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Research Article

Explicit Form of the Inverse Matrices of Tribonacci Circulant Type Matrices

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It is a hot topic that circulant type matrices are applied to networks engineering. The determinants and inverses of Tribonacci circulant type matrices are discussed in the paper. Firstly, Tribonacci circulant type matrices are defined. In addition, we show the invertibility of Tribonacci circulant matrix and present the determinant and the inverse matrix based on constructing the transformation matrices. By utilizing the relation between left circulant, g-circulant matrices and circulant matrix, the invertibility of Tribonacci left circulant and Tribonacci g-circulant matrices is also discussed. Finally, the determinants and inverse matrices of these matrices are given, respectively.

1. Introduction

Circulant type matrices have important applications in various networks engineering. Exploiting the circulant structure of the channel matrices, Eghbali et al. [1] analysed the realistic near fast fading scenarios with circulant frequency selective channels. The optimum sampling in the one- and two-dimensional (1D and 2D) wireless sensor networks (WSNs) with spatial temporally correlated data was studied with circulant matrices in [2]. The repeat space theory (RST) was extended to apply to carbon nanotubes and related molecular networks, where the corresponding matrices are pseudocirculant in [3]. Preconditioners obtained by circulant approximations of stochastic automata networks were considered in [4]. In [5], circulant mutation whose differential equations obtained neither are of repliator-type nor can they be transformed straightway into a linear equation was introduced into autocatalytic reaction networks. Jing and Jafarkhani [6] proposed distributed differential space-time codes that work for networks with any number of relays using circulant matrices. Wang and Cheng [7] studied the existence of doubly periodic travelling waves in cellular networks involving the discontinuous Heaviside step function by circulant matrix. Pais et al. [8] proved conditions for the existence of stable limit cycles arising from multiple distinct Hopf bifurcations of the dynamics in the case of circulant fitness matrices.

Circulant type matrices have been put on the firm basis with the work in [9, 10] and so on. Furthermore, the g-circulant matrices are focused on by many researchers; for the details please refer to [11–13] and the references therein.

Lately, some scholars gave the explicit determinant and inverse of the circulant and skew-circulant involving famous numbers. Jiang et al. [14] discussed the invertibility of circulant type matrices with the sum and product of Fibonacci and Lucas numbers and presented the determinants and the inverses of these matrices. Jiang et al. [15] considered circulant type matrices with the k-Fibonacci and k-Lucas numbers and presented the explicit determinant and inverse matrix by constructing the transformation matrices. Jiang and Hong [16] gave exact form determinants of the RSFPLR circulant matrices and the RSLPFL circulant matrices involving Perrin, Padovan, Tribonacci, and the generalized Lucas number by the inverse factorization of polynomial. Bozkurt and Tam gave determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers in [17]. Cambini presented an explicit form of the inverse of a particular circulant matrix in [18]. Shen et al. considered circulant matrices with Fibonacci and Lucas numbers and presented their explicit determinants and inverses in [19].

The Tribonacci sequences are defined by the following recurrence relations [20–22], respectively:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \tag{1}$$

where $T_0 = 0, T_1 = 1, T_2 = 1, n \geq 3$.

The first few values of the sequences are given by the following table:

$$\begin{array}{c|cccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ \hline T_n & 0 & 1 & 1 & 2 & 4 & 7 & 13 & 24 & 44 & \cdots \end{array}. \tag{2}$$

Let t_1, t_2, and t_3 be the roots of the characteristic equation $x^3 - x^2 - x - 1 = 0$ and then we have

$$\begin{gathered} t_1 + t_2 + t_3 = 1, \\ t_1t_2 + t_1t_3 + t_2t_3 = -1, \\ t_1t_2t_3 = 1. \end{gathered} \tag{3}$$

Hence the Binet formulas of the sequences $\{T_n\}$ have the form

$$T_n = b_1t_1^n + b_2t_2^n + b_3t_3^n, \tag{4}$$

where b_i is the ith root of the polynomial $44y^3 - 2y - 1$ for $i = 1, 2, 3$.

In this paper, we consider circulant type matrices, including the circulant and left circulant and g-circulant matrices. If we suppose T_n is the nth Tribonacci number, then we define a Tribonacci circulant matrix which is an $n \times n$ matrix with the following form:

$$\begin{aligned} & Circ\left(T_1, T_2, \cdots, T_n\right) \\ & = \begin{bmatrix} T_1 & T_2 & \cdots & T_n \\ T_n & T_1 & \cdots & T_{n-1} \\ \vdots & \vdots & & \vdots \\ T_2 & T_3 & \cdots & T_1 \end{bmatrix}. \end{aligned} \tag{5}$$

Besides, a Tribonacci left circulant matrix is given by

$$\begin{aligned} & LCirc\left(T_1, T_2, \cdots, T_n\right) \\ & = \begin{bmatrix} T_1 & T_2 & \cdots & T_n \\ T_2 & T_3 & \cdots & T_1 \\ \vdots & \vdots & & \vdots \\ T_n & T_1 & \cdots & T_{n-1} \end{bmatrix}, \end{aligned} \tag{6}$$

where each row is a cyclic shift of the row above to the left.

A Tribonacci g-circulant matrix is an $n \times n$ matrix with the following form:

$$A_{g,n} = \begin{pmatrix} T_1 & T_2 & \cdots & T_n \\ T_{n-g+1} & T_{n-g+2} & \cdots & T_{n-g} \\ T_{n-2g+1} & T_{n-2g+2} & \cdots & T_{n-2g} \\ \vdots & \vdots & \ddots & \vdots \\ T_{g+1} & T_{g+2} & \cdots & T_g \end{pmatrix}, \tag{7}$$

where g is a nonnegative integer and each of the subscripts is understood to be reduced modulo n. The first row of $A_{g,n}$ is $(T_1, T_2, \ldots, T_n)$, and its $(j+1)$th row is obtained by giving its jth row a right circular shift by g positions (equivalently, g mod n positions). Note that $g = 1$ or $g = n + 1$ yields standard Tribonacci circulant matrix. If $g = n - 1$, then we obtain Tribonacci left circulant matrix.

