Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 974968, 11 pages
http://dx.doi.org/10.1155/2014/974968

Research Article

Positive Solutions for Impulsive Differential Equations with Mixed Monotonicity and Optimal Control

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Received 7 January 2014; Accepted 13 June 2014; Published 7 July 2014

Academic Editor: Yonghong Wu

We consider positive solutions and optimal control problem for a second order impulsive differential equation with mixed monotone terms. Firstly, by using a fixed point theorem of mixed monotone operator, we study positive solutions of the boundary value problem for impulsive differential equations with mixed monotone terms, and sufficient conditions for existence and uniqueness of positive solutions will be established. Also, we study positive solutions of the initial value problem for our system. Moreover, we investigate the control problem of positive solutions to our equations, and then, we prove the existence of an optimal control and its stability. In addition, related examples will be given for illustrations.

1. Introduction

Mixed monotone operators have been introduced by Guo and Lakshmikantham [1] in 1987. Recently, many authors have investigated those kinds of operators in Banach spaces and obtained a lot of interesting and important results (see [2–9]). In this work, by using a fixed point theorem of mixed monotone operator, we study the existence and uniqueness of positive solutions to the boundary value problem of impulsive differential equations with mixed monotone terms:

$$(\mathrm{BP};u)\begin{cases} -x''(t) = a(t)f(t,x(t),x(t)) + u(t), \\ \qquad t\in(0,1)\setminus\{t_1,t_2,\dots,t_m\}, \\ \Delta x|_{t=t_k} = I_k(x(t_k),x(t_k)), \\ \qquad k=1,2,\dots,m, \\ x(0)=b_0, \qquad x'(1)=0. \end{cases} \tag{1}$$

Here, $J=[0,1]$, $R^+=[0,+\infty)$, $f\in C[J\times R^+\times R^+,R^+]$, and $a\in C[J,R^+]$ with $\min_{t\in J}a(t)>0$ on any subinterval of J. A function u is given on $[0,1]$, $0<t_1<t_2<\cdots<t_m<1$, $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t=t_k$ and $\Delta x|_{t=t_k}=x(t_k^+)-x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t=t_k$, respectively. Also, I_k is a given function in $C[R^+\times R^+,R^+]$, $k=1,2,\dots,m$. Furthermore, $b_0>0$ is a given constant.

For convenience, we put $J_0=[0,t_1]$, $J_1=(t_1,t_2],\dots,J_{m-1}=(t_{m-1},t_m]$, $J_m=(t_m,1]$, and $J'=J\setminus\{t_1,t_2,\dots,t_m\}$.

Then, we study the existence and uniqueness of positive solutions to initial value problem as follows:

$$(\mathrm{IP};u)\begin{cases} -x''(t) = a(t)f(t,x(t),x(t)) + u(t), \\ \qquad t\in(0,1)\setminus\{t_1,t_2,\dots,t_m\}, \\ \Delta x|_{t=t_k} = I_k(x(t_k),x(t_k)), \\ \qquad k=1,2,\dots,m, \\ x(0)=b_0, \qquad x'(0)=b_1, \end{cases} \tag{2}$$

where $f\in C[J\times R^+\times R^+,R]$ and $b_0>0$ and $b_1\geqslant 0$ are given constants. Moreover, we consider the related optimal control problem (OP) of (2) as follows:

Problem (OP). Find an optimal control $u^*\in\mathcal{U}_M$ such that

$$\pi(u^*) = \inf_{u\in\mathcal{U}_M}\pi(u). \tag{3}$$

Here, $\mathcal{U}_M$ is a control space defined by

$$\mathcal{U}_M := \left\{u\in L^2(0,1) \mid -M\leqslant u(t)\leqslant 0 \text{ a.e. } t\in[0,1]\right\}, \tag{4}$$

where M is a fixed positive number and $\pi(u)$ is the cost functional defined by

$$\pi(u) := \frac{1}{2}\int_0^1 \left|(x - x_d)(t)\right|^2 dt + x(1) + \frac{1}{2}\int_0^1 |u(t)|^2 dt, \quad (5)$$

where $u \in \mathcal{U}_M$ is the control, function x is a unique positive solution to the state problem (IP; u), and x_d is the given desired target profiles in $L^2(0,1)$.

The existence and uniqueness of solutions for boundary value problem have been discussed by many authors, and the boundary value problem of impulsive differential equation is a new and important branch of the differential equation theory, which has an extensive physical, chemical, biological, and engineering background, realistic mathematical model, and so forth (see [10–14]). The theory on mixed monotone operators has attracted much attention and has been widely studied, such as Guo and Lakshmikantham [1] have applied the monotone iterative technique to discuss an initial value problem of differential equations without impulse:

$$\begin{gathered} u'(t) = f(t, u(t), u(t)), \quad t \in [0, a], \\ u(0) = x_0. \end{gathered} \quad (6)$$

They obtained the existence of the coupled quasisolutions by mixed monotone sequence of coupled quasi upper and lower solutions. Zhai and Zhang [7] showed a new fixed point theorem for differential equations with mixed monotone term. Jinli and Yihai [15] considered the following problem:

$$\begin{gathered} u''(x) - f(x, u, u) = \theta, \quad x \in J,\ x \neq x_i,\ i = 1, 2, \ldots, m, \\ \Delta u|_{x=x_i} = I_i(u(x_i)), \quad i = 1, 2, \ldots, m, \\ \Delta u'\big|_{x=x_i} = \widetilde{I}_i(u(x_i)), \quad i = 1, 2, \ldots, m, \\ u(0) = w_0, \qquad u'(0) = w_1. \end{gathered} \quad (7)$$

They used the coupled fixed point theorem for mixed monotone condensing operators to obtain the existence and uniqueness of solutions.

Also, there is a vast literature on optimal control problems (see [16–19]). For instance, with a fixed point theorem of generalized concave operator, the authors [19] have studied the optimal control problem of positive solutions to the following second order impulsive differential equation:

$$\begin{gathered} -x''(t) = f(t, x(t)) + u(t), \quad t \in (0, T) \setminus \{t_1, t_2, \ldots, t_m\}, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ \Delta x'\big|_{t=t_k} = \overline{I}_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ x(0) = a, \qquad x'(0) = b. \end{gathered} \quad (8)$$

In this paper, we investigate impulsive differential equations with mixed monotone terms, which have variable coefficient nonlinear terms. Then, we prove the existence and uniqueness of positive solutions to (BP; u) and (IP; u). Moreover, we prove the existence of an optimal control to (OP) and its stability.

The plan of this paper is as follows. In Section 2, we recall the fundamentals of a fixed point theorem of mixed monotone operators. In Section 3, we deal with the existence and uniqueness of positive solutions to (BP; u). In Section 4, we show the existence and uniqueness of positive solutions to (IP; u). In Section 5, we prove the existence of an optimal control to (OP) and its stability. In the final Section 6, related examples on the main results are given.

