B imply that

$$\frac{N(t)}{\lambda(t)} \xrightarrow{P} 1. \tag{49}$$

Theorem 13. *Let $\{X_{ij}, j \ge 1\}_{i=1}^{k}$ be LENOD random arrays satisfying the conditions of Theorem 9, and let $\{N_i(t)\}_{i=1}^{k}$ be independent nonnegative integer-valued process independent of $\{X_{ij}, j \ge 1\}_{i=1}^{k}$, satisfying Assumptions A and B; then, for any fixed $\gamma > 0$, we have, for the lower bound,*

$$\liminf_{t \to \infty} \inf_{x \ge \Gamma(k)} \frac{P\left(S(k; t) - \sum_{i=1}^{k} \lambda_i(t)\mu_i > x\right)}{\sum_{i=1}^{k} \rho_{F_i}(|\mu_i|) L_{F_i}^{k+1} \lambda_i(t) \overline{F}_i(x)} \le 1, \tag{50}$$

and, for the upper bound,

$$\limsup_{t \to \infty} \sup_{x \ge \Gamma(k)} \frac{P\left(S(k; t) - \sum_{i=1}^{k} \lambda_i(t)\mu_i > x\right)}{\sum_{i=1}^{k} M_{F_i,\mu_i} L_{F_i}^{-(k+1)} \lambda_i(t) \overline{F}_i(x)} \le 1, \tag{51}$$

where $\Gamma(k) := \max\{\gamma\lambda_i(t), i = 1, \ldots, k\}$ and M_{F_i,μ_i} is defined in Theorem 9.

Remark 14. In Theorem 13, if we assume $F_i \in \mathscr{D} \cap \mathscr{L}$, then (50) and (51) hold with $\rho_{F_i}(|\mu_i|) = M_{F_i,\mu_i} \equiv 1$ for all $i = 1, 2, \ldots, k$. Particularly, if $F_i \in \mathscr{C}$ for all $i = 1, \ldots, k$, then $L_{F_i} = 1$. Hence, Theorem 13 reduces to Theorem 4.1 of S. J. Wang and W. S. Wang [21]. Moreover, if we also assume $\{X_{ij}, j \ge 1\}_{i=1}^{k}$ are nonnegative independent random arrays, one can easily see the condition (28) naturally holds. Therefore, (50) and (51) constitute the results of Theorem 4.1 of S. Wang and W. Wang [17].

Remark 15. If we suppose $k = 1$ or all $F_i(x)$ $(i = 1, \ldots, k)$ are the same distribution function, then Theorem 13 implies Lemma 5. It means that Theorem 13 extends the result of Lemma 5 to multi-risk models. Particularly, if we also assume $F_1 \in \mathscr{C}$, then Theorem 13 indicates Theorem 3.1 of Chen et al. [15] and Theorem 3.2 of S. Wang and X. Wang [30].

Proof. Similarly as the proof of Theorem 9, again by induction, it is sufficient to show that Theorem 13 holds for $k = 2$. We first show that

$$\liminf_{t \to \infty} \inf_{x \ge \Gamma(2)} \frac{P\left(S(2; t) - \lambda_1(t)\mu_1 - \lambda_2(t)\mu_2 > x\right)}{\rho_{F_1}(|\mu_1|) L_{F_1}^{3} \lambda_1(t) \overline{F}_1(x) + \rho_{F_2}(|\mu_2|) L_{F_2}^{3} \lambda_2(t) \overline{F}_2(x)} \ge 1. \tag{52}$$

The similar argument as (32) yields that, for any $0 < \varepsilon < 1$ and any $x > 0$,

$$\begin{aligned}
&P\left(S(2; t) - \lambda_1(t)\mu_1 - \lambda_2(t)\mu_2 > x\right) \\
&\ge P\left(S_{N_1(t)} - \lambda_1(t)\mu_1 > (1+\varepsilon)x,\right. \\
&\qquad \left. S_{N_2(t)} - \lambda_2(t)\mu_2 > -\varepsilon x\right)
\end{aligned}$$

$$\begin{aligned}&+P\left(S_{N_2(t)}-\lambda_2(t)\mu_2>(1+\varepsilon)x,\right.\\&\qquad\left.S_{N_1(t)}-\lambda_1(t)\mu_1>-\varepsilon x\right)\\&-P\left(S_{N_1(t)}-\lambda_1(t)\mu_1>(1+\varepsilon)x,\right.\\&\qquad\left.S_{N_2(t)}-\lambda_2(t)\mu_2>(1+\varepsilon)x\right)\\&:=J_1+J_2-J_3.\end{aligned}\tag{53}$$

To deal with J_1, note that

$$\begin{aligned}J_1\geq{}&P\left(S_{N_1(t)}-\lambda_1(t)\mu_1>(1+\varepsilon)x\right)\\&-P\left(S_{N_2(t)}-\lambda_2(t)\mu_2\leq-\varepsilon x\right).\end{aligned}\tag{54}$$

It follows from Lemmas 2 and 5 that, for any $\delta>0$,

$$\begin{aligned}J_1\geq{}&(1-\delta)\rho_{F_1}(|\mu_1|)L_{F_1}^2\left(L_{F_1}-\delta\right)\lambda_1(t)\overline{F}_1(x)\\&+o\left(\rho_{F_2}(|\mu_2|)L_{F_2}^3\lambda_2(t)\overline{F}_2(x)\right),\end{aligned}\tag{55}$$

$$\begin{aligned}J_2\geq{}&(1-\delta)\rho_{F_2}(|\mu_2|)L_{F_2}^2\left(L_{F_2}-\delta\right)\lambda_2(t)\overline{F}_2(x)\\&+o\left(\rho_{F_1}(|\mu_1|)L_{F_1}^3\lambda_1(t)\overline{F}_1(x)\right).\end{aligned}\tag{56}$$

