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Research Article

The Generalized Weaker (α-ϕ-φ)-Contractive Mappings and Related Fixed Point Results in Complete Generalized Metric Spaces

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We introduce the notion of generalized weaker (α-ϕ-φ)-contractive mappings in the context of generalized metric space. We investigate the existence and uniqueness of fixed point of such mappings. Some consequences on existing fixed point theorems are also derived. The presented results generalize, unify, and improve several results in the literature.

1. Introduction and Preliminaries

In [1], Branciari introduced the notion of generalized metric space by weakening the triangular inequality of metric assumption with quadrilateral inequality. The author [1] characterized and proved the analog of famous Banach fixed point theorem in the setting of generalized metric space. Although the theorem of Branciari [1] is correct, the proofs had gaps [2] since the topology of generalized metric space is not strong enough as the topology of metric space. The disadvantages of generalized metric space can be listed as follows:

($w1$) generalized metric need not be continuous;

($w2$) a convergent sequence in generalized metric space need not be Cauchy;

($w3$) generalized metric space need not be Hausdorff, and hence the uniqueness of limits cannot be guaranteed.

Despite the weakness of the topology of generalized metric space, in [3, 4], the authors suggested some techniques to get a (unique) fixed point in such spaces.

On the other hand, Samet et al. [5] introduced the notion of α-ψ contraction mappings and proved the existence and uniqueness of such mappings in complete metric space. The results of this paper are very impressive since several existing results derived from the main theorem of Samet et al. [5] quiet easily. Later, a number of authors have appreciated these results and have used this technique to get further generalization via α-ψ contraction mappings; see, for example, [6–10].

In this paper, we introduce the generalized weaker α-ψ contraction mappings in the setting of generalized metric spaces. Consequently, we investigate the existence and uniqueness of fixed point by caring the problems ($w1$)–($w3$) mentioned above.

Let us recall basic definitions and notations and interesting results that will be in the sequel.

Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) ψ is nondecreasing;

(ii) there exist $k_0 \in \mathbb{N}$ and $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \leq a\psi^{k}(t) + v_k, \tag{1}$$

for $k \geq k_0$ and any $t \in \mathbb{R}^+$.

In the literature such functions are called either Bianchini-Grandolfi gauge functions (see, e.g., [11–13]) or (c)-comparison functions (see, e.g., [14]).

Lemma 1 (see, e.g., [14]). *If $\psi \in \Psi$, then the following hold:*

(i) $(\psi^n(t))_{n\in\mathbb{N}}$ *converges to* 0 *as* $n \to \infty$ *for all* $t \in \mathbb{R}^+$*;*

(ii) $\psi(t) < t$, *for any* $t \in \mathbb{R}^+$*;*

(iii) ψ *is continuous at* 0*;*

(iv) *the series* $\sum_{k=1}^{\infty} \psi^k(t)$ *converges for any* $t \in \mathbb{R}^+$.

In the following, we recall the notion of generalized metric spaces.

Definition 2 (see [1]). Let X be a nonempty set and let $d : X \times X \to [0,\infty]$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y:

$$\begin{aligned} &(\mathrm{GMS1}) \quad d(x,y) = 0 \quad \text{iff } x = y, \\ &(\mathrm{GMS2}) \quad d(x,y) = d(y,x), \\ &(\mathrm{GMS3}) \quad d(x,y) \leq d(x,u) + d(u,v) + d(v,y). \end{aligned} \tag{2}$$

Then, the map d is called generalized metric. Here, the pair (X,d) is called a generalized metric space and abbreviated as GMS.

In the above definition, if d satisfies only (GMS1) and (GMS2), then it is called semimetric (see, e.g., [15]).

The concepts of convergence, Cauchy sequence, and completeness in a GMS are defined as follows.

Definition 3. (1) A sequence $\{x_n\}$ in a GMS (X,d) is GMS convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(2) A sequence $\{x_n\}$ in a GMS (X,d) is GMS Cauchy if and only if for every $\varepsilon > 0$ there exists positive integer $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.

(3) A GMS (X,d) is called complete if every GMS Cauchy sequence in X is GMS convergent.

The following assumption was suggested by Wilson [15] to replace the triangle inequality with the weakened condition.

(W) For each pair of (distinct) points u, v there is a number $r_{u,v} > 0$ such that, for every $z \in X$,

$$r_{u,v} < d(u,z) + d(z,v). \tag{3}$$

Proposition 4 (see [3]). *In a semimetric space, the assumption (W) is equivalent to the assertion that limits are unique.*

Proposition 5 (see [3]). *Suppose that $\{x_n\}$ is a Cauchy sequence in a GMS (X,d) with $\lim_{n\to\infty} d(x_n,u) = 0$, where $u \in X$. Then $\lim_{n\to\infty} d(x_n,z) = d(u,z)$ for all $z \in X$. In particular, the sequence $\{x_n\}$ does not converge to z if $z \neq u$.*

A function $\phi : [0,\infty) \to [0,\infty)$ is said to be a Meir-Keeler function [16] if, for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0,\infty)$ with $\eta \leq t < \eta + \delta$, we have $\phi(t) < \eta$. Such mapping has been improved and used by several authors [17, 18]. In what follows we recall the notion of weaker Meir-Keeler function.