1.1. Notations. Throughout this paper, we use the following notations.

Let $PC[J, R] := \{x \mid x : J \to R, x(t)$ is continuous at $t \neq t_k$ and left continuous at $t = t_k, x(t_k^+)$ exists, $k = 1, 2, \ldots, m\}$. Then, we can easily find that $PC[J, R]$ is a Banach space with the norm $|x|_{PC} := \sup_{t\in J}|x|$.

We put $H := L^2(J)$ with the usual real Hilbert structure and denote by $|\cdot|_H$ the norm in H, for simplicity, and $W^{2,1}(J, R)$ is a usual Sobolev space, namely,

$$W^{2,1}(J, R) = \left\{f \in L^1(J) : D^k f \in L^1(J), k = 1, 2\right\}, \quad (9)$$

where $D^k f$ denoted the kth derivative of f.

Also, N_i and N_i', $i = 1, 2, 3, \ldots$, denote positive (or nonnegative) constants only depending on their arguments.

2. A Fixed Point Theorem of Mixed Monotone Operator

In this section, we recall the fundamentals of a fixed point theorem of mixed monotone operator.

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, that is, $x \leqslant y$ if and only if $y - x \in P$. If $x \leqslant y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E. Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies

(i) $x \in P,\ \lambda \geq 0 \Rightarrow \lambda x \in P$;

(ii) $x \in P,\ -x \in P \Rightarrow x = \theta$.

Putting $\mathring{P} = \{x \in P \mid x$ is an interior point of $P\}$, a cone P is said to be solid if its interior $\mathring{P}$ is nonempty. Moreover, P is called normal if there exists a constant $M > 0$ such that, for all $x, y \in E, \theta \leqslant x \leqslant y$ implies $\|x\| \leqslant M\|y\|$; in this case M is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leqslant x \leqslant x_2\}$ is called the order interval between x_1 and x_2.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \geqslant y \geqslant \mu x$. Clearly, $\sim$ is an equivalence relation. Giving $h > \theta$ (i.e., $h \geqslant \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$ is convex and $\lambda P_h = P_h$ for all $\lambda > 0$. If $\mathring{P} \neq \emptyset$ and $h \in \mathring{P}$, it is clear that $P_h = P$. For other detailed properties of cones, we refer to the monograph by Guo and Lakshmikantham [20].

Definition 1 (cf. [1, 2]). $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, that is, $u_i, v_i \in P$ $(i = 1, 2)$, $u_1 \leqslant u_2$, $v_1 \geqslant v_2$ implies that $A(u_1, v_1) \leqslant A(u_2, v_2)$. Element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Here, one recalls the following fixed point theorem of mixed monotone operator which has been established by Zhai and Zhang [7].

Proposition 2 (cf. [7, Theorem 2.1]). *Let P be a normal cone of a real Banach space E. Also, let $A : P \times P \to P$ be a mixed monotone operator. Assume that*

(A_1) *there exists $h \in P$ with $h \neq \theta$ such that $A(h, h) \in P_h$;*

(A_2) *for any $u, v \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that $A(tu, t^{-1}v) \geqslant \varphi(t)A(u, v)$.*

Then operator A has a unique fixed point x^* in P_h. Moreover, for any initial $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \qquad n = 1, 2, \ldots, \tag{10}$$

one has $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

By applying Proposition 2, one shows the existence and uniqueness of the positive solution to (BP; u) and (IP; u) on J.

3. Boundary Value Problem (BP; u)

In this section, we show the existence and uniqueness of the positive solution to (BP; u) by applying a fixed point theorem of mixed monotone operator (Proposition 2).

Throughout this section, we assume the following conditions (H_1)–(H_3):

(H_1) $f : J \times R^+ \times R^+ \to R^+$, $f(t, x, y)$ is nondecreasing in x for each $t \in J$ and $y \in R^+$, and is nonincreasing in y for each $t \in J$ and $x \in R^+$. Also, $f(t, (1/2), 1) > 0$ for all $t \in J$.

(H_2) For each $k = 1, 2, \ldots, m$, $I_k : R^+ \times R^+ \to R^+$, $I_k(x, y)$ is nondecreasing in x for each $y \in R^+$ and is nonincreasing in y for each $x \in R^+$.

(H_3) For all $\gamma \in (0, 1)$, there exists a constant $\varphi_1(\gamma), \varphi_2(\gamma) \in (\gamma, 1]$ such that

$$f(t, \gamma x, \gamma^{-1} y) \geqslant \varphi_1(\gamma) f(t, x, y), \qquad I_k(\gamma x, \gamma^{-1} y) \geqslant \varphi_2(\gamma) I_k(x, y). \tag{11}$$

for any $x, y \in R^+$, any $t \in J$, and any $k = 1, 2, \ldots, m$.

We give the definition of solutions to (BP; u).

Definition 3. Let $u \in H$, and let b_0 be a given constant. Then, a function $x \in PC[J, R] \cap W^{2,1}(J, R)$ is called a solution to (BP; u) on J if it satisfies (1).

Now, we mention our first main theorem in this paper, which is concerned with the existence-uniqueness of the positive solution to (BP; u) on J.

Theorem 4. *Assuming the conditions (H_1)–(H_3), and having M has a fixed positive constant, then for each function $u \in H$ with $0 \leqslant u(t) \leqslant M$ a.e. $t \in J$, there exists a unique positive solution to (BP; u) on J.*

Here, we give the key lemma, which is concerned with the characterization of solutions to (BP; u).

Lemma 5. *Assume the same conditions as in Theorem 4. Then, $x \in PC[J, R] \cap W^{2,1}(J, R)$ is a solution to (BP; u) on J if and only if $x \in PC[J, R]$ satisfies the following integral equation:*

$$x(t) = b_0 + \int_0^1 G(t, s) \{a(s) f(s, x(s), x(s)) + u(s)\} ds + \sum_{0<t_k<t} I_k(x(t_k), x(t_k)), \quad \forall t \in J, \tag{12}$$

where

$$G(t, s) = \begin{cases} t, & 0 \leqslant t \leqslant s \leqslant 1, \\ s, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{13}$$

Proof. Firstly, integrating $-x''(s) = a(s) f(s, x(s), x(s)) + u(s)$ from 0 to t, we obtain

$$x'(t) = x'(0) - \int_0^t \{a(s) f(s, x(s), x(s)) + u(s)\} ds, \quad \forall t \in J. \tag{14}$$

Again, integrating (14), we have

$$x(t) = x(0) + x'(0) t - \int_0^t (t - s) \{a(s) f(s, x(s), x(s)) + u(s)\} ds + \sum_{0<t_k<t} \{x(t_k^+) - x(t_k)\}, \quad \forall t \in J. \tag{15}$$