Lemma 1. *The $n \times n$ tridiagonal matrix is given by*

$$A_n = \begin{pmatrix} \tau_2 & \tau_1 & & & & & 0 \\ \tau_3 & \tau_2 & \tau_1 & & & & \\ & \tau_3 & \tau_2 & \tau_1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \tau_3 & \tau_2 & \tau_1 & \\ & & & & \tau_3 & \tau_2 & \tau_1 \\ 0 & & & & & \tau_3 & \tau_2 \end{pmatrix}; \tag{8}$$

then

$$\begin{aligned} & \det A_n \\ & = \begin{cases} \dfrac{\left(\left(\tau_2 + \sqrt{\tau_2^2 - 4\tau_1\tau_3}\right)/2\right)^{n+1} - \left(\left(\tau_2 - \sqrt{\tau_2^2 - 4\tau_1\tau_3}\right)/2\right)^{n+1}}{\sqrt{\tau_2^2 - 4\tau_1\tau_3}}, & \tau_2^2 \neq 4\tau_1\tau_3, \\ (n+1)\left(\dfrac{\tau_2}{2}\right)^n, \quad \tau_2^2 = 4\tau_1\tau_3. \end{cases} \end{aligned} \tag{9}$$

Proof. $\det A_n = \tau_2 \cdot \det A_{n-1} - \tau_1\tau_3 \cdot \det A_{n-2}$; let $x + y = \tau_2$, $xy = \tau_1\tau_3$ and then let x, y be the roots of the equation $x^2 - \tau_2 x + \tau_1\tau_3 = 0$.

We have

$$\begin{aligned} \det A_n & = y^n + xy^{n-1} + \cdots + x^{n-1}y + x^n \\ & = \begin{cases} \dfrac{x^{n+1} - y^{n+1}}{x - y}, & x \neq y, \\ (n+1)\,x^n, & x = y, \end{cases} \end{aligned} \tag{10}$$

where $x = (\tau_2 + \sqrt{\tau_2^2 - 4\tau_1\tau_3})/2$ and $y = (\tau_2 - \sqrt{\tau_2^2 - 4\tau_1\tau_3})/2$.

Hence,

$$\begin{aligned} & \det A_n \\ & = \begin{cases} \dfrac{\left(\left(\tau_2 + \sqrt{\tau_2^2 - 4\tau_1\tau_3}\right)/2\right)^{n+1} - \left(\left(\tau_2 - \sqrt{\tau_2^2 - 4\tau_1\tau_3}\right)/2\right)^{n+1}}{\sqrt{\tau_2^2 - 4\tau_1\tau_3},}, & \tau_2^2 \neq 4\tau_1\tau_3, \\ (n+1)\left(\dfrac{\tau_2}{2}\right)^n, \quad \tau_2^2 = 4\tau_1\tau_3. \end{cases} \end{aligned} \tag{11}$$

□

Lemma 2. *Let*

$$B_n = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} & a_n \\ \tau_2 & \tau_1 & & & & & \\ \tau_3 & \tau_2 & \tau_1 & & & & \\ & \tau_3 & \tau_2 & \tau_1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \tau_3 & \tau_2 & \tau_1 & \\ & & & & \tau_3 & \tau_2 & \tau_1 \end{pmatrix} \tag{12}$$

be an $n \times n$ *matrix; one has*

$$\det B_n = \sum_{i=1}^{n} (-1)^{1+i} \tau_1^{n-i} a_i \cdot \det A_{i-1}, \tag{13}$$

where

$$\det A_{i-1} = \begin{cases} \dfrac{\left(\left(\tau_2 + \sqrt{\tau_2^2 - 4\tau_1\tau_3}\right)/2\right)^i - \left(\left(\tau_2 - \sqrt{\tau_2^2 - 4\tau_1\tau_3}\right)/2\right)^i}{\sqrt{\tau_2^2 - 4\tau_1\tau_3},}, & \tau_2^2 \neq 4\tau_1\tau_3, \\ i\left(\dfrac{\tau_2}{2}\right)^{i-1}, & \tau_2^2 = 4\tau_1\tau_3. \end{cases} \tag{14}$$

Specifically, $\det A_0 = 1$.

Proof. Accoding to the last column determinant expansion and Lemma 1, we obtain

$$\begin{aligned} \det B_n &= \tau_1 \cdot \det\, B_{n-1} + (-1)^{n+1} a_n \cdot \det A_{n-1} \\ &= \tau_1 \left(\tau_1 \cdot \det\, B_{n-2} + (-1)^n a_{n-1} \cdot \det A_{n-2}\right) \\ &\quad + (-1)^{n+1} a_n \cdot \det A_{n-1} \\ &= \tau_1^{n-1} \cdot \det B_1 + (-1)^{1+2} \tau_1^{n-2} a_2 \cdot \det A_1 \\ &\quad + (-1)^{1+3} \tau_1^{n-3} a_3 \cdot \det A_2 \\ &\quad + \cdots + (-1)^{1+n} a_n \cdot \det A_{n-1} \\ &= \sum_{i=1}^{n} (-1)^{1+i} \tau_1^{n-i} a_i \cdot \det A_{i-1}. \end{aligned} \tag{15}$$

□

2. Determinant and Inverse of Tibonacci Circulant Matrix

In this section, let $D_n = Circ(T_1, T_2, \ldots, T_n)$ be a Tribonacci circulant matrix. Firstly, we give the determinant of the matrix D_n. Afterwards, we discuss the invertibility of the matrix D_n, and we find the inverse of the matrix D_n.

Theorem 3. *Let* $D_n = Circ(T_1, T_2, \ldots, T_n)$ *be a Tribonacci circulant matrix; then we have*

$$\begin{aligned} \det D_n = {} & \left[T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right)\right] \delta_1 \\ & - \left[-T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}\right] \delta_2, \end{aligned} \tag{16}$$

where T_n *is the nth Tribonacci number, and*

$$\begin{aligned} \Delta &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \\ \delta_1 &= \left(T_1 - T_n - T_{n-1}\right)\left(T_1 - T_{n+1}\right)^{n-3} \\ &\quad + \sum_{i=2}^{n-2} (-1)^{1+i} \left(T_1 - T_{n+1}\right)^{n-i-2} T_{n-i-1} \cdot \det A_{i-1}, \\ \delta_2 &= \sum_{i=1}^{n-2} (-1)^{1+i} \left(T_1 - T_{n+1}\right)^{n-i-2} \left(T_{n-i+1} - T_{n-i}\right) \cdot \det A_{i-1}, \\ \det A_{i-1} &= \begin{cases} \dfrac{\left(\left(b + \sqrt{b^2 - 4ac}\right)/2\right)^i - \left(\left(b - \sqrt{b^2 - 4ac}\right)/2\right)^i}{\sqrt{b^2 - 4ac}}, & b^2 \neq 4ac, \\ i\left(\dfrac{b}{2}\right)^{i-1}, & b^2 = 4ac, \end{cases} \\ a &= T_1 - T_{n+1}, \\ b &= -T_n - T_{n-1}, \\ c &= -T_n. \end{aligned} \tag{17}$$

