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Research Article

The Best Approximation Theorems and Fixed Point Theorems for Discontinuous Increasing Mappings in Banach Spaces

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We prove that Fan's theorem is true for discontinuous increasing mappings f in a real partially ordered reflexive, strictly convex, and smooth Banach space X. The main tools of analysis are the variational characterizations of the generalized projection operator and order-theoretic fixed point theory. Moreover, we get some properties of the generalized projection operator in Banach spaces. As applications of our best approximation theorems, the fixed point theorems for non-self-maps are established and proved under some conditions. Our results are generalizations and improvements of the recent results obtained by many authors.

1. Introduction

Let X be a real Banach space with the dual space X^* and $C \subset X$ a nonempty subset of X. The set-valued mapping $P_C : X \to C$,

$$P_C(x) = \left\{ z \in C : \|x - z\| = \inf_{y \in C} \|x - y\| \right\}, \quad (1)$$

is called the metric projection operator from X onto C. It is well known that the metric projection operator P_C plays an important role in nonlinear functional analysis, optimization theory, fixed point theory, nonlinear programming, game theory, variational inequality, complementarity problems, and so forth.

In 1994, Alber [1] introduced the generalized projections $\pi_C : X^* \to C$ and $\Pi_C : X \to C$ from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces and studied their properties in detail. In [2], Li extended the generalized projection operator π_C from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces and studied some properties of the generalized projection operator with applications to solving the variational inequality in Banach spaces. Recently, Isac [3] and Nishimura and Ok [4] studied the order-theoretic approach towards establishing the solvability of variational inequality on a Hilbert lattice X which is based on the fact that the metric projection operator P_C is order-preserving if only if C is a sublattice of X. Very recently, Li and Ok [5] obtained the generalized projection operator π_C is order-preserving in partially ordered Banach spaces.

Motivated and inspired by the above mentioned work, in this paper, we get the continuous property of generalized projection operator Π_C and increasing characterizations of Π_C in a partially ordered reflexive, strict convex, and smooth Banach space. Further, we consider the following Fan's approximation theorem (Theorem 2 in [6]) through the variational characterization of Π_C. The normed space version of the theorem is as follows.

Theorem 1. *Let C be a nonempty compact convex set in a normed linear space X. If f is a continuous map from C into X, then there exists a point u in C such that*

$$\|u - f(u)\| = d(f(u), C). \quad (2)$$

The point u is called a best approximation point of f in C.

Fan's theorem has been of great importance in nonlinear analysis, approximation theory, game theory, and minimax theorems. Various aspects of this theorem have been studied by many authors under different assumptions. For some related works, refer to [7–21] and the references therein.

In this paper, we obtain the existence of minimum best approximation point and maximum best approximation point in order interval. As an applications of our best approximation theorems, the fixed point theorems for non-self-maps are established under some conditions which do not need to require any continuous and compact conditions on f.

The content of the present work can be summarized as follows. In Section 2, we review the definition of the generalized projection operator in Banach spaces and its basic properties. We also show some definitions in the partially ordered Banach space and some fundamental results for our theorems. In Section 3, we obtain the properties of the generalized projection operator in the partially ordered Banach space under some assumption. And we combine these results with an order-theoretic fixed point theorem to provide some of the best approximation theorems. Section 4 provides an application of these best approximation theorems to fixed point theory.

2. Preliminaries

2.1. The Partial Order. Suppose that X is a real Banach space and P is a nonempty closed convex cone of X. By θ we denote the zero element of X. We define a partial order $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We will write $x \prec y$ if $x \preceq y$ and $x \neq y$.

The cone P is called normal if there is a number $K > 0$, such that for all $x, y \in X$, $0 \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that $x_1 \preceq x_2 \preceq \cdots \preceq y$ for some $y \in X$, then there is $x \in X$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It has been proved in Theorem 1.2.1 in [22] that every regular cone is normal.

A cone P is called minihedral, if each two-element set $\{x, y\}$ has a least upper bound $\sup\{x, y\}$. Equivalently, the cone P is minihedral if and only if each two-element set $\{x, y\}$ has a greatest lower bound $\inf\{x, y\}$. As is convenient, we denote $\sup\{x, y\}$ as $x \vee y$ and $\inf\{x, y\}$ as $x \wedge y$. And if $\sup M$ exists for every nonempty and bounded from above $M \subset X$, we say the cone P is a strongly minihedral cone. If M is a nonempty subset of X which contains $x \vee y$ and $x \wedge y$ for every $x, y \in M$, then M is said to be subminihedral.