Definition 6 (see, e.g., [19]). We call $\phi : [0,\infty) \to [0,\infty)$ a weaker Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0,\infty)$ with $\eta \leq t < \eta + \delta$, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(t) < \eta$.

Let Φ be the class of all nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ϕ_1) $\phi : [0,\infty) \to [0,\infty)$ is a weaker Meir-Keeler function;

(ϕ_2) $0 < \phi(t) < t$ for all $t > 0$, $\phi(0) = 0$;

(ϕ_3) for all $t \in (0,\infty)$, $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;

(ϕ_4) if $\lim_{n\to\infty} t_n = \gamma$, then $\lim_{n\to\infty} \phi(t_n) \leq \gamma$.

Let Θ be the class of functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(φ_1) φ is continuous;

(φ_2) $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$.

By using the auxiliary functions, defined above, Chen and Sun [19] proved the following theorem.

Theorem 7. *Let (X,d) be a Hausdorff and complete generalized metric space, and let $f : X \to X$ be a function satisfying*

$$d(fx, fy) \leq \phi(d(x,y)) - \varphi(d(x,y)) \tag{4}$$

for all $x, y \in X$ and $\phi \in \Phi$, $\varphi \in \Theta$. Then f has a periodic point μ in X; that is, there exists $\mu \in X$ such that $\mu = f^p\mu$ for some $p \in \mathbb{N}$.

Another interesting auxiliary function, α-admissible, was defined by Samet et al. [5].

Definition 8 (see [5]). For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be mappings. We say that T is α-admissible if

$$\alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx,Ty) \geq 1, \tag{5}$$

for all $x, y \in X$.

Example 9. Let $X = [2,\infty)$ and $T : X \to X$ by $Tx = (x+1)/(x-1)$. Define $\alpha(x,y) : X \times X \to [0,\infty)$ and

$$\alpha(x,y) = \begin{cases} e^{x+1} & \text{if } x \geq y, \\ 0 & \text{if otherwise.} \end{cases} \tag{6}$$

Then T is α-admissible.

Example 10. Let $X = \mathbb{R}$ and $T : X \to X$. Define $\alpha(x, y) : X \times X \to [0, \infty)$ by $Tx = e^{x+1}$ and

$$\alpha(x, y) = \begin{cases} x^2 & \text{if } x \geq y, \\ 0 & \text{if otherwise.} \end{cases} \tag{7}$$

Then T is α-admissible.

Some interesting examples of such mappings were given in [5].

The notion of an α-ψ contractive mapping is defined in the following way.

Definition 11 (see [5]). Let (X, d) be a metric space and let $T : X \to X$ be a given mapping. We say that T is an α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in X. \tag{8}$$

Clearly, any contractive mapping, that is, a mapping satisfying the Banach contraction, is an α-ψ contractive mapping with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$, $k \in (0, 1)$.

Very recently, Karapınar [20] gave the analog of the notion of an α-ψ contractive mapping, in the context of generalized metric spaces as follows.

Definition 12. Let (X, d) be a generalized metric space and let $T : X \to X$ be a given mapping. We say that T is an α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in X. \tag{9}$$

Karapınar [20] also stated the following fixed point theorems.

Theorem 13. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be an α-ψ contractive mapping. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*

(iii) *T is continuous.*

Then there exists a $u \in X$ such that $Tu = u$.

Theorem 14. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be an α-ψ contractive mapping. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then there exists a $u \in X$ such that $Tu = u$.

For the uniqueness, Karapınar [20] (see also [21]) added the following additional conditions.

(U) For all $x, y \in \text{Fix}(T)$, we have $\alpha(x, y) \geq 1$, where $\text{Fix}(T)$ denotes the set of fixed points of T.

(H) For all $x, y \in \text{Fix}(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Theorem 15. *Adding condition (U) to the hypotheses of Theorem 13 (resp., Theorem 14), one obtains that u is the unique fixed point of T.*

Theorem 16. *Adding conditions (H) and (W) to the hypotheses of Theorem 13 (resp., Theorem 14), one obtains that u is the unique fixed point of T.*

Corollary 17. *Adding condition (H) to the hypotheses of Theorem 13 (resp., Theorem 14) and assuming that (X, d) is Hausdorff, one obtains that u is the unique fixed point of T.*

In this paper, we define the notion of weaker generalized α-ψ contractive mappings and prove some fixed point results in the setting of generalized metric spaces by using such mappings. We state some examples to illustrate the validity of the main results of this paper.

2. Main Results

In this section, we will state and prove our main results.

We give an extension of the notion of α-ψ contractive mappings, in the context of generalized metric space as follows.