Proof. Obviously, $\det D_1 = 1$ satisfies (16). In the case where $n > 1$, let

$$\Gamma_1 = \begin{pmatrix} 1 & & & & & & 0 \\ -1 & & & & & & 1 \\ -1 & & & & & 1 & -1 \\ -1 & & & & 1 & -1 & -1 \\ 0 & & & 1 & -1 & -1 & -1 \\ \vdots & & \iddots & \iddots & \iddots & \iddots & \\ 0 & 1 & -1 & -1 & -1 & & \\ 0 \;\; 1 & -1 & -1 & -1 & & & \end{pmatrix}, \tag{18}$$

$$\Pi_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \Delta^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & \Delta^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & \Delta & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

be two $n \times n$ matrices and then we have

$$\Gamma_1 D_n \Pi_1 = \begin{pmatrix} T_1 & p_1 & T_{n-1} & T_{n-2} & \cdots & T_3 & T_2 \\ 0 & p_2 & \varphi_3 & \varphi_4 & \cdots & \varphi_{n-1} & \varphi_n \\ 0 & p_3 & \phi & T_{n-3} & \cdots & T_2 & T_1 \\ 0 & 0 & b & a & & & \\ 0 & 0 & c & b & a & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & c & b & a \end{pmatrix}, \quad (19)$$

where

$$\begin{aligned} p_1 &= \sum_{i=1}^{n-1} T_{i+1} \Delta^{n-(i+1)}, \\ p_2 &= T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right), \\ p_3 &= -T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}, \\ \varphi_i &= T_{n+3-i} - T_{n+2-i}, \quad (i = 3, \ldots, n), \\ \phi &= T_1 - T_n - T_{n-1}, \\ \Delta &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \\ a &= T_1 - T_{n+1}, \qquad b = -T_n - T_{n-1}, \qquad c = -T_n. \end{aligned} \quad (20)$$

We obtain

$$\begin{aligned} &\det \Gamma_1 \det D_n \det \Pi_1 \\ &\quad = T_1 \cdot \left(p_2 \delta_1 - p_3 \delta_2\right) \\ &\quad = \left[T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right)\right] \delta_1 \\ &\qquad - \left[-T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}\right] \delta_2, \end{aligned} \quad (21)$$

where

$$\begin{aligned} \delta_1 &= \left(T_1 - T_n - T_{n-1}\right)\left(T_1 - T_{n+1}\right)^{n-3} \\ &\quad + \sum_{i=2}^{n-2} (-1)^{1+i} \left(T_1 - T_{n+1}\right)^{n-i-2} \cdot T_{n-i-1} \cdot \det A_{i-1}, \\ \delta_2 &= \sum_{i=1}^{n-2} (-1)^{1+i} \left(T_1 - T_{n+1}\right)^{n-i-2} \left(T_{n-i+1} - T_{n-i}\right) \cdot \det A_{i-1}. \end{aligned} \quad (22)$$

Let

$$\mathscr{B}_n = \begin{pmatrix} \phi & T_{n-3} & \cdots & T_3 & T_2 & T_1 \\ b & a & & & & \\ c & b & a & & & \\ & \ddots & \ddots & \ddots & & \\ & & c & b & a & \\ & & & c & b & a \end{pmatrix},$$
$$\mathscr{C}_n = \begin{pmatrix} \varphi_3 & \varphi_4 & \cdots & \varphi_{n-2} & \varphi_{n-1} & \varphi_n \\ b & a & & & & \\ c & b & a & & & \\ & \ddots & \ddots & \ddots & & \\ & & c & b & a & \\ & & & c & b & a \end{pmatrix} \quad (23)$$

be two $(n-2) \times (n-2)$ matrices, and $\delta_1 = \det \mathscr{B}_n$, and $\delta_2 = \det \mathscr{C}_n$.

Aoccording to Lemma 2, thus

$$\begin{aligned} \delta_1 &= \left(T_1 - T_n - T_{n-1}\right)\left(T_1 - T_{n+1}\right)^{n-3} \\ &\quad + \sum_{i=2}^{n-2} (-1)^{1+i} \left(T_1 - T_{n+1}\right)^{n-i-2} \cdot T_{n-i-1} \cdot \det A_{i-1}, \\ \delta_2 &= \sum_{i=1}^{n-2} (-1)^{1+i} \left(T_1 - T_{n+1}\right)^{n-i-2} \left(T_{n-i+1} - T_{n-i}\right) \cdot \det A_{i-1}. \end{aligned} \quad (24)$$

While

$$\det \Gamma_1 = \det \Pi_1 = (-1)^{(n-1)(n-2)/2}, \quad (25)$$

we have

$$\begin{aligned} \det D_n = &\left[T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right)\right] \delta_1 \\ &- \left[-T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}\right] \delta_2. \end{aligned} \quad (26)$$

□

Theorem 4. *Let $D_n = Circ(T_1, T_2, \ldots, T_n)$ be a Tribonacci circulant matrix; if $n \neq 2$ and $n \neq 2k\pi(\arctan(\pm\sqrt{4ac - b^2}/-b))^{-1}$ $(k = 1, 2, \ldots, n-1)$, then D_n is an invertible matrix.*

Proof. When $n = 2$ in Theorem 3, then we have $\det D_2 = 0$. Hence, D_2 is not invertible.