Let $(X, \preceq)$ be a real partially ordered Banach space. Given $u_0, v_0 \in X$ such that $u_0 \prec v_0$, the set $[u_0, v_0] = \{z \in X : u_0 \preceq z \preceq v_0\}$ is called ordered interval. If the cone P is minihedral, it is easy to see that $[u_0, v_0]$ is a subminihedral set of X.

Definition 2 (see [5]). For any partially ordered spaces $(X, \preceq_X)$ and $(Y, \preceq_Y)$, we say that a map $F : X \to Y$ is order-preserving if

$$x \preceq_X y \text{ implies } F(x) \preceq_Y F(y). \tag{3}$$

Definition 3 (see [23]). Let $(X, \preceq)$ be a partially ordered space and $D \subset X$ is convex; we say that a map $F : D \to X$ is convex if

$$F(tx + (1-t)y) \preceq tF(x) + (1-t)F(y), \quad \forall x, y \in D, \quad x \preceq y, \quad 0 \leq t \leq 1. \tag{4}$$

2.2. Order-Dual. Let $(X, \preceq)$ be a real partially ordered Banach space whose (topological) dual we denote by X^* and P a cone in X. Recall that $P^* = \{\phi \in X^* : \phi(x) \geq 0,\ \forall x \in P\}$ is called the dual cone of P. The dual of $\preceq$ is the partial order $\preceq^*$ on X^* defined as follows:

$$\phi \preceq^* \varphi \quad \text{iff } \varphi - \phi \in P^*. \tag{5}$$

If P is a minihedral cone, it is well known that P^* is a minihedral cone in X^*. We now show that $x \in P$ if and only if $\langle \varphi, x\rangle \geq 0$ for every $\varphi \in P^*$ (see [24, Proposition 1.4.2]).

We denote by $(H, \|\cdot\|_1)$ a Hilbert space H whose norm $\|\cdot\|_1$ satisfies

$$|x| \preceq |y| \text{ implies } \|x\|_1 \leq \|y\|_1, \quad \forall x, y \in H, \tag{6}$$

where $|x|$ is defined by $|x| = x \vee (-x)$ for each $x \in H$.

2.3. The Generalized Projection Operator. Let X be a real Banach space with the dual X^*. We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$Jx = \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\|\|x\|,\ \|x\| = \|x^*\|\}, \tag{7}$$

for all $x \in X$, where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between X^* and X. See [1] for basic characterizations of the normalized duality mapping.

Recall that a Banach space X has the Kadec-Klee property, if for any sequence $\{x_n\} \subset X$ and $x \in X$ with $x_n \rightharpoonup x$ (weak convergence) and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$, as $n \to \infty$. It is well known that if X is a uniformly convex Banach space, then X has the Kadec-Klee property.

Let X be a reflexive, strictly convex, and smooth Banach space and C a nonempty closed convex subset of X. Consider the Lyapunov functional defined by

$$W(x, y) = \|x\|^2 - 2\langle Jx, y\rangle + \|y\|^2, \quad \forall x, y \in X. \tag{8}$$

Following Alber [1], the generalized projection $\Pi_C : X \to C$ is a map that assigns to an arbitrary point $x \in X$ the minimum point of the functional $W(x, y)$; that is, $\Pi_C(x) = \widehat{x}$, where $\widehat{x} \in C$ is the solution to the minimization problem:

$$W(x, \widehat{x}) = \inf_{y\in C} W(x, y); \tag{9}$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $W(x, y)$ and strict monotonicity of the mapping J. It is obvious from the definition of functional W that

$$(\|x\| - \|y\|)^2 \leq W(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in X. \tag{10}$$

If X is a Hilbert space, then $W(x, y) = (\|x - y\|)^2$ and $\Pi_C = P_C$.

If X is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in X$, $W(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $W(x, y) = 0$ then $x = y$. From (10), we have $\|x\| = \|y\|$. This implies that $\langle Jx, y\rangle = \|y\|^2 = \|Jx\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$. See [25, 26] for more details.

In [1], the generalized projection operators on arbitrary convex closed sets C satisfy the following property.

The point $\Pi_C(x) = \hat{x}$ is a generalized projection of x on $C \subset X$ if and only if the following inequality is satisfied:

$$\langle Jx - J\hat{x}, \hat{x} - y\rangle \geq 0, \quad \forall y \in C. \tag{11}$$

We denote $d_W(x, C) = \inf\{W(x, y) : y \in C\}$, where $x \in X$ and W is Lyapunov functional in X.