From (14) with $x'(1) = 0$, we infer that

$$x'(0) = \int_0^1 \{a(s) f(s, x(s), x(s)) + u(s)\} ds. \tag{16}$$

Hence, from $x(0) = b_0$, (15), and (16), we can find that

$$x(t) = b_0 + \int_0^1 G(t, s) \{a(s) f(s, x(s), x(s)) + u(s)\} ds + \sum_{0<t_k<t} I_k(x(t_k), x(t_k)), \quad \forall t \in J, \tag{17}$$

where $G(t, s)$ is the function defined as in (13). Thus, the proof is completed. □

By Lemma 5, we can show the solvability of (BP; u). In fact, we define an operator $A : PC[J,R] \times PC[J,R] \to PC[J,R]$ by

$$\begin{aligned} A(x,y)(t) &= b_0 + \int_0^1 G(t,s)\{a(s) f(s,x(s),y(s)) + u(s)\} ds \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k), y(t_k)) \quad \forall x, y \in PC[J,R],\ t \in J, \end{aligned} \tag{18}$$

where $G(t,s)$ is the function defined as in (13).

We can easily find that the following lemma holds.

Lemma 6. *Assume the same conditions as in Theorem 4. Then, $x \in PC[J,R] \cap W^{2,1}(J,R)$ is a solution to (BP; u) on J if and only if $x \in PC[J,R]$ is the fixed point of the operator $A : PC[J,R] \times PC[J,R] \to PC[J,R]$ defined by* (18).

Taking account of Proposition 2 and Lemmas 5 and 6, one can prove Theorem 4 concerning the existence and uniqueness of the positive solution to (BP; u) on J.

Proof of Theorem 4. By applying a fixed point theorem of mixed monotone operator (Proposition 2), we show the existence and uniqueness of the positive solution to (BP; u) on J.

To do so, set

$$\widetilde{P} := \{x \in PC[J,R]; x(t) \geqslant 0, \forall t \in J\}. \tag{19}$$

Clearly, $\widetilde{P}$ is a normal cone in $PC[J,R]$ and the normality constant is 1.

Let $A : PC[J,R] \times PC[J,R] \to PC[J,R]$ be the operator defined by (18). Then, we infer from (H_1), (H_2), (13), and $u(t) \geqslant 0$ a.e. $t \in J$ that

$$A(x,y)(t) \geqslant 0 \quad \forall x, y \in \widetilde{P},\ t \in J. \tag{20}$$

Thus, we see that $A : \widetilde{P} \times \widetilde{P} \to \widetilde{P}$.

Firstly, by (H_1), (H_2), and (18), we can easily prove that $A : \widetilde{P} \times \widetilde{P} \to \widetilde{P}$ is a mixed monotone operator.

Next, we show (A_2). Put

$$\varphi(\gamma) = \min\{\varphi_1(\gamma), \varphi_2(\gamma)\}, \quad \gamma \in (0,1). \tag{21}$$

Then, we see from (H_3) that $\varphi(\gamma) \in (\gamma, 1]$. Therefore, for any $\gamma \in (0,1)$ and $x, y \in \widetilde{P}$, we observe from (H_1)–(H_3), (13), and $u(t) \geqslant 0$ a.e. $t \in J$ that

$$\begin{aligned} A(\gamma x, \gamma^{-1} y)(t) &= b_0 + \int_0^1 G(t,s)\{a(s) f(s, \gamma x(s), \gamma^{-1} y(s)) \\ &\qquad + u(s)\} ds \\ &\quad + \sum_{0<t_k<t} I_k(\gamma x(t_k), \gamma^{-1} y(t_k)) \\ &\geqslant b_0 + \varphi_1(\gamma) \int_0^1 G(t,s)\{a(s) f(s, x(s), y(s)) \\ &\qquad + u(s)\} ds \\ &\quad + \varphi_2(\gamma) \sum_{0<t_k<t} I_k(x(t_k), y(t_k)) \\ &\geqslant \varphi(\gamma) A(x,y)(t), \quad \forall t \in J, \end{aligned} \tag{22}$$

which implies that

$$A(\gamma x, \gamma^{-1} y) \geqslant \varphi(\gamma) A(x,y) \quad \forall x, y \in \widetilde{P},\ \gamma \in (0,1). \tag{23}$$

Thus, the condition (A_2) holds.

Now, we show (A_1), defining a function h by

$$h(t) := \frac{1}{2} + \int_0^1 G(t,s) ds, \quad \forall t \in J; \tag{24}$$

hence, $h(t) = -(t^2/2) + t + (1/2)$ for all $t \in J$, then we can easily see that $(1/2) \leqslant h(t) \leqslant 1$ for all $t \in J$.

Now we show that $A(h,h) \in \widetilde{P}_h$. Set

$$r_1 = \min_{t \in J} f\left(t, \frac{1}{2}, 1\right), \qquad r_2 = \max_{t \in J} f\left(t, 1, \frac{1}{2}\right), \tag{25}$$

then, $0 < r_1 \leqslant r_2$.

Note that $a(t)$ has maximum and minimum on J, since $a(t) \in C[J, R^+]$ with $\min_{t \in J} a(t) > 0$ on any subinterval of J. So, let

$$a_{\min} = \min_{t \in J} a(t), \qquad a_{\max} = \max_{t \in J} a(t). \tag{26}$$

Here, put $r_3 := \min\{2b_0, r_1 a_{\min}\}$. Then, from ($H_1$), ($H_2$), (13), and $u(t) \geqslant 0$ a.e. $t \in J$, it follows that

$$\begin{aligned} A(h,h)(t) &= b_0 + \int_0^1 G(t,s)\{a(s) f(s, h(s), h(s)) + u(s)\} ds \\ &\quad + \sum_{0<t_k<t} I_k(h(t_k), h(t_k)) \\ &\geqslant b_0 + a_{\min} \int_0^1 G(t,s) f\left(s, \frac{1}{2}, 1\right) ds \\ &\geqslant b_0 + r_1 a_{\min} \int_0^1 G(t,s) ds \\ &\geqslant r_3 h(t), \quad \forall t \in J. \end{aligned} \tag{27}$$

Also, we have

$$\begin{aligned} A(h,h)(t) &= b_0 + \int_0^1 G(t,s)\{a(s) f(s,h(s),h(s)) + u(s)\}\,ds \\ &\quad + \sum_{0<t_k<t} I_k(h(t_k),h(t_k)) \\ &\leqslant b_0 + a_{\max}\int_0^1 G(t,s) f\left(s,1,\frac{1}{2}\right) ds \\ &\quad + M\int_0^1 G(t,s)\,ds + \sum_{0<t_k<t} I_k\left(1,\frac{1}{2}\right) \\ &\leqslant \left[2b_0 + r_2 a_{\max} + M + 2\sum_{k=1}^{m} I_k\left(1,\frac{1}{2}\right)\right] h(t), \\ &\qquad \forall t \in J. \end{aligned} \tag{28}$$

Thus, we observe that

$$r_3 h \leqslant A(h,h) \leqslant \left[2b_0 + r_2 a_{\max} + M + 2\sum_{k=1}^{m} I_k\left(1,\frac{1}{2}\right)\right] h, \tag{29}$$

which implies that $A(h,h) \in \widetilde{P}_h$.