In the case where $n > 2$, since $T_n = b_1 t_1^n + b_2 t_2^n + b_3 t_3^n$, where b_i is the ith root of the polynomial $44y^3 - 2y - 1$, we have

$$\begin{aligned} f\left(\omega^k\right) &= \sum_{j=1}^{n} T_j \left(\omega^k\right)^{j-1} \\ &= \sum_{j=1}^{n} \left(b_1 t_1^j + b_2 t_2^j + b_3 t_3^j\right) \left(\omega^k\right)^{j-1} \end{aligned}$$

$$= \frac{b_1 t_1 (1 - t_1^n)}{1 - t_1 \omega^k} + \frac{b_2 t_2 (1 - t_2^n)}{1 - t_2 \omega^k} + \frac{b_3 t_3 (1 - t_3^n)}{1 - t_3 \omega^k}$$
$$= \frac{1 - T_{n+1} + (-T_n - T_{n-1}) \omega^k - T_n \omega^{2k}}{1 - \omega^k - \omega^{2k} - \omega^{3k}} \quad (k = 1, 2, \ldots, n-1). \tag{27}$$

If there exists ω^l $(l = 1, 2, \ldots, n-1)$ such that $f(\omega^l) = 0$, we obtain $1 - T_{n+1} + (-T_n - T_{n-1})\omega^k - T_n\omega^{2k} = 0$ for $1 - \omega^k - \omega^{2k} - \omega^{3k} \neq 0$. Thus, $\omega^l = (T_n + T_{n-1} \pm \sqrt{(-T_n - T_{n-1})^2 + 4T_n(T_1 - T_{n+1})})/-2T_n$.

Let $a = T_1 - T_{n+1}$, $b = -T_n - T_{n-1}$, and $c = -T_n$; if $b^2 - 4ac \geq 0$, we have ω^l is a real number.

While

$$\omega^l = \exp\left(\frac{2l\pi i}{n}\right) = \cos\left(\frac{2l\pi}{n}\right) + i \sin\left(\frac{2l\pi}{n}\right), \tag{28}$$

$\sin(2l\pi/n) = 0$. We have $\omega^l = -1$ for $0 < (2l\pi/n) < 2\pi$, but $1 - \omega^k - \omega^{2k} - \omega^{3k} = 0$. We obtain $f(\omega^k) \neq 0$ for any ω^k $(k = 1, 2, \ldots, n-1)$, while $f(1) = \sum_{j=1}^{n} T_j = -(1/2)(1 - T_n - T_{n+2}) \neq 0$.

If $b^2 - 4ac < 0$, ω^l is an imaginary number.

If

$$\cos\left(\frac{2l\pi}{n}\right) = \frac{T_n + T_{n-1}}{-2T_n}$$
$$\sin\left(\frac{2l\pi}{n}\right) = \frac{\sqrt{-4T_n (T_1 - T_{n+1}) - (-T_n - T_{n-1})^2}}{-2T_n} \tag{29}$$

or

$$\cos\left(\frac{2l\pi}{n}\right) = \frac{T_n + T_{n-1}}{-2T_n},$$
$$\sin\left(\frac{2l\pi}{n}\right) = -\frac{\sqrt{-4T_n (T_1 - T_{n+1}) - (-T_n - T_{n-1})^2}}{-2T_n}, \tag{30}$$

we obtain $n = 2k\pi(\arctan(\pm\sqrt{4ac - b^2}/-b))^{-1}$, such that $f(\omega^l) = 0$. If $1 - \omega^k - \omega^{2k} - \omega^{3k} = 0$, we have $\omega^k = -1$ and if n is an even number, then $f(\omega^k) = \sum_{j=1}^{n} T_j (\omega^k)^{j-1} = T_1 - T_2 + \cdots - T_n < 0$. By Lemma 1 in [15], the proof is completed. □

Lemma 5. *Let $\Phi = \begin{pmatrix} a & V \\ U & A \end{pmatrix}$ be an $n \times n$ matrix; then*

$$\Phi^{-1} = \begin{pmatrix} \frac{1}{\ell} & -\frac{1}{\ell} V A^{-1} \\ -\frac{1}{\ell} A^{-1} U & A^{-1} + \frac{1}{\ell} A^{-1} U V A^{-1} \end{pmatrix}, \tag{31}$$

where $\ell = a - VA^{-1}U$, V is a row vector, and U is a column vector.

Proof. Consider

$$\begin{pmatrix} a & V \\ U & A \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\ell} & -\frac{1}{\ell} V A^{-1} \\ -\frac{1}{\ell} A^{-1} U & A^{-1} + \frac{1}{\ell} A^{-1} U V A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix} = I_n,$$
$$\begin{pmatrix} \frac{1}{\ell} & -\frac{1}{\ell} V A^{-1} \\ -\frac{1}{\ell} A^{-1} U & A^{-1} + \frac{1}{\ell} A^{-1} U V A^{-1} \end{pmatrix} \cdot \begin{pmatrix} a & V \\ U & A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix} = I_n. \tag{32}$$

□

Lemma 6. *Let the matrix $\mathcal{G} = [g_{i,j}]_{i,j=1}^{n-3}$ be of the form*

$$g_{i,j} = \begin{cases} T_1 - T_{n+1}, & i = j, \\ -T_n - T_{n-1}, & i = j+1, \\ -T_n, & i = j+2, \\ 0, & \textit{otherwise}; \end{cases} \tag{33}$$

then the inverse $\mathcal{G}^{-1} = [g'_{i,j}]_{i,j=1}^{n-3}$ of the matrix $\mathcal{G}$ is equal to

$$g'_{i,j} = \begin{cases} \frac{1}{T_1 - T_{n+1}} \left(\frac{\beta^{i-j+1} - \alpha^{i-j+1}}{\beta - \alpha}\right), & i \geq j, \\ 0, & i < j, \end{cases} \tag{34}$$

where

$$\alpha = -\frac{(-T_n - T_{n-1}) + \sqrt{(-T_n - T_{n-1})^2 + 4T_n (T_1 - T_{n+1})}}{2(T_1 - T_{n+1})},$$
$$\beta = -\frac{(-T_n - T_{n-1}) - \sqrt{(-T_n - T_{n-1})^2 + 4T_n (T_1 - T_{n+1})}}{2(T_1 - T_{n+1})}. \tag{35}$$

Proof. Let $c_{i,j} = \sum_{k=1}^{n-3} g_{i,k} g'_{k,j}$. Obviously, $c_{i,j} = 0$ for $i < j$. In the case where $i = j$, we obtain

$$c_{i,i} = g_{i,i} g'_{i,i} = (T_1 - T_{n+1}) \cdot \frac{1}{T_1 - T_{n+1}} = 1. \tag{36}$$