Definition 18. Let (X, d) be a generalized metric space and let $T : X \to X$ be a given mapping. We say that T is a (α-ϕ-φ)-contractive mapping of type I if there exist functions $\alpha : X \times X \to [0, \infty)$, $\varphi \in \Theta$, and $\phi \in \Phi$ such that

$$\alpha(x, y) d(Tx, Ty) \leq \phi(M(x, y)) - \varphi(M(x, y)) \tag{10}$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}. \tag{11}$$

Definition 19. Let (X, d) be a generalized metric space and let $T : X \to X$ be a given mapping. We say that T is a (α-ϕ-φ)-contractive mapping of type II if there exist functions $\alpha : X \times X \to [0, \infty)$, $\varphi \in \Theta$, and $\phi \in \Phi$ such that

$$\alpha(x, y) d(Tx, Ty) \leq \phi(N(x, y)) - \varphi(N(x, y)) \tag{12}$$

for all $x, y \in X$, where

$$N(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}\right\}. \tag{13}$$

Next, we introduce the notion of *triangular α-admissible* as follows.

Definition 20. Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. The mapping T is said to be weak *triangular α-admissible* if for all $x \in X$, one has

$$\alpha(x, Tx) \geq 1, \qquad \alpha(Tx, T^2x) \geq 1 \Longrightarrow \alpha(x, T^2x) \geq 1. \tag{14}$$

Now, we state the first fixed point theorem.

Theorem 21. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a (α-ϕ-φ)-contractive mapping of type I. Suppose that*

(i) *T is weak triangular α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous.*

Then, T has a fixed point $u \in X$; that is $Tu = u$.

Proof. Due to statement (ii) of the theorem, there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$. First, we define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. Notice that if $x_{n_0} = x_{n_0+1}$ for some n_0, then the proof is completed. Indeed, we have $u = x_{n_0} = x_{n_0+1} = Tx_{n_0} = Tu$. Thus, for the rest of the proof, we assume that

$$x_n \neq x_{n+1} \quad \forall n. \tag{15}$$

Owing to the fact that T is α-admissible, we derive that

$$\begin{aligned} \alpha(x_0, x_1) &= \alpha(x_0, Tx_0) \geq 1 \\ &\Longrightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1. \end{aligned} \tag{16}$$

Utilizing the expression above, we find that

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall n = 0, 1, \ldots. \tag{17}$$

Since T is a weak triangular α-admissible mapping, we obtain that

$$\alpha(x_0, x_2) \geq 1. \tag{18}$$

Since $\alpha(x_0, T_0^x) \geq 1$ and $\alpha(Tx_0, T^2x_0) = \alpha(x_0, x_2)$, iteratively, we conclude that

$$\alpha(x_n, x_{n+k}) \geq 1, \quad \forall n, k = 0, 1, \ldots. \tag{19}$$

Taking (10) and (17) into account, we observe that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \alpha(x_n, x_{n-1})\, d(Tx_n, Tx_{n-1}) \\ &\leq \phi(M(x_n, x_{n-1})) - \varphi(M(x_n, x_{n-1})), \end{aligned} \tag{20}$$

for all $n \geq 1$, where

$$\begin{aligned} &M(x_n, x_{n-1}) \\ &\quad = \max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} \\ &\quad = \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &\quad = \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}. \end{aligned} \tag{21}$$

If $M(x_n, x_{n-1}) = d(x_n, x_{n+1})$, then by (15) and property of the function φ, inequality (20) turns into

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \phi(M(x_n, x_{n-1})) - \varphi(M(x_n, x_{n-1})) \\ &= \phi(d(x_n, x_{n+1})) - \varphi(d(x_n, x_{n+1})) \\ &< \phi(d(x_n, x_{n+1})). \end{aligned} \tag{22}$$

Since $\{\phi^n(t)\}$ is decreasing, the inequality above yields a contradiction. Hence, we conclude that $M(x_n, x_{n-1}) = d(x_n, x_{n-1})$ and (20) becomes

$$d(x_{n+1}, x_n) \leq \phi(d(x_n, x_{n-1})), \tag{23}$$

for all $n \geq 1$. Recursively, we derive that

$$d(x_{n+1}, x_n) \leq \phi^n(d(x_1, x_0)), \quad \forall n \geq 1. \tag{24}$$

Owing to the fact that the sequence $\{\phi^n(d(x_0, x_1))\}_{n\in\mathbb{N}}$ is decreasing, it converges to some $\eta \geq 0$. We will show that $\eta = 0$. Suppose, on the contrary, that $\eta > 0$. Taking the definition of weaker Meir-Keeler function ϕ into account, there exists $\delta > 0$ such that for $x_0, x_1 \in X$ with $\eta \leq d(x_0, x_1) < \delta + \eta$, and there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(d(x_0, x_1)) < \eta$. Regarding $\lim_{n\to\infty}\phi^n(d(x_0, x_1)) = \eta$, there exists $p_0 \in \mathbb{N}$ such that $\eta \leq \phi^p(d(x_0, x_1)) < \delta + \eta$, for all $p \geq p_0$. Hence, we deduce that $\phi^{p_0+n_0}(d(x_0, x_1)) < \eta$, which is a contradiction. Thus, $\lim_{n\to\infty}\phi^n(d(x_0, x_1)) = 0$, and hence