By arguments as above, we see that the operator $A : \widetilde{P} \times \widetilde{P} \to \widetilde{P}$ defined by (18) satisfies conditions (A_1) and (A_2) in Proposition 2. Therefore, by applying Proposition 2, we conclude that an operator equation $x = A(x,x)$ has a unique solution in $\widetilde{P}_h$; hence there exists a unique positive solution to (BP; u) on J. □

4. Initial Value Problem (IP; u)

In this section, we show the existence-uniqueness of the positive solution to (IP; u) on J by arguments similar to (BP; u).

Throughout this section, we assume the following conditions $(H_1)'$, $(H_3)'$:

$(H_1)'$ $f : J \times R^+ \times R^+ \to R$, such that $f(t,x,y) \leqslant 0$ for all $t \in J$ and $x, y \in R^+$. Also, $f(t,x,y)$ is nonincreasing in x for each $t \in J$ and $y \in R^+$ and is nondecreasing in y for each $t \in J$ and $x \in R^+$. Moreover, $f(t,(1/2),1) < 0$ for all $t \in J$.

$(H_3)'$ For all $\gamma \in (0,1)$, there exists a constant $\varphi_1(\gamma), \varphi_2(\gamma) \in (\gamma,1]$ such that

$$\begin{aligned} f\left(t,\gamma x,\gamma^{-1} y\right) &\leqslant \varphi_1(\gamma) f(t,x,y), \\ I_k\left(\gamma x,\gamma^{-1} y\right) &\geqslant \varphi_2(\gamma) I_k(x,y), \end{aligned} \tag{30}$$

for any $x, y \in R^+$, any $t \in J$, and any $k = 1,2,\ldots,m$.

Here, we give the definition of solutions to (IP; u).

Definition 7. Let $u \in H$ and b_0 and b_1 as given constants. Then, a function $x \in PC[J,R] \cap W^{2,1}(J,R)$ is called a solution to (IP; u) on J if it satisfies (2).

Now, we mention our second main theorem in this paper, which is concerned with the existence-uniqueness of the positive solution to (IP; u) on J.

Theorem 8. *Assuming the conditions (H_2), $(H_1)'$, and $(H_3)'$ and having M has a fixed positive constant. Then, for each function $u \in H$ with $-M \leqslant u(t) \leqslant 0$ a.e. $t \in J$, there exists a unique positive solution to (IP; u) on J.*

Based on the proof of Lemma 5 (cf. (15)), one can get the following key lemma concerning the characterization of solutions to (IP; u).

Lemma 9. *Assume the same conditions as in Theorem 8. Then, $x \in PC[J,R] \cap W^{2,1}(J,R)$ is a solution to (IP; u) on J if and only if $x \in PC[J,R]$ satisfies the following integral equations:*

$$\begin{aligned} x(t) &= b_0 + b_1 t - \int_0^t (t-s)\{a(s) f(s,x(s),x(s)) + u(s)\}\,ds \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k),x(t_k)), \quad \forall t \in J. \end{aligned} \tag{31}$$

By Lemma 9 and Proposition 2, one can show Theorem 8 concerning the existence-uniqueness of the positive solution to (IP; u) on J.

Proof of Theorem 8. Now, we define an operator $A : PC[J,R] \times PC[J,R] \to PC[J,R]$ by

$$\begin{aligned} A(x,y)(t) &= b_0 + b_1 t - \int_0^t (t-s)\{a(s) f(s,x(s),y(s)) + u(s)\}\,ds \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k),y(t_k)) \quad \forall x,y \in PC[J,R],\ t \in J. \end{aligned} \tag{32}$$

Then, we have to find the fixed point of the operator A in order to show the existence-uniqueness of the solution to (IP; u) on J.

Let $\widetilde{P}$ be the same space defined by (19). Then, we infer from $(H_1)'$, (H_2), (32), and $u(t) \leqslant 0$ a.e. $t \in J$ that

$$A(x,y)(t) \geqslant 0 \quad \forall x,y \in \widetilde{P},\ t \in J. \tag{33}$$

Thus, we see that $A : \widetilde{P} \times \widetilde{P} \to \widetilde{P}$.

Also, we observe from $(H_1)'$, (H_2), and (32) that $A : \widetilde{P} \times \widetilde{P} \to \widetilde{P}$ is a mixed monotone operator.

Next, we show (A_2). Put

$$\varphi(\gamma) = \min\{\varphi_1(\gamma), \varphi_2(\gamma)\}, \quad \gamma \in (0,1). \tag{34}$$

Then, we see from $(H_3)'$ that $\varphi(\gamma) \in (\gamma, 1]$. Therefore, for any $\gamma \in (0,1)$ and $x, y \in \widetilde{P}$, we observe from (H_2), $(H_1)'$, $(H_3)'$, (32), and $u(t) \leqslant 0$ a.e. $t \in J$ that

$$\begin{aligned} &A\left(\gamma x, \gamma^{-1} y\right)(t) \\ &\quad = b_0 + b_1 t - \int_0^t (t-s)\left\{a(s) f\left(s, \gamma x(s), \gamma^{-1} y(s)\right) \right. \\ &\qquad\qquad \left. + u(s)\right\} ds \\ &\qquad + \sum_{0<t_k<t} I_k\left(\gamma x(t_k), \gamma^{-1} y(t_k)\right) \\ &\quad \geqslant b_0 + b_1 t - \varphi_1(\gamma) \int_0^t (t-s)\{a(s) f(s, x(s), y(s)) \\ &\qquad\qquad + u(s)\} ds \\ &\qquad + \varphi_2(\gamma) \sum_{0<t_k<t} I_k(x(t_k), y(t_k)) \\ &\quad \geqslant \varphi(\gamma) A(x,y)(t), \quad \forall t \in J, \end{aligned} \tag{35}$$

which implies that

$$A\left(\gamma x, \gamma^{-1} y\right) \geqslant \varphi(\gamma) A(x,y) \quad \forall x, y \in \widetilde{P},\ \gamma \in (0,1). \tag{36}$$

Thus, the condition (A_2) holds.