For $i \geq j+1$, we obtain

$$\begin{aligned}
c_{i,j} &= \sum_{k=1}^{n-3} g_{i,k} g'_{k,j} \\
&= g_{i,i-2} g'_{i-2,j} + g_{i,i-1} g'_{i-1,j} + g_{i,i} g'_{i,j} \\
&= -T_n \cdot \frac{1}{T_1 - T_{n+1}} \left(\frac{\beta^{i-j-1} - \alpha^{i-j-1}}{\beta - \alpha} \right) \\
&\quad + \left(-T_n - T_{n-1}\right) \cdot \frac{1}{T_1 - T_{n+1}} \left(\frac{\beta^{i-j} - \alpha^{i-j}}{\beta - \alpha} \right) \\
&\quad + \left(T_1 - T_{n+1}\right) \cdot \frac{1}{T_1 - T_{n+1}} \left(\frac{\beta^{i-j+1} - \alpha^{i-j+1}}{\beta - \alpha} \right) \\
&= 0.
\end{aligned} \tag{37}$$

Hence, we verify $\mathscr{G}\mathscr{G}^{-1} = I_{n-3}$, where I_{n-3} is $(n-3) \times (n-3)$ identity matrix. Similarly, we can verify $\mathscr{G}^{-1}\mathscr{G} = I_{n-3}$. Thus, the proof is completed. □

Theorem 7. *Let $D_n = Circ(T_1, T_2, \ldots, T_n)$ be an invertible Tribonacci circulant matrix; then one has*

$$\begin{aligned}
D_n^{-1} = Circ\Bigg(& x'_2 + \left(-1 - \frac{p_3}{p_2}\right) x'_3 - x'_4 - x'_5, \\
& -x'_2 + \left(-1 + \frac{p_3}{p_2}\right) x'_3 - x'_4, x'_n, x'_{n-1} - x'_n, \\
& x'_{n-2} - x'_{n-1} - x'_n, \ldots, x'_3 - x'_4 - x'_5 - x'_6 \Bigg),
\end{aligned} \tag{38}$$

where

$$\begin{aligned}
p_2 &= T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right), \\
p_3 &= -T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}, \\
x'_1 &= 0, \\
x'_2 &= \frac{1}{p_2}, \\
x'_3 &= \frac{\gamma_3}{\ell} + \frac{\left(T_1 - T_{n+1}\right) \sum_{i=2}^{n-2} g_{i1} \gamma_{i+2}}{\ell}, \\
x'_4 &= -\frac{\gamma_3 \sum_{i=2}^{n-3} g_{i1} \rho_{i+3}}{\ell} + \sum_{i=1}^{n-3} g_{i1} \gamma_{i+3} \\
&\quad - \frac{\left(T_1 - T_{n+1}\right) \sum_{i=1}^{n-3} g_{i1} \rho_{i+3} \sum_{i=2}^{n-2} g_{i1} \gamma_{i+2}}{\ell}, \\
&\ \ \vdots \\
x'_k &= -\frac{\gamma_3 \sum_{i=2}^{n-k+1} g_{i1} \rho_{i+k-1}}{\ell} + \sum_{i=1}^{n-k+1} g_{i1} \gamma_{i+k-1} \\
&\quad - \frac{\left(T_1 - T_{n+1}\right) \sum_{i=1}^{n-k+1} g_{i1} \rho_{i+k-1} \sum_{i=2}^{n-2} g_{i1} \gamma_{i+2}}{\ell}, \\
&\qquad (k \geq 4).
\end{aligned} \tag{39}$$

Proof. Let

$$\Gamma_2 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & & & \\
0 & -\dfrac{p_3}{p_2} & 1 & 0 & 0 \\
\cdots & \cdots & \cdots & \ddots & \cdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}; \tag{40}$$

thus

$$\Gamma_2 \Gamma_1 D_n \Pi_1 = \begin{pmatrix}
T_1 & p_1 & T_{n-1} & T_{n-2} & \cdots & T_3 & T_2 \\
0 & p_2 & \varphi_3 & \varphi_4 & \cdots & \varphi_{n-1} & \varphi_n \\
0 & 0 & \rho_3 & \rho_4 & \cdots & \rho_{n-1} & \rho_n \\
0 & 0 & b & a & & & \\
0 & 0 & c & b & a & & \\
 & & & \ddots & \ddots & \ddots & \\
 & & & & c & b & a
\end{pmatrix}, \tag{41}$$

where

$$\begin{aligned}
\rho_3 &= T_1 - T_n - T_{n-1} - \frac{p_3}{p_2} \left(T_n - T_{n-1}\right), \\
\rho_i &= T_{n-i+1} - \frac{p_3}{p_2} \left(T_{n-i+3} - T_{n-i+2}\right) \\
&\qquad (i = 4, 5, \cdots, n), \\
\varphi_i &= T_{n+3-i} - T_{n+2-i}, \quad (i = 3, \ldots, n).
\end{aligned} \tag{42}$$

According to Lemma 5, let

$$F = \begin{pmatrix} \rho_3 & V \\ U & \mathscr{G} \end{pmatrix} \tag{43}$$

be an $(n-2) \times (n-2)$ matrix and we obtain

$$F^{-1} = \begin{pmatrix}
\dfrac{1}{\ell} & -\dfrac{V\mathscr{G}^{-1}}{\ell} \\
-\dfrac{\mathscr{G}^{-1}U}{\ell} & \mathscr{G}^{-1} + \dfrac{UV\mathscr{G}^{-1}}{\ell}
\end{pmatrix}, \tag{44}$$

where

$$\begin{aligned}
\rho_3 &= T_1 - T_n - T_{n-1} - \frac{p_3}{p_2}\left(T_n - T_{n-1}\right),\\
U &= \left(-T_n - T_{n-1}, -T_n, 0, \ldots, 0\right)^T,\\
V &= \left(\rho_4, \rho_5, \ldots, \rho_n\right),\\
\ell &= \rho_3 - b\sum_{i=1}^{n-3} g_{i1}\rho_{i+3} - c\sum_{i=1}^{n-4}\rho_{i+4}.
\end{aligned} \tag{45}$$