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{25}$$

Regarding (10) and (19), we deduce that

$$\begin{aligned} d(x_{n+2}, x_n) &= d(Tx_{n+1}, Tx_{n-1}) \\ &\leq \alpha(x_{n+1}, x_{n-1})\, d(Tx_{n+1}, Tx_{n-1}) \\ &\leq \phi(M(x_{n+1}, x_{n-1})) - \varphi(M(x_{n+1}, x_{n-1})), \end{aligned} \tag{26}$$

for all $n \geq 1$, where

$$\begin{aligned} &M(x_{n+1}, x_{n-1}) \\ &\quad = \max\{d(x_{n+1}, x_{n-1}), d(x_{n+1}, Tx_{n+1}), d(x_{n-1}, Tx_{n-1})\} \\ &\quad = \max\{d(x_{n+1}, x_{n-1}), d(x_{n+1}, x_{n+2}), d(x_{n-1}, x_n)\}. \end{aligned} \tag{27}$$

If $M(x_n, x_{n-1}) = d(x_{n-1}, x_{n+1})$ then inequality (26) turns into

$$\begin{aligned} d(x_{n+2}, x_n) &\leq \phi(d(x_{n+1}, x_{n-1})) - \varphi(d(x_{n+1}, x_{n-1})) \\ &\leq \phi(d(x_{n+1}, x_{n-1})) \end{aligned} \tag{28}$$

for all $n \geq 1$. By repeating the same argument, inequality (15) implies that

$$d(x_{n+2}, x_n) \leq \phi^n(d(x_2, x_0)), \quad \forall n \geq 1. \tag{29}$$

Due to the fact that the sequence $\{\phi^n(d(x_0,x_2))\}_{n\in\mathbb{N}}$ is decreasing, we conclude that

$$\lim_{n\to\infty} d(x_{n+2},x_n)=0, \tag{30}$$

by following the lines at the proof of (25).

If either $M(x_n,x_{n-1}) = d(x_{n-1},x_n)$ or $M(x_{n+1},x_{n+2}) = d(x_{n+1},x_{n+2})$, then inequality (26) becomes either

$$\begin{aligned} d(x_{n+2},x_n) &\le \phi(d(x_{n-1},x_n)) - \varphi(d(x_{n-1},x_n)) \\ &< \phi(d(x_{n-1},x_n)) \end{aligned} \tag{31}$$

or

$$\begin{aligned} d(x_{n+2},x_n) &\le \phi(d(x_{n+1},x_{n+2})) - \varphi(d(x_{n+1},x_{n+2})) \\ &< \phi(d(x_{n+1},x_{n+2})), \end{aligned} \tag{32}$$

for all $n\ge 1$. Letting $n\to\infty$ in any of the cases, (31) or (32), together with (25), we have

$$\begin{gathered} \lim_{n\to\infty} d(x_{n+2},x_n) \le \lim_{n\to\infty}\phi(d(x_{n-1},x_n)) \le 0 \\ \text{or} \\ \lim_{n\to\infty} d(x_{n+2},x_n) \le \lim_{n\to\infty}\phi(d(x_{n+1},x_{n+2})) \le 0. \end{gathered} \tag{33}$$

Let $x_n = x_m$ for some $m,n\in\mathbb{N}$ with $m\neq n$. Without loss of generality, assume that $m>n$. Thus, $x_m = T^{m-n}(T^n x_0) = T^n x_0 = x_n$. Regarding (15), we consider now

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,x_n) = d(Tx_m,x_m) \\ &= d(Tx_m,Tx_{m-1}) \\ &\le \alpha(x_m,x_{m-1})\, d(Tx_m,Tx_{m-1}) \\ &\le \phi(M(x_m,x_{m-1})) - \varphi(M(x_m,x_{m-1})), \end{aligned} \tag{34}$$

where

$$\begin{aligned} &M(x_m,x_{m-1}) \\ &\quad = \max\{d(x_m,x_{m-1}), d(x_m,Tx_m), d(x_{m-1},Tx_{m-1})\} \\ &\quad = \max\{d(x_m,x_{m-1}), d(x_m,x_{m+1}), d(x_{m-1},x_m)\} \\ &\quad = \max\{d(x_{m-1},x_m), d(x_m,x_{m+1})\}. \end{aligned} \tag{35}$$