3. Best Approximation Theorems

First we give the following properties of the generalized projection operators.

Lemma 4 (see [27]). *Let $(X, \preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to a minihedral cone P. Suppose P^* is the dual cone of P. The following statements are equivalent:*

(H_1) the normalized duality mapping J is order-preserving;

(H_2) $\|Jx \wedge Jy\|^2 + \|Jx \vee Jy\|^2 \leq \|x\|^2 + \|y\|^2, \forall x, y \in X, x \preceq y$.

Lemma 5 (see [27]). *Let $(X, \preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to a minihedral cone P and satisfy condition (H_2). Suppose that C is closed convex subminihedral set of X. Moreover, C satisfies the condition:*

(H_3) $\|x \wedge y\|^2 + \|x \vee y\|^2 \leq \|x\|^2 + \|y\|^2$, $\forall x, y \in C$.

Then, Π_C is increasing.

Remark 6. The minihedral cones of many Banach spaces satisfy (H_3). For example, if $p \geq 2$, every subminihedral set M of $(\ell^p, \preceq)$ (here partial order $\preceq$ is defined coordinatewise) such that $x \succeq \theta$, $\forall x \in M$, then M satisfies (H_3); if $p \geq 2$, every subminihedral set M of $(R^{n,p}, \preceq)$ (here $\preceq$ stands again for the coordinatewise ordering), such that $x \succeq \theta$, $\forall x \in M$, then M satisfies (H_3). See [5] for more details.

Lemma 7. *If X is a uniformly convex and smooth Banach space and C is a nonempty, closed, and convex subset of X, then the generalized projection operator $\Pi_C : X \to C$ is continuous.*

Proof. Since X is a uniformly convex and smooth Banach space, Π_C is single valued. Suppose $x_n \to x$, as $n \to \infty$, and suppose $\Pi_C(x_n) = \hat{x}_n$ $(n = 1, 2, 3, \ldots)$, and $\Pi_C(x) = \hat{x}$. From the inequalities

$$\begin{aligned}(\|x_n\| - \|\hat{x}_n\|)^2 &\leq W(x_n, \hat{x}_n) \\ &\leq W(x_n, \hat{x}) \\ &\leq (\|x_n\| + \|\hat{x}\|)^2\end{aligned} \tag{12}$$

and the hypothesis that $x_n \to x$, as $n \to \infty$, it yields $\{\hat{x}_n\}$ is a bounded subset of X. Since X is reflexive, there exists a subsequence of $\{\hat{x}_n\}$; without loss of the generality, we may assume it is itself, such that $\{\hat{x}_n\}$ converges weakly to x'. From the properties of weakly convergence, we have $\|x'\| \leq \liminf_{n\to\infty} \|\hat{x}_n\|$. Moreover, $W(x, \hat{x}) \leq W(x, \hat{x}_n)$ and $W(x_n, \hat{x}_n) \leq W(x_n, \hat{x})$, which implies $W(x, \hat{x}_n) \to W(x, \hat{x})$, as $n \to \infty$. Now we have

$$\begin{aligned}W(x, x') &= \|x\|^2 - 2\langle Jx, x'\rangle + \|x'\|^2 \\ &= \lim_{n\to\infty} \left(\|x\|^2 - 2\langle Jx, \hat{x}_n\rangle + \|x'\|^2\right) \\ &\leq \liminf_{n\to\infty} \left(\|x\|^2 - 2\langle Jx, \hat{x}_n\rangle + \|\hat{x}_n\|^2\right) \\ &= \liminf_{n\to\infty} W(x, \hat{x}_n) \\ &= \lim_{n\to\infty} W(x, \hat{x}_n) \\ &= \inf_{y\in C} W(x, y).\end{aligned} \tag{13}$$

Thus we have $x' = \hat{x}$.