Now, we show (A_1), defining a function h by

$$h(t) := \frac{1}{2} + \int_0^t (t-s)\, ds, \quad \forall t \in J; \tag{37}$$

hence, $h(t) = (t^2/2) + (1/2)$ for all $t \in J$. Then, we can easily see that $(1/2) \leqslant h(t) \leqslant 1$ for all $t \in J$.

Now we show that $A(h,h) \in \widetilde{P}_h$. Set

$$r_1 = \min_{t\in J}\left[-f\left(t, \frac{1}{2}, 1\right)\right], \qquad r_2 = \max_{t\in J}\left[-f\left(t, 1, \frac{1}{2}\right)\right], \tag{38}$$

then, $0 < r_1 \leqslant r_2$.

From $(H_1)'$, (H_2), (32), and $u(t) \leqslant 0$ a.e. $t \in J$, it follows that

$$\begin{aligned} &A(h,h)(t) \\ &\quad = b_0 + b_1 t - \int_0^t (t-s)\{a(s) f(s, h(s), h(s)) + u(s)\} ds \\ &\qquad + \sum_{0<t_k<t} I_k(h(t_k), h(t_k)) \\ &\quad \geqslant b_0 + a_{\min}\left[-\int_0^t (t-s) f\left(s, \frac{1}{2}, 1\right) ds\right] \\ &\quad \geqslant b_0 + r_1 a_{\min} \int_0^t (t-s)\, ds \\ &\quad \geqslant r_3 h(t), \quad \forall t \in J. \end{aligned} \tag{39}$$

Also, we have

$$\begin{aligned} &A(h,h)(t) \\ &\quad = b_0 + b_1 t - \int_0^t (t-s)\{a(s) f(s, h(s), h(s)) + u(s)\} ds \\ &\qquad + \sum_{0<t_k<t} I_k(h(t_k), h(t_k)) \\ &\quad \leqslant b_0 + b_1 + a_{\max}\left[-\int_0^t (t-s) f\left(s, 1, \frac{1}{2}\right) ds\right] \\ &\qquad + M\int_0^t (t-s)\, ds + \sum_{0<t_k<t} I_k\left(1, \frac{1}{2}\right) \\ &\quad \leqslant 2\left[b_0 + b_1 + r_2 a_{\max} + M + \sum_{k=1}^m I_k\left(1, \frac{1}{2}\right)\right] h(t), \\ &\qquad\qquad \forall t \in J. \end{aligned} \tag{40}$$

Thus, we observe that

$$r_3 h \leqslant A(h,h) \leqslant 2\left[b_0 + b_1 + r_2 a_{\max} + M + \sum_{k=1}^m I_k\left(1, \frac{1}{2}\right)\right] h, \tag{41}$$

which implies that $A(h,h) \in \widetilde{P}_h$.

By arguments as above, we see that the operator $A : \widetilde{P} \times \widetilde{P} \to \widetilde{P}$ defined by (32) satisfies conditions (A_1) and (A_2) in Proposition 2. Therefore, by applying Proposition 2, we conclude that an operator equation $x = A(x,x)$ has a unique solution in $\widetilde{P}_h$; hence there exists a unique positive solution to (IP; u) on J. □

5. Optimal Control Problem (OP)

In this section, we consider an optimal control problem (OP) to (IP; u). Throughout this section, we assume all the conditions of Theorem 8. Also, we assume the following additional conditions.

(H_4) There is a constant $C_f > 0$ such that

$$|f(t,x,x) - f(t,y,y)| \leqslant C_f |x-y| \quad \forall t \in J,\ x, y \in R^+. \tag{42}$$

Also, for each $k = 1, 2, \ldots, m$, there exists a positive constant $C_k > 0$ such that

$$|I_k(x,x) - I_k(y,y)| \leqslant C_k |x-y| \quad \forall x, y \in R^+. \tag{43}$$

(H_5) x_d is a given desired target profile in H.

At first, we give the key lemma in order to show the result of continuous dependence of positive solutions to (IP; u).

Lemma 10 (cf. [19, Lemma 5.1]). *Let $\{u_n\} \subset H$, and let $Q : H \to C[J,R]$ be an operator given by*

$$(Qz)(t) := \int_0^t (t-s) z(s)\, ds, \quad \forall z \in H,\ \forall t \in J. \tag{44}$$

Assume that $u_n \to u$ weakly in H as $n \to \infty$ for some $u \in H$. Then

$$Qu_n \longrightarrow Qu \quad \text{in } C[J,R] \text{ as } n \longrightarrow \infty. \tag{45}$$

For the detailed proof of Lemma 10, we refer to [19, Lemma 5.1].

Taking account of Lemma 10, one can show the following proposition concerning the result of continuous dependence of positive solutions to (IP; u).

Proposition 11 (cf. [19, Proposition 5.2]). *Assume the same conditions as in Theorem 8, (H_4), and (H_5). Let $\{u_n\} \subset \mathcal{U}_M$ and $u \in \mathcal{U}_M$, where $\mathcal{U}_M$ is the control space defined by (4). Assume $u_n \to u$ weakly in H as $n \to \infty$. Then, the unique positive solution x_n to (IP; u_n) on J converges to one x to (IP; u) on J in the sense that*

$$x_n \longrightarrow x \quad \text{in } PC[J,R] \text{ as } n \longrightarrow \infty. \tag{46}$$

Proof. By arguments similar to [19, Proposition 5.2], we can prove (46). In fact, note from Lemma 9 that x_n is a solution to (IP; u_n) on J if and only if

$$\begin{aligned} x_n(t) = b_0 + b_1 t &- \int_0^t (t-s)\, a(s)\, f(s, x_n(s), x_n(s))\, ds \\ &- \int_0^t (t-s)\, u_n(s)\, ds + \sum_{0<t_k<t} I_k(x_n(t_k), x_n(t_k)), \\ &\forall t \in J. \end{aligned} \tag{47}$$

Now, let $t \in J_0 = [0, t_1] \subset J$, then, we obtain from ($H_4$) that

$$\begin{aligned} &|x_n(t) - x(t)| \\ &\leqslant \left| \int_0^t (t-s)\, a(s)\, f(s, x_n(s), x_n(s))\, ds \right. \\ &\quad \left. - \int_0^t (t-s)\, a(s)\, f(s, x(s), x(s))\, ds \right| \\ &\quad + \left| \int_0^t (t-s)\, u_n(s)\, ds - \int_0^t (t-s)\, u(s)\, ds \right| \\ &\leqslant C_f |a|_{C[J,R]} \int_0^t |x_n(s) - x(s)|\, ds + |Qu_n - Qu|_{C[J,R]}, \\ &\qquad \forall t \in J_0 \end{aligned} \tag{48}$$

for all $n = 1, 2, \ldots$, where Q is a function defined in (44).