If $M(x_m,x_{m-1}) = d(x_{m-1},x_m)$, then from (34) and (23) we get that

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,x_n) = d(Tx_m,x_m) \\ &= d(Tx_m,Tx_{m-1}) \\ &\le \alpha(x_m,x_{m-1})\, d(Tx_m,Tx_{m-1}) \\ &\le \phi(d(x_m,x_{m-1})) - \varphi(d(x_m,x_{m-1})) \\ &\le \phi(d(x_m,x_{m-1})) \\ &\le \phi^{m-n}(d(x_{n+1},x_n)). \end{aligned} \tag{36}$$

If $M(x_m,x_{m-1}) = d(x_m,x_{m+1})$, inequalities (34) and (23) become

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,x_n) = d(Tx_m,x_m) \\ &= d(Tx_m,Tx_{m-1}) \\ &\le \alpha(x_m,x_{m-1})\, d(Tx_m,Tx_{m-1}) \\ &\le \phi(d(x_m,x_{m+1})) - \varphi(d(x_m,x_{m+1})) \\ &\le \phi(d(x_m,x_{m+1})) \\ &\le \phi^{m-n+1}(d(x_{n+1},x_n)). \end{aligned} \tag{37}$$

Due to (ϕ_2), inequalities (36) and (37) yield that

$$\begin{aligned} d(x_{n+1},x_n) &\le \phi^{m-n}(d(x_{n+1},x_n)) < d(x_{n+1},x_n), \\ d(x_{n+1},x_n) &\le \phi^{m-n+1}(d(x_{n+1},x_n)) < d(x_{n+1},x_n), \end{aligned} \tag{38}$$

which is a contradiction. Hence $\{x_n\}$ has no periodic point.

In what follows we will prove that the sequence $\{x_n\}$ is Cauchy by standard technique. Suppose, on the contrary, that there exists $\varepsilon>0$ such that for any $k\in\mathbb{N}$, there are $m(k),n(k)\in\mathbb{N}$ with $n(k)>m(k)>k$ satisfying

$$d(x_{m(k)},x_{n(k)}) \ge \varepsilon. \tag{39}$$

Furthermore, corresponding to $m(k)$, one can choose $n(k)$ in a way that it is the smallest integer $n(k)>m(k)$ with $d(x_{m(k)},x_{n(k)})\ge\varepsilon$. Consequently, we have $d(x_{m(k)},x_{n(k)-1})<\varepsilon$. Consider

$$\begin{aligned} \varepsilon &\le d(x_{n(k)},x_{m(k)}) \\ &\le d(x_{n(k)},x_{n(k)-2}) + d(x_{n(k)-2},x_{n(k)-1}) + d(x_{n(k)-1},x_{m(k)}) \\ &\le d(x_{n(k)},x_{n(k)-2}) + d(x_{n(k)-2},x_{n(k)-1}) + \varepsilon. \end{aligned} \tag{40}$$

Letting $k\to\infty$, we get that

$$d(x_{n(k)},x_{m(k)}) \longrightarrow \varepsilon. \tag{41}$$

On the other hand, again by using the quadrilateral inequality, we find

$$\begin{aligned} &d(x_{n(k)},x_{m(k)}) \\ &\quad \le d(x_{n(k)},x_{n(k)-1}) + d(x_{n(k)-1},x_{m(k)-1}) \\ &\qquad + d(x_{m(k)-1},x_{m(k)}), \\ &d(x_{n(k)-1},x_{m(k)-1}) \\ &\quad \le d(x_{n(k)-1},x_{n(k)}) + d(x_{n(k)},x_{m(k)}) + d(x_{m(k)},x_{m(k)-1}). \end{aligned} \tag{42}$$

Letting $n\to\infty$, in the inequalities above, we get that

$$d(x_{n(k)-1},x_{m(k)-1}) \longrightarrow \varepsilon. \tag{43}$$

On account of (10), we have

$$\begin{aligned} d&(x_{n(k)}, x_{m(k)}) \\ &= d(Tx_{n(k)-1}, Tx_{m(k)-1}) \\ &\le \alpha(x_{n(k)-1}, x_{m(k)-1}) d(Tx_{n(k)-1}, Tx_{m(k)-1}) \\ &\le \phi(M(x_{n(k)-1}, x_{m(k)-1})) - \varphi(M(x_{n(k)-1}, x_{m(k)-1})), \end{aligned} \tag{44}$$

where

$$\begin{aligned} M&(x_{n(k)-1}, x_{m(k)-1}) \\ &= \max\{d(x_{n(k)-1}, x_{m(k)-1}), d(x_{n(k)-1}, x_{n(k)}), \\ &\qquad d(x_{m(k)-1}, x_{m(k)})\}. \end{aligned} \tag{45}$$

Letting $n \to \infty$, in (44), and regarding definitions of auxiliary functions ϕ, φ and (45), we conclude that

$$\varepsilon \le \varepsilon - \varphi(\varepsilon), \tag{46}$$

which yields that $\varphi(\varepsilon) = 0$. By definition of φ, we derive that $\varepsilon = 0$, which is a contradiction. Hence, we conclude that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{47}$$

Since T is continuous, we obtain from (47) that

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} d(Tx_n, Tu) = 0. \tag{48}$$

From (47) and (48) we get immediately that $\lim_{n\to\infty} T^n x_0 = \lim_{n\to\infty} Tx_n = Tu$. Taking Proposition 5 into account, we conclude that $Tu = u$. □

The following result is deduced from the obvious inequality $N(x, y) \le M(x, y)$.