For any $\lambda \in [0, 1]$, one has $\lambda\hat{x} + (1 - \lambda)\hat{x}_n \in C$. From the inequality $W(x, \hat{x}) \leq W(x, \lambda\hat{x} + (1 - \lambda)\hat{x}_n)$, we have

$$\begin{aligned}&\|x\|^2 - 2\langle Jx, \hat{x}\rangle + \|\hat{x}\|^2 \leq \|x\|^2 \\ &\quad \leq \|x\|^2 - 2\langle Jx, \lambda\hat{x} + (1 - \lambda)\hat{x}_n\rangle + \|\lambda\hat{x} + (1 - \lambda)\hat{x}_n\|^2.\end{aligned} \tag{14}$$

Therefore,

$$2\langle Jx, (1 - \lambda)(\hat{x}_n - \hat{x})\rangle \leq \|\lambda\hat{x} + (1 - \lambda)\hat{x}_n\|^2 - \|\hat{x}\|^2. \tag{15}$$

Similar to the above argument, from inequality $W(x_n, \hat{x}_n) \leq W(x_n, \hat{x})$, we obtain

$$2\langle Jx_n, \hat{x} - \hat{x}_n\rangle \leq \|\hat{x}\|^2 - \|\hat{x}_n\|^2. \tag{16}$$

Adding the above two inequalities side by side, we obtain

$$\begin{aligned}&2\langle Jx - Jx_n, \hat{x}_n - \hat{x}\rangle \\ &\quad \leq \|\lambda\hat{x} + (1 - \lambda)\hat{x}_n\|^2 \\ &\qquad - \|\hat{x}_n\|^2 + 2\lambda\langle Jx, \hat{x}_n - \hat{x}\rangle \\ &\quad \leq \lambda^2\|\hat{x}\|^2 \\ &\qquad + 2\lambda(1 - \lambda)\|\hat{x}\|\|\hat{x}_n\| + (1 - \lambda)^2\|\hat{x}_n\|^2\end{aligned}$$

$$\begin{aligned} &- \|\hat{x}_n\|^2 + 2\lambda \langle Jx, \hat{x}_n - \hat{x}\rangle \\ &\leq \lambda^2 \|\hat{x}\|^2 + \lambda (1-\lambda) \left(\|\hat{x}\|^2 + \|\hat{x}_n\|^2\right) \\ &\quad + (1-\lambda)^2 \|\hat{x}_n\|^2 - \|\hat{x}_n\|^2 + 2\lambda \langle Jx, \hat{x}_n - \hat{x}\rangle \\ &= \lambda \left(\|\hat{x}\|^2 - \|\hat{x}_n\|^2\right) + 2\lambda \langle Jx, \hat{x}_n - \hat{x}\rangle . \end{aligned} \tag{17}$$

So

$$2\langle Jx - Jx_n, \hat{x} - \hat{x}_n\rangle \geq \lambda \left(\|\hat{x}_n\|^2 - \|\hat{x}\|^2\right) + 2\lambda \langle Jx, \hat{x} - \hat{x}_n\rangle . \tag{18}$$

If we use the inequalities $W(x,\hat{x}) \leq W(x,\hat{x}_n)$ and $W(x_n,\hat{x}_n) \leq W(x_n,\lambda\hat{x} + (1-\lambda)\hat{x}_n)$, similar to the above argument, we obtain

$$\begin{aligned} 2\langle Jx - Jx_n, \hat{x} - \hat{x}_n\rangle &\geq (1-\lambda) \left(\|\hat{x}\|^2 - \|\hat{x}_n\|^2\right) \\ &\quad + 2(1-\lambda) \langle Jx_n, \hat{x}_n - \hat{x}\rangle . \end{aligned} \tag{19}$$

In (18) and (19), taking $\lambda = 1/2$, we have

$$\begin{aligned} 4\langle Jx - Jx_n, \hat{x} - \hat{x}_n\rangle &\geq \left(\|\hat{x}_n\|^2 - \|\hat{x}\|^2\right) + 2\langle Jx, \hat{x} - \hat{x}_n\rangle , \\ 4\langle Jx - Jx_n, \hat{x} - \hat{x}_n\rangle &\geq \left(\|\hat{x}\|^2 - \|\hat{x}_n\|^2\right) + 2\langle Jx_n, \hat{x}_n - \hat{x}\rangle . \end{aligned} \tag{20}$$

From the conditions that $x_n \to x$, as $n \to \infty$ and X is a smooth Banach space, we have $Jx_n \to Jx$, as $n \to \infty$. Using $\hat{x}_n \rightharpoonup \hat{x}$, as $n \to \infty$ and combining (20), it yields $\|\hat{x}_n\| \to \|\hat{x}\|$, as $n \to \infty$. Since X is a uniformly convex Banach space, then X has the Kadec-Klee property. Therefore, we obtain $\hat{x}_n \to \hat{x}$, as $n \to \infty$. Thus this lemma is proved. □

Lemma 8. *Let $(X, \preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to P and satisfy condition (H_2). Suppose that P is a minihedral cone and satisfies the condition:*

(H_4) $\|x \wedge y\|^2 + \|x \vee y\|^2 \leq \|x\|^2 + \|y\|^2$, $\forall x \in X$, $y \in P$.