Let

$$\Pi_2 = \begin{pmatrix} 1 & -p_1 & -T_{n-1} + \dfrac{p_1\varphi_3}{p_2} & \cdots & -T_2 + \dfrac{p_1\varphi_n}{p_2} \\ 0 & 1 & \dfrac{-\varphi_3}{p_2} & \cdots & \dfrac{-\varphi_n}{p_2} \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{46}$$

where

$$\begin{aligned}
p_1 &= \sum_{i=1}^{n-1} T_{i+1}\Delta^{n-(i+1)},\\
p_2 &= T_1 - T_n + \sum_{i=1}^{n-2}\Delta^i\left(T_{n-i+1} - T_{n-i}\right),\\
\varphi_i &= T_{n+3-i} - T_{n+2-i}, \quad (i = 3, \ldots, n),\\
\Delta &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\\
a &= T_1 - T_{n+1}, \quad b = -T_n - T_{n-1}, \quad c = -T_n.
\end{aligned} \tag{47}$$

We have

$$\Gamma_2\Gamma_1 D_n \Pi_1\Pi_2 = \mathscr{D}_1 \oplus F, \tag{48}$$

where $\mathscr{D}_1 = \operatorname{diag}(T_1, p_2)$ is a diagonal matrix and $\mathscr{D}_1 \oplus F$ is the direct sum of $\mathscr{D}_1$ and F. If we denote $\Gamma = \Gamma_2\Gamma_1$ and $\Pi = \Pi_1\Pi_2$, we obtain

$$D_n^{-1} = \Pi\left(\mathscr{D}_1^{-1} \oplus F^{-1}\right)\Gamma. \tag{49}$$

Since the last row elements of the matrix Π are $0, 1, (T_{n-1} - T_n)/p_2, (T_{n-2} - T_{n-1})/p_2, \ldots, (T_3 - T_4)/p_2, (T_2 - T_3)/p_2$, then the elements of last row of $\Pi(\mathscr{D}_1^{-1} \oplus F^{-1})$ are given by the following equations:

$$\begin{aligned}
x_1' &= 0,\\
x_2' &= \frac{1}{p_2},\\
x_3' &= \frac{\gamma_3}{\ell} + \frac{\left(T_1 - T_{n+1}\right)\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell},\\
x_4' &= -\frac{\gamma_3\sum_{i=2}^{n-3} g_{i1}\rho_{i+3}}{\ell} + \sum_{i=1}^{n-3} g_{i1}\gamma_{i+3}\\
&\quad - \frac{\left(T_1 - T_{n+1}\right)\sum_{i=1}^{n-3} g_{i1}\rho_{i+3}\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell},\\
&\vdots\\
x_k' &= -\frac{\gamma_3\sum_{i=2}^{n-k+1} g_{i1}\rho_{i+k-1}}{\ell} + \sum_{i=1}^{n-k+1} g_{i1}\gamma_{i+k-1}\\
&\quad - \frac{\left(T_1 - T_{n+1}\right)\sum_{i=1}^{n-k+1} g_{i1}\rho_{i+k-1}\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell},\\
&\qquad (k \geq 4),
\end{aligned} \tag{50}$$

where

$$\begin{aligned}
\gamma_i &= \frac{T_{n-i+2} - T_{n-i+3}}{p_2}, \quad (i = 3, 4 \ldots, n),\\
\rho_3 &= T_1 - T_n - T_{n-1} - \frac{p_3}{p_2}\left(T_n - T_{n-1}\right),\\
\rho_i &= T_{n-i+1} - \frac{p_3}{p_2}\left(T_{n-i+3} - T_{n-i+2}\right)\\
&\qquad (i = 4, 5, \ldots, n).
\end{aligned} \tag{51}$$

By Lemma 6, if $D_n^{-1} = Circ(x_1, x_2, \ldots, x_n)$, then its last row elements are given by the following equations:

$$\begin{aligned}
x_2 &= -x_2' + \left(-1 + \frac{p_3}{p_2}\right)x_3' - x_4',\\
x_3 &= x_n',\\
x_4 &= x_{n-1}' - x_n',\\
x_5 &= x_{n-2}' - x_{n-1}' - x_n',\\
&\vdots\\
x_k &= x_{n-k+3}' - x_{n-k+4}' - x_{n-k+5}' - x_{n-k+6}'\\
&\qquad (5 < k \leq n),\\
x_1 &= x_2' + \left(-1 - \frac{p_3}{p_2}\right)x_3' - x_4' - x_5'.
\end{aligned} \tag{52}$$

Hence, the proof is compeleted. □

3. Determinant and Inverse of Tibonacci Left Circulant Matrix

In this section, let $D'_n = LCirc(T_1, T_2, \ldots, T_n)$ be a Tibonacci left circulant matrix. By using the obtained conclusions, we give a determinant formula for the matrix D'_n. Afterwards, we discuss the invertibility of the matrix D'_n. The inverse of the matrix D'_n is also presented. According to Lemma 2 in [15] and Theorems 3, 4, and 7, we can obtain the following theorems.

Theorem 8. *Let $D'_n = LCirc(T_1, T_2, \ldots, T_n)$ be a Tribonacci left circulant matrix; then one has*

$$\begin{aligned}\det D'_n = {} & (-1)^{(n-1)(n-2)/2} \\ & \times \left[\left[T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right)\right]\delta_1 \right. \\ & - \left[-T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) \right. \\ & \left.\left. + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}\right]\delta_2\right],\end{aligned} \tag{53}$$

where T_n is the nth Tribonacci number.

Theorem 9. *Let $D'_n = LCirc(T_1, T_2, \ldots, T_n)$ be a Tribonacci left circulant matrix. If $n \neq 2$ and $n \neq 2k\pi(\arctan(\pm\sqrt{4ac-b^2}/-b))^{-1}$ $(k = 1, 2, \ldots, n-1)$, then D'_n is an invertible matrix.*

Theorem 10. *Let $D'_n = LCirc(T_1, T_2, \ldots, T_n)$ be a Tribonacci left circulant matrix. If D_n is an invertible matrix, then we have*

$$\begin{aligned}D'^{-1}_n = Circ\Bigg(& x'_2 + \left(-1 - \frac{p_3}{p_2}\right)x'_3 - x'_4 - x'_5, \\ & x'_3 - x'_4 - x'_5 - x'_6, \ldots, x'_{n-2} - x'_{n-1} - x'_n, \\ & x'_{n-1} - x'_n, x'_n, -x'_2 + \left(-1 + \frac{p_3}{p_2}\right)x'_3 - x'_4\Bigg),\end{aligned} \tag{54}$$

where

$$\begin{aligned}x'_1 &= 0, \\ x'_2 &= \frac{1}{p_2}, \\ x'_3 &= \frac{\gamma_3}{\ell} + \frac{\left(T_1 - T_{n+1}\right)\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell}, \\ x'_4 &= -\frac{\gamma_3 \sum_{i=2}^{n-3} g_{i1}\rho_{i+3}}{\ell} + \sum_{i=1}^{n-3} g_{i1}\gamma_{i+3} \\ & \quad - \frac{\left(T_1 - T_{n+1}\right)\sum_{i=1}^{n-3} g_{i1}\rho_{i+3}\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell}, \\ & \vdots \\ x'_k &= -\frac{\gamma_3 \sum_{i=2}^{n-k+1} g_{i1}\rho_{i+k-1}}{\ell} + \sum_{i=1}^{n-k+1} g_{i1}\gamma_{i+k-1} \\ & \quad - \frac{\left(T_1 - T_{n+1}\right)\sum_{i=1}^{n-k+1} g_{i1}\rho_{i+k-1}\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell}, \\ & \qquad (k \geq 4).\end{aligned} \tag{55}$$