Theorem 22. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a (α-ϕ-φ)-contractive mapping of type II. Suppose that*

(i) *T is a weak triangular α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*

(iii) *T is continuous.*

Then there exists a $u \in X$ such that $Tu = u$.

Theorem 23. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a (α-ϕ-φ)-contractive mapping of type I. Suppose that*

(i) *T is a weak triangular α-admissible and ϕ is upper semicontinuous function;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \ge 1$ for all n.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof. Following the proof of Theorem 21, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \ge 0$ converges for some $u \in X$. We will show that $Tu = u$. Suppose, on the contrary, that $Tu \ne u$. From (17) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \ge 1$ for all k. By applying the quadrilateral inequality together with (10) and (15), for all k, we get that

$$\begin{aligned} d&(u, Tu) \\ &\le d(u, x_{n(k)+2}) + d(x_{n(k)+2}, x_{n(k)+1}) + d(x_{n(k)+1}, Tu) \\ &\le d(u, x_{n(k)+2}) + d(x_{n(k)+2}, x_{n(k)+1}) + d(Tx_{n(k)}, Tu) \\ &\le d(u, x_{n(k)+2}) + d(x_{n(k)+2}, x_{n(k)+1}) \\ &\quad + \alpha(x_{n(k)}, u) d(Tx_{n(k)}, Tu) \\ &\le d(u, x_{n(k)+2}) + d(x_{n(k)+2}, x_{n(k)+1}) + \phi(M(x_{n(k)}, u)), \end{aligned} \tag{49}$$

where

$$M(x_{n(k)}, u) = \max\{d(x_{n(k)}, u), d(x_{n(k)}, Tx_{n(k)}), d(u, Tu)\}. \tag{50}$$

Letting $k \to \infty$ in the above equality and regarding that the ϕ is an upper semicontinuous mapping, we find that

$$d(u, Tu) \le \phi(d(u, Tu)). \tag{51}$$

It implies that from (ϕ_2)

$$d(u, Tu) \le \phi(d(u, Tu)) < d(u, Tu), \tag{52}$$

which is a contradiction. Hence, we obtain that u is a fixed point of T; that is, $Tu = u$. □

In the following theorem, we remove the semicontinuity of ϕ by weakening the contractive mapping type.

Theorem 24. *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a (α-ϕ-φ)-contractive mapping of type II. Suppose that*

(i) *T is a weak triangular α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \ge 1$ for all n.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof. Following the proof of Theorem 21, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \ge 0$ converges for some $u \in X$. We will show that $Tu = u$. Suppose, on the contrary, that $Tu \ne u$. From (17) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \ge 1$

for all k. By applying the quadrilateral inequality together with (10) and (15), for all k, we get that

$$\begin{aligned} d(u,Tu) &\leq d(u,x_{n(k)+2}) + d(x_{n(k)+2},x_{n(k)+1}) + d(x_{n(k)+1},Tu) \\ &\leq d(u,x_{n(k)+2}) + d(x_{n(k)+2},x_{n(k)+1}) + d(Tx_{n(k)},Tu) \\ &\leq d(u,x_{n(k)+2}) + d(x_{n(k)+2},x_{n(k)+1}) \\ &\quad + \alpha(x_{n(k)},u)\, d(Tx_{n(k)},Tu) \\ &\leq d(u,x_{n(k)+2}) + d(x_{n(k)+2},x_{n(k)+1}) + \phi(N(x_{n(k)},u)), \end{aligned} \tag{53}$$

where

$$N(x_{n(k)},u) = \max\left\{d(x_{n(k)},u), \frac{d(x_{n(k)},Tx_{n(k)}) + d(u,Tu)}{2}\right\}. \tag{54}$$

Letting $k \to \infty$ in the above equality and regarding (ϕ_4), we find that

$$d(u,Tu) \leq \phi\left(\frac{d(u,Tu)}{2}\right) \leq \frac{d(u,Tu)}{2} \tag{55}$$

which is a contradiction. Hence, we obtain that u is a fixed point of T; that is, $Tu = u$. □

Theorem 25. *Adding condition (U) to the hypotheses of Theorem 21 (resp., Theorem 23), one obtains that u is the unique fixed point of T.*

Proof. In what follows we will show that u is a unique fixed point of T. We will use the *reductio ad absurdum*. Let v be another fixed point of T with $v \neq u$. It is evident that $\alpha(u,v) = \alpha(Tu,Tv)$.