Then, Π_P is increasing, and $\Pi_P(x+y) \preceq \Pi_P(x) + \Pi_P(y)$, $\forall x, y \in X$.

Proof. Since (H_4) implies (H_3) and P is subminihedral, from Lemma 5, Π_P is increasing. Next, we prove $x \preceq \Pi_P(x)$, $\forall x \in X$. To derive a contradiction, assume that there exists x_0 which does not satisfy $x_0 \preceq \Pi_P(x_0)$; that is, $x_0 \wedge \Pi_P(x_0) \neq x_0$ and $x_0 \vee \Pi_P(x_0) \neq \Pi_P(x_0)$. Then we have

$$W(x_0, \Pi_P(x_0)) < W(x_0, x_0 \vee \Pi_P(x_0)); \tag{21}$$

that is,

$$\begin{aligned} &\|x_0\|^2 - 2\langle Jx_0, \Pi_P(x_0)\rangle + \|\Pi_P(x_0)\|^2 \\ &\quad < \|x_0\|^2 - 2\langle Jx_0, x_0 \vee \Pi_P(x_0)\rangle + \|x_0 \vee \Pi_P(x_0)\|^2 . \end{aligned} \tag{22}$$

Hence,

$$\begin{aligned} &2\langle Jx_0, x_0 \vee \Pi_P(x_0) - \Pi_P(x_0)\rangle \\ &\quad < \|x_0 \vee \Pi_P(x_0)\|^2 - \|\Pi_P(x_0)\|^2 . \end{aligned} \tag{23}$$

As $x_0 \wedge \Pi_P(x_0) \neq x_0$, we have

$$\begin{aligned} W(x_0, x_0 \wedge \Pi_P(x_0)) &= \|x_0\|^2 - 2\langle Jx_0, x_0 \wedge \Pi_P(x_0)\rangle \\ &\quad + \|x_0 \wedge \Pi_P(x_0)\|^2 > 0, \end{aligned} \tag{24}$$

and then,

$$2\langle Jx_0, x_0 \wedge \Pi_P(x_0)\rangle < \|x_0\|^2 + \|x_0 \wedge \Pi_P(x_0)\|^2 . \tag{25}$$

Since $x_0 \wedge \Pi_P(x_0) + x_0 \vee \Pi_P(x_0) = x_0 + \Pi_P(x_0)$, from (23) and (25), we have

$$\begin{aligned} 2\langle Jx_0, x_0\rangle &< \|x_0\|^2 + \|x_0 \wedge \Pi_P(x_0)\|^2 \\ &\quad + \|x_0 \vee \Pi_P(x_0)\|^2 - \|\Pi_P(x_0)\|^2 . \end{aligned} \tag{26}$$

And hence $\|x_0 \wedge \Pi_P(x_0)\|^2 + \|x_0 \vee \Pi_P(x_0)\|^2 - \|\Pi_P(x_0)\|^2 - \|x_0\|^2 > 0$. This contradicts (H_4). Thus, $x \preceq \Pi_P(x)$, $\forall x \in X$. And hence,

$$x + y \preceq \Pi_P(x) + \Pi_P(y), \quad \forall x, y \in X. \tag{27}$$

As Π_P is increasing, we have

$$\Pi_P(x+y) \preceq \Pi_P(x) + \Pi_P(y), \quad \forall x, y \in X. \tag{28}$$

The assertion is proved. □

Lemma 9. *Let $(X, \preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to a minihedral cone P and satisfy condition (H_2). Suppose $u_0, v_0 \in X$ with $u_0 \prec v_0$ and the following condition is satisfied:*

(H_5) $\|x \wedge y\|^2 + \|x \vee y\|^2 \leq \|x\|^2 + \|y\|^2$, $\forall x \in X$, $y \in [u_0, v_0]$.