Applying a Gronwall-type inequality (e.g., [21, Proposition 0.4.1]) to (48), we obtain

$$\int_0^t |x_n(s) - x(s)|\, ds \leqslant e^{C_f |a|_{C[J,R]}} |Qu_n - Qu|_{C[J,R]}, \quad \forall t \in J_0, \tag{49}$$

for all $n = 1, 2, \ldots$. Therefore, it follows from (48) and (49) that

$$\begin{aligned} &|x_n(t) - x(t)| \\ &\leqslant C_f |a|_{C[J,R]} e^{C_f |a|_{C[J,R]}} |Qu_n - Qu|_{C[J,R]} + |Qu_n - Qu|_{C[J,R]} \\ &\equiv N_1 |Qu_n - Qu|_{C[J,R]}, \quad \forall t \in J_0 = [0, t_1], \end{aligned} \tag{50}$$

for all $n = 1, 2, \ldots$.

By (50) and the assumption (H_4), we also have

$$\begin{aligned} &|x_n(t_1^+) - x(t_1^+)| \\ &\quad = |x_n(t_1) + I_1(x_n(t_1), x_n(t_1)) - x(t_1) \\ &\qquad - I_1(x(t_1), x(t_1))| \\ &\quad \leqslant (1 + C_1) |x_n(t_1) - x(t_1)| \\ &\quad \leqslant (1 + C_1) N_1 |Qu_n - Qu|_{C[J,R]} \\ &\quad \equiv N_1' |Qu_n - Qu|_{C[J,R]}, \end{aligned} \tag{51}$$

for all $n = 1, 2, \ldots$.

Next, we consider the time interval $J_1 = (t_1, t_2]$. Then, we see from (50) and (H_4) that

$$\begin{aligned} &|x_n(t) - x(t)| \\ &\leqslant \left| \int_0^t (t-s)\, a(s)\, f(s, x_n(s), x_n(s))\, ds \right. \\ &\quad \left. - \int_0^t (t-s)\, a(s)\, f(s, x(s), x(s))\, ds \right| \\ &\quad + \left| \int_0^t (t-s)\, u_n(s)\, ds - \int_0^t (t-s)\, u(s)\, ds \right| \\ &\quad + |I_1(x_n(t_1), x_n(t_1)) - I_1(x(t_1), x(t_1))| \\ &\leqslant C_f |a|_{C[J,R]} \int_0^t |x_n(s) - x(s)|\, ds \\ &\quad + |Qu_n - Qu|_{C[J,R]} + C_1 |x_n(t_1) - x(t_1)| \\ &\leqslant C_f |a|_{C[J,R]} \int_0^t |x_n(s) - x(s)|\, ds \\ &\quad + (1 + C_1 N_1) |Qu_n - Qu|_{C[J,R]}, \end{aligned} \tag{52}$$

for any $t \in J_1$ and $n = 1, 2, \ldots$. By the same arguments as before (cf. (49) and (50)), we can take some constants $N_2 > 0$ so that

$$|x_n(t) - x(t)| \leqslant N_2 |Qu_n - Qu|_{C[J,R]}, \quad \forall t \in J_1 = (t_1, t_2], \tag{53}$$

for all $n = 1, 2, \ldots$.

Also, we obtain from (H_4) and (53) that

$$\begin{aligned}
&\left|x_n\left(t_2^+\right)-x\left(t_2^+\right)\right| \\
&\quad\leqslant\left|x_n\left(t_2\right)-x\left(t_2\right)\right|+\left|I_2\left(x_n\left(t_2\right),x_n\left(t_2\right)\right)\right. \\
&\qquad\qquad\left.-I_2\left(x\left(t_2\right),x\left(t_2\right)\right)\right| \\
&\quad\leqslant\left(1+C_2\right)\left|x_n\left(t_2\right)-x\left(t_2\right)\right| \\
&\quad\leqslant N_2'\left|Qu_n-Qu\right|_{C[J,R]},\quad \forall n=1,2,\ldots,
\end{aligned}\tag{54}$$

for some positive constants $N_2'>0$.

By repeating this procedure, we can take positive constants $N_k>0$ and $N_k'>0$ such that

$$\begin{gathered}
\left|x_n(t)-x(t)\right|\leqslant N_k\left|Qu_n-Qu\right|_{C[J,R]}, \\
\forall t\in J_{k-1},\quad k=1,2,\ldots,m+1, \\
\left|x_n\left(t_k^+\right)-x\left(t_k^+\right)\right|\leqslant N_k'\left|Qu_n-Qu\right|_{C[J,R]},\quad k=1,2,\ldots,m,
\end{gathered}\tag{55}$$

for all $n=1,2,\ldots$.

Here, put $N:=\max\{N_1,N_1',N_2,N_2',\ldots,N_m,N_m',N_{m+1}\}$. Then, we infer from (55) that

$$\left|x_n-x\right|_{PC}\leqslant N\left|Qu_n-Qu\right|_{C[J,R]},\quad \forall n=1,2,\ldots. \tag{56}$$

Since $u_n\to u$ weakly in H as $n\to\infty$, we observe from Lemma 10 that

$$Qu_n\longrightarrow Qu\quad\text{in } C[J,R]\ \text{ as } n\longrightarrow\infty. \tag{57}$$

Hence, we see from (56) and (57) that

$$x_n\longrightarrow x\quad\text{in } PC[J,R]\ \text{ as } n\longrightarrow\infty. \tag{58}$$

Thus, the proof of Proposition 11 has been completed. □

Now, we mention our main result concerning the existence of an optimal control to (OP).

Theorem 12. *Assume the same conditions as in Theorem 8, (H_4), and (H_5). Then, the problem (OP) has at least one optimal control $u^*\in\mathcal{U}_M$ such that*

$$\pi\left(u^*\right)=\inf_{u\in\mathcal{U}_M}\pi(u), \tag{59}$$

where $\mathcal{U}_M$ is a control space defined by (4) *and $\pi(\cdot)$ is the cost functional defined in* (5).

Proof. By the quite standard method, we can prove Theorem 12. In fact, let $\{u_n\}\subset\mathcal{U}_M$ be a minimizing sequence so that

$$\lim_{n\to\infty}\pi\left(u_n\right)=\inf_{u\in\mathcal{U}_M}\pi(u). \tag{60}$$

By the definition (5) of $\pi(\cdot)$, we see that $\{u_n\}$ is bounded in H. Hence, there is a subsequence $\{n_k\}\subset\{n\}$ and a function $u^*\in\mathcal{U}_M$ such that $n_k\to\infty$ and

$$u_{n_k}\longrightarrow u^*\quad\text{weakly in } H \text{ as } k\longrightarrow\infty. \tag{61}$$

For any $k\in\mathbb{N}$, let x_{n_k} be a unique positive solution to (IP; u_{n_k}) on J. Then, from (61) and Proposition 11, we observe that

$$x_{n_k}\longrightarrow x\quad\text{in } PC[J,R]\ \text{ as } k\longrightarrow\infty, \tag{62}$$

where x is a unique positive solution to (IP; u) on J.