Now, due to (10) and (ϕ_2), we have

$$\begin{aligned} d(u,v) &\leq \alpha(u,v)\, d(u,v) \\ &= \alpha(u,v)\, d(Tu,Tv) \\ &\leq \phi(M(u,v)) - \varphi(M(u,v)) \\ &= \phi(d(u,v)) - \varphi(d(u,v)) \\ &\leq \phi(d(u,v)) \\ &< d(u,v) \end{aligned} \tag{56}$$

which is a contradiction, where

$$M(u,v) = \max\{d(u,v), d(u,Tu), d(v,Tv)\} = d(u,v). \tag{57}$$

Hence, $u = v$. □

Theorem 26. *Adding condition (U) to the hypotheses of Theorem 22 (resp., Theorem 24), one obtains that u is the unique fixed point of T.*

Proof. The proof is analog of the proof of Theorem 25 which will be concluded by using the *reductio ad absurdum*. Suppose, on the contrary, that v is another fixed point of T with $v \neq u$. It is evident that $\alpha(u,v) = \alpha(Tu,Tv)$.

Now, due to (12) and (ϕ_2), we have

$$\begin{aligned} d(u,v) &\leq \alpha(u,v)\, d(u,v) \\ &= \alpha(u,v)\, d(Tu,Tv) \\ &\leq \phi(N(u,v)) - \varphi(N(u,v)) \\ &= \phi(d(u,v)) - \varphi(d(u,v)) \\ &= \phi(d(u,v)) \\ &< d(u,v) \end{aligned} \tag{58}$$

which is a contradiction, where

$$N(u,v) = \max\left\{d(u,v), \frac{d(u,Tu) + d(v,Tv)}{2}\right\} = d(u,v). \tag{59}$$

Hence, $u = v$. □

For the uniqueness, we can also consider the following condition.

(H^*) For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$. Further, $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$, where $z_1 = z$ and $z_{n+1} = Tz_n$ for $n = 1, 2, 3, \ldots$.

Theorem 27. *Adding conditions (H^*) and (W) to the hypotheses of Theorem 21 (resp., Theorem 23), one obtains that u is the unique fixed point of T.*

Proof. Suppose that v is another fixed point of T. From (H^*), there exists $z \in X$ such that

$$\alpha(u,z) \geq 1, \qquad \alpha(v,z) \geq 1. \tag{60}$$

Since T is α-admissible, from (60), we have

$$\alpha(u,T^n z) \geq 1, \quad \alpha(v,T^n z) \geq 1, \quad \forall n. \tag{61}$$

Define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$ for all $n \geq 0$ and $z_0 = z$. From (61), for all n, we have

$$\begin{aligned} d(u,z_{n+1}) = d(Tu,Tz_n) &\leq \alpha(u,z_n)\, d(Tu,Tz_n) \\ &\leq \phi(M(u,z_n)) - \varphi(M(u,z_n)), \end{aligned} \tag{62}$$

where

$$\begin{aligned} M(u,z_n) &= \max\{d(u,z_n), d(u,Tu), d(z_n,Tz_n)\} \\ &= \max\{d(u,z_n), d(z_n,Tz_n)\}. \end{aligned} \tag{63}$$

If $M(u,z_n) = d(z_n,Tz_n)$ then by letting $n \to \infty$ in (62) we get that

$$\lim_{n\to\infty} d(z_n,u) = 0, \tag{64}$$

due to the continuity of φ, (ϕ_4) and the fact that $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$. If $M(u, z_n) = d(u, z_n)$ then (62) turns into

$$d(u, z_{n+1}) \le \phi(d(u, z_n)) - \varphi(d(u, z_n)) \le \phi(d(u, z_n)). \tag{65}$$

Iteratively, by using inequality (62), we get that

$$d(u, z_{n+1}) \le \phi^n(d(u, z_0)), \tag{66}$$

for all n. Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} d(z_n, u) = 0. \tag{67}$$

Similarly, one can show that

$$\lim_{n\to\infty} d(z_n, v) = 0. \tag{68}$$

Regarding (W) together with (67) and (68), it follows that $u = v$. Thus we proved that u is the unique fixed point of T. □

Theorem 28. *Adding conditions (H^*) and (W) to the hypotheses of Theorem 22 (resp., Theorem 24), one obtains that u is the unique fixed point of T.*

The proof is the analog of the proof of Theorem 27; hence we omit it.

Corollary 29. *Adding condition (H^*) to the hypotheses of Theorem 21 (resp., Theorems 23, 22, and 24) and assuming that (X, d) is Hausdorff, one obtains that u is the unique fixed point of T.*

The proof is clear, and hence it is omitted. Indeed, Hausdorffness implies the uniqueness of the limit. Thus, the theorem above yields the conclusions.

3. Consequences

Now, we will show that many existing results in the literature can be deduced easily from Theorems 13 and 14.

Definition 30. Let (X, d) be a generalized metric space and let $T : X \to X$ be a given mapping. We say that T is a (α-ϕ-φ)-contractive mapping of type III if there exist functions $\alpha : X \times X \to [0, \infty)$, $\varphi \in \Theta$, and $\phi \in \Phi$ such that

$$\alpha(x, y) d(Tx, Ty) \le \phi(d(x, y)) - \varphi(d(x, y)) \tag{69}$$

for all $x, y \in X$.