Then, $\Pi_{[u_0,v_0]}$ is increasing, and

$$\begin{aligned} &\Pi_{[u_0,v_0]}(tx + (1-t)y) \preceq t\Pi_{[u_0,v_0]}(x) + (1-t)\Pi_{[u_0,v_0]}(y) \\ &\qquad \forall t \in [0,1], \quad \forall x, y \preceq v_0. \end{aligned} \tag{29}$$

Proof. Following a similar argument as in the proof of Lemma 8, we obtain that $\Pi_{[u_0,v_0]}$ is increasing and $x \preceq \Pi_{[u_0,v_0]}(x)$, $\forall x \preceq v_0$. And hence,

$$\begin{aligned} &tx + (1-t)y \preceq t\Pi_{[u_0,v_0]}(x) + (1-t)\Pi_{[u_0,v_0]}(y), \\ &\qquad \forall t \in [0,1], \quad x, y \preceq v_0. \end{aligned} \tag{30}$$

As $\Pi_{[u_0,v_0]}$ is increasing and $t\Pi_{[u_0,v_0]}(x) + (1-t)\Pi_{[u_0,v_0]}(y) \in [u_0, v_0]$, we have

$$\Pi_{[u_0,v_0]}(tx + (1-t)y) \preceq t\Pi_{[u_0,v_0]}(x) + (1-t)\Pi_{[u_0,v_0]}(y). \tag{31}$$

The proof is completed. □

Remark 10. If $(H, \|\cdot\|_1)$ is a partially ordered Hilbert space with respect to P and P a minihedral cone, (H_4) and (H_5) are satisfied.

From the above properties of the generalized projection operators and order-theoretic fixed point theorems, we can obtain the following best approximation theorems.

Theorem 11. *Let $(X, \preceq)$ be a real partially ordered uniformly convex and smooth Banach space with respect to a minihedral cone P and satisfy condition (H_2). Suppose that $f : [u_0, v_0] \to X$ is an increasing map. Moreover, $[u_0, v_0]$ satisfies the condition (H_3) and $f([u_0, v_0])$ is relatively compact. Then, f has a minimum best approximation point x_* and a maximum best approximation point x^* with respect to $W(x, y)$ in $[u_0, v_0]$, such that*

$$\begin{aligned} u_0 &\preceq u_1 \preceq \cdots \preceq u_n \preceq \cdots \preceq x_* \preceq x^* \\ &\preceq \cdots \preceq v_n \preceq \cdots \preceq v_1 \preceq v_0, \end{aligned} \tag{32}$$

where $u_n = \Pi_{[u_0,v_0]}(f(u_{n-1}))$, $v_n = \Pi_{[u_0,v_0]}(f(v_{n-1}))$ $(n = 1, 2, 3, \ldots)$.

Proof. Define $F : [u_0, v_0] \to [u_0, v_0]$ by $F(x) = \Pi_{[u_0,v_0]}(f(x))$. From Lemma 5, we get F is increasing. It is easy to see $u_0 \preceq F(u_0)$ and $F(v_0) \preceq v_0$. By Lemma 7, we know $\Pi_{[u_0,v_0]}$ is continuous and $F([u_0, v_0])$ is relatively compact. Thus F satisfies all conditions of Theorem 2.1.4 in [22]. Then, F has a minimum fixed point x_* and a maximum fixed point x^* and satisfies (32). Now we consider $F(x_*) = x_*$, $F(x^*) = x^*$; that is, $\Pi_{[u_0,v_0]}(f(x_*)) = x_*$ and $\Pi_{[u_0,v_0]}(f(x^*)) = x^*$. By the definition of $\Pi_{[u_0,v_0]}$, we get

$$\begin{aligned} W(f(x_*), x_*) &= \inf_{y \in [u_0,v_0]} W(f(x_*), y) \\ &= d_W(f(x_*), [u_0, v_0]), \\ W(f(x^*), x^*) &= \inf_{y \in [u_0,v_0]} W(f(x^*), y) \\ &= d_W(f(x^*), [u_0, v_0]). \end{aligned} \tag{33}$$

The assertion is proved. □

Theorem 12. *Let $(X, \preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to a normal and minihedral cone P and satisfy condition (H_2). Suppose that $f : [u_0, v_0] \to X$ is an increasing map. Moreover, $[u_0, v_0]$ satisfies the condition (H_3). Then, f has a minimum best approximation point x_* and a maximum best approximation point x^* with respect to $W(x, y)$ in $[u_0, v_0]$. Moreover, if $u_n = \Pi_{[u_0,v_0]}(f(u_{n-1}))$, $v_n = \Pi_{[u_0,v_0]}(f(v_{n-1}))$ $(n = 1, 2, 3, \ldots)$, (32) holds.*

Proof. Define $F : [u_0, v_0] \to [u_0, v_0]$ by $F(x) = \Pi_{[u_0,v_0]}(f(x))$. From Lemma 5, we get F is increasing. It is easy to see $u_0 \preceq F(u_0)$ and $F(v_0) \preceq v_0$. Since X is reflexive and P is normal, P is regular. Thus F satisfies all conditions of Theorem 3.1.4 in [23]. Then, F has a minimum fixed point x_* and a maximum fixed point x^* and satisfies (32). By the definition of $\Pi_{[u_0,v_0]}$, the assertion is proved. □

Remark 13. In the above Theorem 11, f is discontinuous map. And in Theorem 12, f is discontinuous map and has no compact conditions.