Next we turn to estimate J_3. Note that $\{N_i(t)\}_{i=1}^2$ being independent and $\{X_{ij},\ j\geq1\}_{i=1}^2$ being LENOD, by Lemma 5, one gets

$$\begin{aligned}J_3\leq{}&MP\left(S_{N_1(t)}-\lambda_1(t)\mu_1>(1+\varepsilon)x\right)\\&\times P\left(S_{N_2(t)}-\lambda_2(t)\mu_2>(1+\varepsilon)x\right)\\\leq{}&M(1+\delta)^2M_{F_1,\mu_1}L_{F_1}^{-2}\lambda_1(t)\overline{F}_1(x)M_{F_2,\mu_2}\\&\times L_{F_2}^{-2}\lambda_2(t)\overline{F}_2(x)\\={}&o\left(\rho_{F_1}(|\mu_1|)L_{F_1}^3\lambda_1(t)\overline{F}_1(x)\right.\\&\left.+\rho_{F_2}(|\mu_2|)L_{F_2}^3\lambda_2(t)\overline{F}_2(x)\right),\end{aligned}\tag{57}$$

where in the last step we use the fact that $\lim_{t\to\infty}\sup_{x\geq\gamma\lambda_i(t)}\lambda_i(t)\overline{F}_i(x)=0$, $i=1,2$. Therefore, by (53)–(57), for any sufficiently large t and uniformly for $x\geq\Gamma(2)$,

$$\begin{aligned}&P\left(S(2;t)-\lambda_1(t)\mu_1-\lambda_2(t)\mu_2>x\right)\\&\geq(1-\delta)\left[\rho_{F_1}(|\mu_1|)L_{F_1}^2\left(L_{F_1}-\delta\right)\lambda_1(t)\overline{F}_1(x)\right.\\&\qquad\left.+\rho_{F_2}(|\mu_2|)L_{F_2}^2\left(L_{F_2}-\delta\right)\lambda_2(t)\overline{F}_2(x)\right]\\&\quad+o\left(\rho_{F_1}(|\mu_1|)L_{F_1}^3\lambda_1(t)\overline{F}_1(x)\right.\\&\qquad\left.+\rho_{F_2}(|\mu_2|)L_{F_2}^3\lambda_2(t)\overline{F}_2(x)\right).\end{aligned}\tag{58}$$

Letting $\delta\downarrow0$, (52) derives directly from the inequality above.

Next we show that

$$\limsup_{t\to\infty}\sup_{x\geq\Gamma(2)}\frac{P\left(S(2;t)-\lambda_1(t)\mu_1-\lambda_2(t)\mu_2>x\right)}{M_{F_1,\mu_1}L_{F_1}^{-3}\lambda_1(t)\overline{F}_1(x)+M_{F_2,\mu_2}L_{F_2}^{-3}\lambda_2(t)\overline{F}_2(x)}\leq1.\tag{59}$$

For any $\in(0,1/2)$, $x>0$, $0<\delta<1$, by Lemma 5, (41), and (42), we arrive at

$$\begin{aligned}&P\left(S(2;t)-\lambda_1(t)\mu_1-\lambda_2(t)\mu_2>x\right)\\&\leq P\left(S_{N_1(t)}-\lambda_1(t)\mu_1>(1-\varepsilon)x\right)\\&\quad+P\left(S_{N_2(t)}-\lambda_2(t)\mu_2>(1-\varepsilon)x\right)\\&\quad+MP\left(S_{N_1(t)}-\lambda_1(t)\mu_1>\varepsilon x\right)\\&\quad\times P\left(S_{N_2(t)}-\lambda_2(t)\mu_2>\varepsilon x\right)\\&\leq(1+\delta)\left[M_{F_1,\mu_1}L_{F_1}^{-2}\lambda_1(t)\overline{F}_1((1-\varepsilon)x)\right.\\&\qquad\left.+M_{F_2,\mu_2}L_{F_2}^{-2}\lambda_2(t)\overline{F}_2((1-\varepsilon)x)\right]\\&\quad+(1+\delta)^2MM_{F_1,\mu_1}L_{F_1}^{-2}\lambda_1(t)\overline{F}_1(\varepsilon x)M_{F_2,\mu_2}\\&\quad\times L_{F_2}^{-2}\lambda_2(t)\overline{F}_2(\varepsilon x)\\&\leq(1+\delta)\left[M_{F_1,\mu_1}L_{F_1}^{-2}\left(L_{F_1}^{-1}+\delta\right)\lambda_1(t)\overline{F}_1(x)\right.\\&\qquad\left.+M_{F_2,\mu_2}L_{F_2}^{-2}\left(L_{F_2}^{-1}+\delta\right)\lambda_2(t)\overline{F}_2(x)\right]\\&\quad+o\left(M_{F_1,\mu_1}L_{F_1}^{-3}\lambda_1(t)\overline{F}_1(x)\right.\\&\qquad\left.+M_{F_2,\mu_2}L_{F_2}^{-3}\lambda_2(t)\overline{F}_2(x)\right).\end{aligned}\tag{60}$$

Letting $\delta\downarrow0$, thus we get (59).

Combining (52) and (59), it indicates that Theorem 13 holds for $k=2$. Finally, by induction similar as Theorem 9, the proof of Theorem 13 is now completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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