Hence, it follows from (61), (62), and the weak lower semicontinuity of H-norm that

$$\pi\left(u^*\right)\leqslant\lim_{k\to\infty}\pi\left(u_{n_k}\right)=\inf_{u\in\mathcal{U}_M}\pi(u), \tag{63}$$

which implies that $u^*\in\mathcal{U}_M$ is an optimal control to (OP). □

Now, we mention our final main result in this paper, which is concerned with the stability of the optimal control to (OP).

Theorem 13. *Assume the same conditions as in Theorem 12. Let $u\in\mathcal{U}_M$ and $u+\varepsilon u_0\in\mathcal{U}_M$ for some $u_0\in H$ and small positive constant ε. Also, let x and x_ε be unique positive solutions to (IP; u) and (IP;$u+\varepsilon u_0$) on J, respectively. Then,*

$$\left|x_\varepsilon-x\right|_{PC}=\mathcal{O}(\varepsilon),\quad(\varepsilon\longrightarrow 0). \tag{64}$$

Proof. Note from Lemma 9 that x is a solution of (IP; u) on J if and only if

$$\begin{aligned}
x(t)&=b_0+b_1t-\int_0^t(t-s)\,a(s)\,f(s,x(s),x(s))\,ds \\
&\quad-\int_0^t(t-s)\,u(s)\,ds+\sum_{0<t_k<t}I_k\left(x\left(t_k\right),x\left(t_k\right)\right), \\
&\qquad\qquad\forall t\in J.
\end{aligned}\tag{65}$$

Now, let $t\in J_0=[0,t_1]\subset J$. Then, we obtain from ($H_4$) that

$$\begin{aligned}
&\left|x_\varepsilon(t)-x(t)\right| \\
&\quad\leqslant\left|\int_0^t(t-s)\,a(s)\,f\left(s,x_\varepsilon(s),x_\varepsilon(s)\right)ds\right. \\
&\qquad\left.-\int_0^t(t-s)\,a(s)\,f(s,x(s),x(s))\,ds\right| \\
&\qquad+\left|\int_0^t(t-s)\left(u+\varepsilon u_0\right)(s)\,ds-\int_0^t(t-s)\,u(s)\,ds\right| \\
&\quad\leqslant C_f|a|_{C[J,R]}\int_0^t\left|x_\varepsilon(s)-x(s)\right|ds+\varepsilon\left|u_0\right|_H,\quad\forall t\in J_0.
\end{aligned}\tag{66}$$

Applying a Gronwall-type inequality (e.g., [21, Proposition 0.4.1]) to (66), we obtain

$$\int_0^t\left|x_\varepsilon(s)-x(s)\right|ds\leqslant\varepsilon e^{C_f|a|_{C[J,R]}}\left|u_0\right|_H,\quad\forall t\in J_0. \tag{67}$$

Therefore, it follows from (66) and (67) that

$$\begin{aligned}|x_\varepsilon(t)-x(t)| &\leqslant \varepsilon C_f|a|_{C[J,R]}e^{C_f|a|_{C[J,R]}}|u_0|_H+\varepsilon|u_0|_H \\ &\equiv \varepsilon\widetilde{N}_1|u_0|_H, \quad \forall t\in J_0=[0,t_1].\end{aligned} \tag{68}$$

By (68) and the assumption (H_4), we also have

$$\begin{aligned}&|x_\varepsilon(t_1^+)-x(t_1^+)| \\ &\quad=|x_\varepsilon(t_1)+I_1(x_\varepsilon(t_1),x_\varepsilon(t_1))-x(t_1) \\ &\qquad -I_1(x(t_1),x(t_1))| \\ &\quad\leqslant(1+C_1)|x_\varepsilon(t_1)-x(t_1)| \\ &\quad\leqslant(1+C_1)\varepsilon\widetilde{N}_1|u_0|_H \\ &\quad\equiv\varepsilon\widetilde{N}_1'|u_0|_H.\end{aligned} \tag{69}$$

Next, we consider the time interval $J_1=(t_1,t_2]$. Then, we see from (68) and (H_4) that

$$\begin{aligned}&|x_\varepsilon(t)-x(t)| \\ &\quad\leqslant\left|\int_0^t(t-s)a(s)f(s,x_\varepsilon(s),x_\varepsilon(s))\,ds\right. \\ &\qquad\left.-\int_0^t(t-s)a(s)f(s,x(s),x(s))\,ds\right| \\ &\qquad+\left|\int_0^t(t-s)(u+\varepsilon u_0)(s)\,ds-\int_0^t(t-s)u(s)\,ds\right| \\ &\qquad+|I_1(x_\varepsilon(t_1),x_\varepsilon(t_1))-I_1(x(t_1),x(t_1))| \\ &\quad\leqslant C_f|a|_{C[J,R]}\int_0^t|x_\varepsilon(s)-x(s)|\,ds+\varepsilon|u_0|_H \\ &\qquad+C_1|x_\varepsilon(t_1)-x(t_1)| \\ &\quad\leqslant C_f|a|_{C[J,R]}\int_0^t|x_\varepsilon(s)-x(s)|\,ds \\ &\qquad+\varepsilon\left(1+C_1\widetilde{N}_1\right)|u_0|_H,\end{aligned} \tag{70}$$

for any $t\in J_1$. By the same arguments as before (cf. (67) and (68)), we can take some constant $\widetilde{N}_2>0$ so that

$$|x_\varepsilon(t)-x(t)|\leqslant\varepsilon\widetilde{N}_2|u_0|_H, \quad \forall t\in J_1=(t_1,t_2]. \tag{71}$$

Also, we obtain from (H_4) and (71) that

$$\begin{aligned}&|x_\varepsilon(t_2^+)-x(t_2^+)| \\ &\quad\leqslant|x_\varepsilon(t_2)-x(t_2)|+|I_2(x_\varepsilon(t_2),x_\varepsilon(t_2)) \\ &\qquad -I_2(x(t_2),x(t_2))| \\ &\quad\leqslant(1+C_2)|x_\varepsilon(t_2)-x(t_2)| \\ &\quad\leqslant\varepsilon\widetilde{N}_2'|u_0|_H,\end{aligned} \tag{72}$$

for some positive constant $\widetilde{N}_2'>0$.