Example 14. Let $(X, \preceq) = (\ell^2, \preceq)$. Here $\preceq$ stands for the coordinatewise ordering. It is easy to prove that all conditions in Theorem 12 hold. Given $u_0, v_0 \in \ell^2$ such that $u_0 \prec v_0$. Then, every increasing $f : [u_0, v_0] \to \ell^2$ has a minimum best approximation point and a maximum best approximation point with respect to $W(x, y)$ in $[u_0, v_0]$.

Theorem 15. *Let $(X, \preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to P. If $u_0 \prec v_0$ and the following conditions are satisfied,*

(i) *P is a normal, minihedral cone with satisfying (H_2) and (H_5);*

(ii) *$f : [u_0, v_0] \to X$ is an increasing and convex map;*

(iii) *there exists a $0 < \varepsilon < 1$ such that $f(v_0) \preceq \varepsilon u_0 + (1-\varepsilon) v_0$,*

then, f has a unique approximation point $\hat{x}$ with respect to $W(x, y)$ in $[u_0, v_0]$. Moreover, if we take $x_n = \Pi_{[u_0,v_0]}(f(x_{n-1}))$ $(n = 1, 2, 3, \ldots)$ for $\forall x_0 \in [u_0, v_0]$,

$$\|x_n - \hat{x}\| \longrightarrow 0 \quad (n \longrightarrow \infty), \tag{34}$$

$$\|x_n - \hat{x}\| \le M(1-\varepsilon)^n \quad (n = 1, 2, 3, \ldots), \tag{35}$$

where $M > 0$ has nothing to do with x_0.

Proof. Define $F : [u_0, v_0] \to [u_0, v_0]$ by $F(x) = \Pi_{[u_0,v_0]}(f(x))$. Since f is convex and $\Pi_{[u_0,v_0]}$ is increasing, for $\forall t \in [0, 1]$, we have

$$\begin{aligned} F(tx + (1-t)y) &= \Pi_{[u_0,v_0]}(f(tx + (1-t)y)) \\ &\preceq \Pi_{[u_0,v_0]}(tf(x) + (1-t)f(y)). \end{aligned} \tag{36}$$

Using Lemma 9 and $f(x) \preceq f(v_0) \preceq v_0$, we obtain

$$\begin{aligned} F(tx + (1-t)y) &\preceq t\Pi_{[u_0,v_0]}(f(x)) + (1-t)\Pi_{[u_0,v_0]}(f(y)) \\ &= tF(x) + (1-t)F(y). \end{aligned} \tag{37}$$

Thus F is convex. And $F(v_0) \preceq \varepsilon u_0 + (1-\varepsilon)v_0$. Thus F satisfies all conditions of Theorem 3.1.6 in [23]. Then, F has a unique fixed point $\hat{x}$ and satisfies (35). By the definition of $\Pi_{[u_0,v_0]}$, the assertion is proved. □

4. Fixed Point Theorems

In this section, we will prove some new fixed point theorems for non-self-maps by using results of Section 3.

Theorem 16. *Let $(X, \preceq)$ be a real partially ordered uniformly convex and smooth Banach space with respect to a minihedral cone P and satisfy condition (H_2). Suppose that $f : [u_0, v_0] \to X$ is an increasing map and $f([u_0, v_0])$ is relative compact. Moreover, $[u_0, v_0]$ satisfies the condition (H_3) and*

$$|\mathrm{co}\{x, f(x)\} \cap [u_0, v_0]| \ge 2, \quad \forall x \in [u_0, v_0]. \tag{38}$$

Then, f has at least one fixed point in $[u_0, v_0]$.