4. Determinant and Inverse of Tibonacci g-Circulant Matrix

In this section, let $D_{g,n} = g\text{-}Circ(T_1, T_2, \ldots, T_n)$ be a Tibonacci g-circulant matrix. By using the obtained conclusions, we give a determinant formula for the matrix $D_{g,n}$. Afterwards, we discuss the invertibility of the matrix $D_{g,n}$. The inverse of the matrix $D_{g,n}$ is also presented. From Lemmas 3 and 4 in [15] and Theorems 3, 4, and 7, we deduce the following results.

Theorem 11. *Let $D_{g,n} = g\text{-}Circ(T_1, T_2, \ldots, T_n)$ be a Tribonacci g-circulant matrix and $(n, g) = 1$; then one has*

$$\begin{aligned}\det D_{g,n} = \det \mathbb{Q}_g \cdot & \left[\left[T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right)\right]\delta_1 \right. \\ & - \left[-T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) \right. \\ & \left.\left. + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}\right]\delta_2\right],\end{aligned} \tag{56}$$

where T_n is the nth Tribonacci number.

Theorem 12. *Let $D_{g,n} = g\text{-}Circ(T_1, T_2, \ldots, T_n)$ be a Tribonacci g-circulant matrix and $(n, g) = 1$. If $n \neq 2$ and $n \neq 2k\pi(\arctan(\pm\sqrt{4ac-b^2}/-b))^{-1}$ $(k = 1, 2, \ldots, n-1)$, then $D_{g,n}$ is an invertible matrix.*

Theorem 13. *Let $D_{g,n} = g\text{-}Circ(T_1, T_2, ..., T_n)$ be a Tribonacci g-circulant matrix and $(n, g) = 1$. If D_n is an invertible matrix, then one has*

$$\begin{aligned}D^{-1}_{g,n} = \Bigg[Circ\Bigg(& x'_2 + \left(-1 - \frac{p_3}{p_2}\right)x'_3 - x'_4 - x'_5, \\ & -x'_2 + \left(-1 + \frac{p_3}{p_2}\right)x'_3 - x'_4, x'_n, x'_{n-1} - x'_n, x'_{n-2} \\ & - x'_{n-1} - x'_n, \ldots, x'_3 - x'_4 - x'_5 - x'_6\Bigg)\Bigg]\mathbb{Q}^T_g,\end{aligned} \tag{57}$$

where

$$
\begin{aligned}
x_1' &= 0, \\
x_2' &= \frac{1}{p_2}, \\
x_3' &= \frac{\gamma_3}{\ell} + \frac{(T_1 - T_{n+1}) \sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell}, \\
x_4' &= -\frac{\gamma_3 \sum_{i=2}^{n-3} g_{i1}\rho_{i+3}}{\ell} + \sum_{i=1}^{n-3} g_{i1}\gamma_{i+3} \\
&\quad - \frac{(T_1 - T_{n+1}) \sum_{i=1}^{n-3} g_{i1}\rho_{i+3} \sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell}, \\
&\vdots \\
x_k' &= -\frac{\gamma_3 \sum_{i=2}^{n-k+1} g_{i1}\rho_{i+k-1}}{\ell} + \sum_{i=1}^{n-k+1} g_{i1}\gamma_{i+k-1} \\
&\quad - \frac{(T_1 - T_{n+1}) \sum_{i=1}^{n-k+1} g_{i1}\rho_{i+k-1} \sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell}, \\
&\qquad (k \geq 4).
\end{aligned}
\tag{58}
$$

5. Conclusion

The related problem of Tribonacci circualnt type matrices is studied in this paper. We not only discuss nonsingularity of Tibonacci circulant type matrices, but also give the explicit determinant and inverse of Tribonacci circulant matrix, Tribonacci left circualant matrix, and Tribonacci g-circualant matrix. Furthermore, according to Theorem 11 in [23] and the result in Theorem 3 in the paper, identities can be easily obtained:

$$
\begin{aligned}
&\frac{(1 - T_{n+1})^n - (c_1^n + d_1^n) + (-T_n)^n}{\mathbb{L}_{-n} - \mathbb{L}_n} \\
&\quad = \left[T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i (T_{n-i+1} - T_{n-i}) \right] \delta_1 \\
&\qquad - \left[-T_n + \Delta (T_1 - T_n - T_{n-1}) + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2} \right] \delta_2,
\end{aligned}
\tag{59}
$$

where T_n is the nth Tribonacci number, $\mathbb{L}_n$ is the nth generalized Lucas number, and

$$
\begin{aligned}
c_1 &= \frac{(T_{n+2} - T_{n+1}) + \mu_1}{2}, \\
d_1 &= \frac{(T_{n+2} - T_{n+1}) - \mu_1}{2}, \\
\mu_1 &= \sqrt{(T_{n+2} - T_{n+1})^2 - 4T_n (T_{n+1} - 1)}, \\
\Delta &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \\
\delta_1 &= (T_1 - T_n - T_{n-1}) (T_1 - T_{n+1})^{n-3} \\
&\quad + \sum_{i=2}^{n-2} (-1)^{1+i} (T_1 - T_{n+1})^{n-i-2} \cdot T_{n-i-1} \det A_{i-1}, \\
\delta_2 &= \sum_{i=1}^{n-2} (-1)^{1+i} (T_1 - T_{n+1})^{n-i-2} (T_{n-i+1} - T_{n-i}) \det A_{i-1}, \\
&\det A_{i-1} \\
&= \begin{cases} \dfrac{\left(\left(b + \sqrt{b^2 - 4ac}\right)/2\right)^i - \left(\left(b - \sqrt{b^2 - 4ac}\right)/2\right)^i}{\sqrt{b^2 - 4ac}}, \\ \qquad\qquad b^2 \neq 4ac, \\ i\left(\dfrac{b}{2}\right)^{i-1}, \quad b^2 = 4ac. \end{cases} \\
a &= T_1 - T_{n+1}, \qquad b = -T_n - T_{n-1}, \qquad c = -T_n.
\end{aligned}
\tag{60}
$$