Now, we state the first fixed point theorem.

Theorem 31. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a (α-ϕ-φ)-contractive mapping of type III. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$ and $\alpha(x_0, T^2x_0) \ge 1$;*

(iii) *T is continuous.*

Then, T has a fixed point $u \in X$; that is, $Tu = u$.

We omit the proof of Theorem 31, since it can be derived easily by following the lines in the proof of Theorem 21, analogously.

Theorem 32. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a (α-ϕ-φ)-contractive mapping of type III. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$ and $\alpha(x_0, T^2x_0) \ge 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \ge 1$ for all n.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof. Following the proof of Theorem 21 (resp., Theorem 31), we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \ge 0$ converges for some $u \in X$. We will show that $Tu = u$. Suppose, on the contrary, that $Tu \ne u$. From (17) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \ge 1$ for all k. By applying the quadrilateral inequality together with (10) and (15), for all k, we get that

$$\begin{aligned} d(u, Tu) &\le d(u, x_{n(k)+2}) + d(x_{n(k)+2}, x_{n(k)+1}) + d(x_{n(k)+1}, Tu) \\ &\le d(u, x_{n(k)+2}) + d(x_{n(k)+2}, x_{n(k)+1}) + d(Tx_{n(k)}, Tu) \\ &\le d(u, x_{n(k)+2}) + d(x_{n(k)+2}, x_{n(k)+1}) \\ &\quad + \alpha(x_{n(k)}, u) d(Tx_{n(k)}, Tu) \\ &\le d(u, x_{n(k)+2}) + d(x_{n(k)+2}, x_{n(k)+1}) + \phi(d(x_{n(k)}, u)). \end{aligned} \tag{70}$$

Letting $k \to \infty$ in the above equality and regarding (ϕ_4), we find that

$$d(u, Tu) \le \lim_{n\to\infty} \phi(d(x_{n(k)}, u)) \le 0, \tag{71}$$

which is a contradiction. Hence, we obtain that u is a fixed point of T; that is, $Tu = u$. □

Theorem 33. *Adding condition (U) to the hypotheses of Theorem 31 (resp., Theorem 32), one obtains that u is the unique fixed point of T.*

Proof. In what follows we will show that u is a unique fixed point of T. We will use the *reductio ad absurdum*. Let v be another fixed point of T with $v \ne u$. It is evident that $\alpha(u, v) = \alpha(Tu, Tv)$.

Now, due to (10) and (ϕ_2), we have

$$\begin{aligned} d(u,v) &\le \alpha(u,v)\, d(u,v) \\ &= \alpha(Tu,Tv)\, d(Tu,Tv) \\ &\le \phi(d(u,v)) - \varphi(d(u,v)) \\ &\le \phi(d(u,v)) \\ &< d(u,v) \end{aligned} \tag{72}$$

which is a contradiction. □

Theorem 34. *Adding conditions (H) and (W) to the hypotheses of Theorem 31 (resp., Theorem 32), one obtains that u is the unique fixed point of T.*

Corollary 35. *Let (X,d) be a complete generalized metric space and let $T : X \to X$ be a given mapping. Suppose that there exist $\phi \in \Phi$ and $\varphi \in \Theta$ such that*

$$d(Tx,Ty) \le \phi(M(x,y)) - \varphi(M(x,y)), \tag{73}$$

for all $x, y \in X$, where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}. \tag{74}$$

Then T has a unique fixed point.

Proof. Let $\alpha : X \times X \to [0,\infty)$ be the mapping defined by $\alpha(x,y) = 1$, for all $x, y \in X$. Then T is a $(\alpha$-ϕ-$\varphi)$-contractive mapping of type I. It is evident that all conditions of Theorem 21 are satisfied. Hence, T has a unique fixed point. □

Corollary 36. *Let (X,d) be a complete generalized metric space and let $T : X \to X$ be a given mapping. Suppose that there exist $\phi \in \Phi$ and $\varphi \in \Theta$ such that*

$$d(Tx,Ty) \le \phi(N(x,y)) - \varphi(N(x,y)), \tag{75}$$

for all $x, y \in X$, where

$$N(x,y) = \max\left\{d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}\right\}. \tag{76}$$

Then T has a unique fixed point.

The following Corollary is stronger than the main result of [19]. Notice that we do not need the Hausdorffness condition although it was required in [19].

Corollary 37. *Let (X,d) be a complete generalized metric space and let $T : X \to X$ be a given mapping. Suppose that there exist $\phi \in \Phi$ and $\varphi \in \Theta$ such that*

$$d(Tx,Ty) \le \phi(d(x,y)) - \varphi(d(x,y)), \tag{77}$$

for all $x, y \in X$. Then T has a unique fixed point.

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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