By repeating this procedure, we can take positive constants $\widetilde{N}_k>0$ and $\widetilde{N}_k'>0$ such that

$$\begin{aligned}&|x_\varepsilon(t)-x(t)|\leqslant\varepsilon\widetilde{N}_k|u_0|_H, \quad \forall t\in J_{k-1},\ k=1,2,\ldots,m+1, \\ &|x_\varepsilon(t_k^+)-x(t_k^+)|\leqslant\varepsilon\widetilde{N}_k'|u_0|_H, \quad k=1,2,\ldots,m.\end{aligned} \tag{73}$$

Here, put $\widetilde{N}:=\max\{\widetilde{N}_1,\widetilde{N}_1',\widetilde{N}_2,\widetilde{N}_2',\ldots,\widetilde{N}_m,\widetilde{N}_m',\widetilde{N}_{m+1}\}$. Then, we infer from (73) that

$$|x_\varepsilon-x|_{PC}\leqslant\varepsilon\widetilde{N}|u_0|_H. \tag{74}$$

Thus, the proof of Theorem 13 has been completed. □

By Theorem 13 and the definition of π (cf. (5)), we easily see that the following corollary holds.

Corollary 14. *Assume the same conditions as in Theorem 12. Let $u\in\mathscr{U}_M$ and $u+\varepsilon u_0\in\mathscr{U}_M$ for some $u_0\in H$ and small positive constant ε. Then,*

$$|\pi(u+\varepsilon u_0)-\pi(u)|=\mathscr{O}(\varepsilon), \quad (\varepsilon\longrightarrow 0). \tag{75}$$

6. Examples

In this section, we give an example of the main results.

Example 1. Consider the following boundary value problem of second order impulsive differential equation:

$$\begin{aligned}&-x''(t)=(2t+1)\left[(1+x(t))^{(1/2)}+(1+x(t))^{-(1/4)}\right]+u(t), \\ &\qquad t\in(0,1),\ t\neq\frac{1}{3}, \\ &\Delta x|_{t=(1/3)}=\left(1+x\left(\frac{1}{3}\right)\right)^{(1/2)}+\left(1+x\left(\frac{1}{3}\right)\right)^{-(1/4)}, \\ &\qquad x(0)=1, \qquad x'(1)=0.\end{aligned} \tag{76}$$

Conclusion. The boundary value problem (76) admits a unique positive solution, which is continuously differentiable on $[0,(1/3))\cup((1/3),1]$.

Proof. Let $J=[0,1]$, $t_1=(1/3)$, $f(t,x,y):=f(x,y)=(1+x)^{(1/2)}+(1+y)^{-(1/4)}$, $a(t)=2t+1$, and $I_1(x,y)=(1+x)^{(1/2)}+(1+y)^{-(1/4)}$. Evidently, $f(x,y)$ and $I_1(x,y)$ are increasing in x for $y\geqslant 0$ and are decreasing in y for $x\geqslant 0$.

Set $\varphi(\gamma)=\gamma^{(1/2)}$, $\gamma\in(0,1)$, then,

$$\begin{aligned}&f(\gamma x,\gamma^{-1}y) \\ &\quad=(1+\gamma x)^{(1/2)}+\left(1+\gamma^{-1}y\right)^{-(1/4)}\geqslant\varphi(\gamma)f(x,y) \\ &\qquad\qquad \forall x,y\geqslant 0, \\ &I_1(\gamma x,\gamma^{-1}y)\geqslant\varphi(\gamma)I_1(x,y) \quad \forall x,y\geqslant 0.\end{aligned} \tag{77}$$

Therefore, we easily see that (H_1)–(H_5) hold. Hence, applying Theorem 4 to (76), we get a unique positive solution to (76) on J for each $u \in H$ with $0 \leqslant u(t) \leqslant M$ a.e. $t \in J$, where $M > 0$ is a given constant. □

Example 2. Consider the following initial value problem of the second order impulsive differential equation:

$$\begin{aligned} &-x''(t) \\ &\quad = (2t+1)\left[(1+x(t))^{(1/2)} + (1+x(t))^{-(1/4)}\right] + u(t), \\ &\qquad\qquad t \in (0,1), \quad t \neq \frac{1}{3}, \\ &\Delta x|_{t=(1/3)} = \left(1 + x\left(\frac{1}{3}\right)\right)^{(1/2)} + \left(1 + x\left(\frac{1}{3}\right)\right)^{-(1/4)}, \\ &\qquad x(0) = 1, \qquad x'(0) = 0. \end{aligned} \tag{78}$$

Conclusion. The initial value problem (78) admits a unique positive solution, which is continuously differentiable on $[0,(1/3)) \cup ((1/3),1]$. Moreover, the problem (OP) to (78) has at least one optimal control, and the stability result of optimal control holds.

Proof. Let $J = [0,1]$, $t_1 = (1/3)$, $f(t,x,y) := f(x,y) = -(1+x)^{(1/2)} - (1+y)^{-(1/4)}$, and $a(t) = 2t+1$. Clearly, $f(x,y)$ is decreasing in x for $y \geqslant 0$ and is increasing in y for $x \geqslant 0$.

Also, let $I_1(x,y) = (1+x)^{(1/2)} + (1+y)^{-(1/4)}$, evidently, $I_1(x,y)$ is increasing in x for $y \geqslant 0$ and is decreasing in y for $x \geqslant 0$.

Set $\varphi(\gamma) = \gamma^{(1/2)}$, $\gamma \in (0,1)$, then,

$$\begin{aligned} &f\left(\gamma x, \gamma^{-1} y\right) \\ &\quad = -(1+\gamma x)^{(1/2)} + \left(1+\gamma^{-1} y\right)^{-(1/4)} \leqslant \varphi(\gamma) f(x,y) \\ &\qquad\qquad \forall x, y \geqslant 0, \\ &I_1\left(\gamma x, \gamma^{-1} y\right) \geqslant \varphi(\gamma) I_1(x,y) \quad \forall x, y \geqslant 0. \end{aligned} \tag{79}$$

Therefore, we easily see that conditions (H_2), (H_4), $(H_1)'$, $(H_3)'$, and $(H_5)'$ hold. Hence, applying Theorem 8 to (78), we get a unique positive solution to (78) on J for each $u \in H$ with $-M \leqslant u(t) \leqslant 0$ a.e. $t \in J$, where $M > 0$ is a given constant.

In addition, let $C_f = C_1 = 1$. Then, we easily see that (H_6) holds. Hence, applying Theorem 12, we see that Problem (OP) to (78) has at least one optimal control for each desired target profile x_d in H. Also, applying Theorem 13, we get the result on the stability of optimal control to (OP). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors would like to thank the referees for their valuable suggestions and comments. The research was partially supported by the National Natural Science Foundation of China (no. 61250011), the Postdoctoral Science Foundation of China (no. 2012M510786) and the Science Foundation of Shanxi Province (no. 2012011004-4).

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