Proof. By Theorem 11, f has at least one best approximation point $\hat{x}$ in $[u_0, v_0]$; that is, $\Pi_{[u_0,v_0]}(f(\hat{x})) = \hat{x}$. From (11), we have

$$\langle J(f(\hat{x})) - J\hat{x}, \hat{x} - y\rangle \geq 0, \quad \forall y \in [u_0, v_0]. \tag{39}$$

We may use (38) to find a $\lambda \in (0, 1]$ such that $(1-\lambda)\hat{x}+\lambda f(\hat{x}) \in [u_0, v_0]$, and hence

$$\langle J(f(\hat{x})) - J\hat{x}, \hat{x} - [(1-\lambda)\hat{x} + \lambda f(\hat{x})]\rangle \geq 0; \tag{40}$$

that is,

$$\langle J(f(\hat{x})) - J\hat{x}, \hat{x} - f(\hat{x})\rangle \geq 0. \tag{41}$$

Moreover,

$$\begin{aligned}
&\langle J(f(\hat{x})) - J\hat{x}, f(\hat{x}) - \hat{x}\rangle \\
&\quad = \|f(\hat{x})\|^2 - \langle J(f(\hat{x})), \hat{x}\rangle - \langle J\hat{x}, f(\hat{x})\rangle + \|\hat{x}\|^2 \\
&\quad \geq \|f(\hat{x})\|^2 - 2\|f(\hat{x})\|\|\hat{x}\| + \|\hat{x}\|^2 \\
&\quad = (\|f(\hat{x})\| - \|\hat{x}\|)^2 \geq 0.
\end{aligned} \tag{42}$$

So we conclude that $\langle J(f(\hat{x})) - J\hat{x}, f(\hat{x}) - \hat{x}\rangle = 0$. It follows that $\|f(\hat{x})\| = \|\hat{x}\|$. Moreover, as $\langle J(f(\hat{x})), \hat{x}\rangle \leq \|f(\hat{x})\|\|\hat{x}\|$, and the inequality above must hold as an equality. We have $\langle J(f(\hat{x})), \hat{x}\rangle = \|f(\hat{x})\|\|\hat{x}\|$. Therefore, $J(f(\hat{x})) = J\hat{x}$. And thus $f(\hat{x}) = \hat{x}$. The assertion is proved. □

Following a similar argument as in the proof of Theorem 16, we can obtain the following fixed point theorems.

Theorem 17. *Let $(X, \preceq)$ be a real partially ordered uniformly convex and smooth Banach space with respect to P and satisfy condition (H_2). Suppose that P is a normal, minihedral cone and $f : [u_0, v_0] \to X$ is an increasing map. Moreover, $[u_0, v_0]$ satisfies the condition (H_3) and (38). Then, f has at least one fixed point in $[u_0, v_0]$.*

Example 18. Let $(X, \preceq) = (L^2(\Omega), \preceq)$, the space of measurable functions which are the 2nd power summable on Ω. Endow $L^2(\Omega)$ with the following norm and the cone P:

$$\begin{gathered}
\|x\| = \left(\int_\Omega |x(t)|^2 \, d\mu\right)^{1/2}, \\
P = \left\{x \in L^2(\Omega) : x(t) \geq 0, \forall \text{a.e. } t \in \Omega\right\}.
\end{gathered} \tag{43}$$

Given $u_0, v_0 \in L^2(\Omega)$ such that $u_0 \prec v_0$. It is easy to see that $(L^2(\Omega), \preceq)$ satisfies (H_2) and (H_3) holds in $[u_0, v_0]$. Thus, by Theorem 17, every increasing $f : [u_0, v_0] \to L^2(\Omega)$ satisfying (38) has at least one fixed point in $[u_0, v_0]$.

Theorem 19. *Let $(X, \preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to P. If $u_0 \prec v_0$ and the following conditions are satisfied,*

(i) *P is a normal, minihedral cone with satisfying (H_2), (H_5) and (38);*

(ii) *$f : [u_0, v_0] \to X$ is an increasing and convex map;*

(iii) *there exists $0 < \varepsilon < 1$ such that $f(v_0) \preceq \varepsilon u_0 + (1-\varepsilon)v_0$,*

then, f has a unique fixed point $\hat{x}$ in $[u_0, v_0]$. Moreover, if we take $x_n = \Pi_{[u_0,v_0]}(f(x_{n-1}))$ $(n = 1, 2, 3, \ldots)$ for $\forall x_0 \in [u_0, v_0]$,

$$\begin{gathered}
\|x_n - \hat{x}\| \longrightarrow 0 \quad (n \longrightarrow \infty), \\
\|x_n - \hat{x}\| \leq M(1-\varepsilon)^n \quad (n = 1, 2, 3, \ldots),
\end{gathered} \tag{44}$$

where $M > 0$ has nothing to do with x_0.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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