In addition, we will develop solving the problem in [24–26] by circulant matrices technology.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] H. Eghbali, S. Muhaidat, S. A. Hejazi, and Y. Ding, "Relay selection strategies for single-carrier frequency-domain equalization multi-relay cooperative networks," *IEEE Transactions on Wireless Communications*, vol. 12, no. 5, pp. 2034–2045, 2013.

[2] N. Sun and J. Wu, "Optimum sampling in spatial-temporally correlated wireless sensor networks," *EURASIP Journal on Wireless Communications and Networking*, vol. 2013, article 5, 2013.

[3] S. Arimoto, "Repeat space theory applied to carbon nanotubes and related molecular networks. I," *Journal of Mathematical Chemistry*, vol. 41, no. 3, pp. 231–269, 2007.

[4] R. H. Chan and W. K. Ching, "Circulant preconditioners for stochastic automata networks," *Numerische Mathematik*, vol. 87, no. 1, pp. 35–57, 2000.

[5] P. F. Stadler, W. Schnabl, C. V. Forst, and P. Schuster, "Dynamics of small autocatalytic reaction networks-II. Replication, mutation and catalysis," *Bulletin of Mathematical Biology*, vol. 57, no. 1, pp. 21–61, 1995.

[6] Y. Jing and H. Jafarkhani, "Distributed differential space-time coding for wireless relay networks," *IEEE Transactions on Communications*, vol. 56, no. 7, pp. 1092–1100, 2008.

[7] G. Q. Wang and S. S. Cheng, "6-periodic travelling waves in an artificial neural network with bang-bang control," *Journal of*

Difference Equations and Applications, vol. 18, no. 2, pp. 261–304, 2012.

[8] D. Pais, C. H. Caicedo-Nunez, and N. E. Leonard, "Hopf bifurcations and limit cycles in evolutionary network dynamics," *SIAM Journal on Applied Dynamical Systems*, vol. 11, no. 4, pp. 1754–1784, 2012.

[9] P. J. Davis, *Circulant Matrices*, John Wiley & Sons, New York, NY, USA, 1979.

[10] Z. L. Jiang and Z. X. Zhou, *Circulant Matrices*, Chengdu Technology University Publishing Company, Chengdu, China, 1999.

[11] C. Erbas and M. M. Tanik, "Generating solutions to the N-queens problem using 2-circulants," *Mathematics Magazine*, vol. 68, no. 5, pp. 343–356, 1995.

[12] Y. K. Wu, R. Z. Jia, and Q. Li, "g-circulant solutions to the $(0,1)$ matrix equation $A^m = J_n^*$," *Linear Algebra and Its Applications*, vol. 345, no. 1–3, pp. 195–224, 2002.

[13] E. Ngondiep, S. Serra-Capizzano, and D. Sesana, "Spectral features and asymptotic properties for g-circulants and g-Toeplitz sequences," *SIAM Journal on Matrix Analysis and Applications*, vol. 31, no. 4, pp. 1663–1687, 2010.

[14] Z. L. Jiang, Y. P. Gong, and Y. Gao, "Circulant type matrices with the sum and product of Fibonacci and Lucas numbers," *Abstract and Applied Analysis*, vol. 2014, Article ID 375251, 12 pages, 2014.

[15] Z. Jiang, Y. Gong, and Y. Gao, "Invertibility and explicit inverses of circulant-type matrices with k-Fibonacci and k-Lucas numbers," *Abstract and Applied Analysis*, vol. 2014, Article ID 238953, 9 pages, 2014.

[16] X. Jiang and K. Hong, "Exact determinants of some special circulant matrices involving four kinds of famous numbers," *Abstract and Applied Analysis*, vol. 2014, Article ID 273680, 12 pages, 2014.

[17] D. Bozkurt and T. Y. Tam, "Determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas Numbers," *Applied Mathematics and Computation*, vol. 219, no. 2, pp. 544–551, 2012.

[18] A. Cambini, "An explicit form of the inverse of a particular circulant matrix," *Discrete Mathematics*, vol. 48, no. 2-3, pp. 323–325, 1984.

[19] S. Q. Shen, J. M. Cen, and Y. Hao, "On the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers," *Applied Mathematics and Computation*, vol. 217, no. 23, pp. 9790–9797, 2011.

[20] M. Elia, "Derived sequences, the Tribonacci recurrence and cubic forms," *The Fibonacci Quarterly*, vol. 39, no. 2, pp. 107–115, 2001.

[21] B. Balof, "Restricted tilings and bijections," *Journal of Integer Sequences*, vol. 15, Article ID 12.2.3, 2012.

[22] S. Rabinowitz, "Algorithmic manipulation of third-order linear recurrences," *The Fibonacci Quarterly*, vol. 34, no. 5, pp. 447–464, 1996.

[23] J. Li, Z. L. Jiang, and F. Lu, "Determinants, norms, and the spread of circulant matrices with Tribonacci and generalized Lucas numbers," *Abstract and Applied Analysis*, vol. 2014, Article ID 381829, 9 pages, 2014.

[24] D. Ding, Z. Wang, J. Hu, and H. Shu, "Dissipative control for state-saturated discrete time-varying systems with randomly occurring nonlinearities and missing measurements," *International Journal of Control*, vol. 86, no. 4, pp. 674–688, 2013.

[25] Z. Wang, H. Dong, B. Shen, and H. Gao, "Finite-horizon H_∞ filtering with missing measurements and quantization effects," *IEEE Transactions on Automatic Control*, vol. 58, no. 7, pp. 1707–1718, 2013.

[26] J. Hu, Z. Wang, B. Shen, and H. Gao, "Quantised recursive filtering for a class of nonlinear systems with multiplicative noises and missing measurements," *International Journal of Control*, vol. 86, no. 4, pp. 650–663, 2013.