Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 258436, 8 pages
http://dx.doi.org/10.1155/2015/258436

Research Article

JPD-Coloring of the Monohedral Tiling for the Plane

S. A. El-Shehawy[1] and M. Basher[2]

[1] *Department of Mathematics, Faculty of Science, Menoufia University, Shebin El-Kom 32511, Egypt*
[2] *Department of Mathematics and Computer Science, Faculty of Science, Suez University, Suez 43518, Egypt*

Correspondence should be addressed to S. A. El-Shehawy; shshehawy64@yahoo.com

Received 24 November 2014; Revised 5 January 2015; Accepted 19 January 2015

Academic Editor: Gaston Mandata N'guérékata

We introduce a definition of coloring by using joint probability distribution "JPD-coloring" for the plane which is equipped by tiling $\mathfrak{I}$. We investigate the JPD-coloring of the r-monohedral tiling for the plane by mutually congruent regular convex polygons which are equilateral triangles at $r = 3$ or squares at $r = 4$ or regular hexagons at $r = 6$. Moreover we present some computations for determining the corresponding probability values which are used to color in the three studied cases by MAPLE-Package.

1. Introduction

A tiling of the plane is a family of sets—called tiles—that cover the plane without gaps or overlaps. Tilings are known as tessellations or pavings; they have appeared in human activities since prehistoric times. Their mathematical theory is mostly elementary, but nevertheless it contains a rich supply of interesting problems at various levels. The same is true for the special class of tiling called tiling by regular polygons [1]. The notions of tiling by regular polygons in the plane are introduced by Grünbaum and Shephard in [2]. For more details see [3–5].

Definition 1 (see [1, 6]). A tiling of the plane is a collection $\mathfrak{I} = \{T_s : s = 1, 2, 3, \ldots\}$ of closed topological discs (tiles) which covers the Euclidean plane R^2 and is such that the interiors of its tiles are disjoint.

More explicitly, the union of the sets $T_1, T_2, T_3, \ldots$, tiles, is to be the whole plane, and the interiors of the sets T_s are pairwise disjoint. We will restrict our interest to the case where each tile is a topological disc; that is, it has a boundary that is a single simple closed curve. Two tiles are called adjacent if they have an edge in common, and then each is called an adjacent of the other. Two distinct edges are adjacent if they have a common endpoint. The word incident is used to denote the relation of a tile to each of its edges or vertices and also of an edge to each of its endpoints. Two tilings $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are congruent if $\mathfrak{I}_1$ may be made to coincide with $\mathfrak{I}_2$ by a rigid motion of the plane, possibly including reflection [6].

Definition 2 (see [1, 6]). A tiling is called edge-to-edge if the relation of any two tiles is one of the following three possibilities:

(a) they are disjoint,

(b) they have precisely one common point which is a vertex of each of the polygons,

(c) they share a segment that is an edge of each of the two polygons.

Definition 3 (see [6]). A regular tiling $\mathfrak{I}$ will be called r-monohedral tiling if every tile in $\mathfrak{I}$ is congruent to one fixed set T. The set T is called the prototile of $\mathfrak{I}$, where r is the number of vertices for each tile.

Hence a point of the plane that is a vertex of one of the polygons in an edge-to-edge tiling is also a vertex of every other polygon to which it belongs and it is called a vertex of the tiling. Similarly, each edge of one of the polygons, regular tiling, is an edge of precisely one other polygon and it is called an edge of the tiling. It should be noted that the only possible edge-to-edge tilings of the plane by mutually congruent regular convex polygons are the three regular tilings by equilateral triangles, by squares, or by regular hexagons.

The notions of the coloring of the monohedral tiling for the plane have been introduced by Grünbaum and Shephard [2]. The σ-coloring and the perfect σ-coloring for the plane equipped by the r-monohedral tiling $\mathfrak{J}$ have been introduced by Basher [7].

In this paper we redefine the coloring of the r-monohedral tiling for the plane by using joint probability distribution (JPD). We aim to investigate the three regular tilings by equilateral triangles, squares, and regular hexagons using JPD. These three tilings are shown graphically and computationally. Some computations by MAPLE-Package to determine the probability values (vertices) for the three studied tilings are presented. We introduce this alternative technique to expand and update the coloring technique to implement tiling according to a probabilistic approach. The probability values refer to percentages in the coloring process and this contributes to convert the coloring process into a computational process in the future.

Throughout this paper we consider two discrete random variables X and Y with a joint probability mass function $f_{X,Y}(x, y) = P(X = x, Y = y)$ which satisfies

(1) $f_{X,Y}(x_i, y_j) \geq 0$, for all points (x_i, y_j) in the range of (X, Y),

(2) $\sum_{x_i} \sum_{y_j} f_{X,Y}(x_i, y_j) = 1$.

The value $f_{X,Y}(x_i, y_j)$ is usually written as p_{ij} for each point (x_i, y_j) in the range of (X, Y); see [8, 9]. In this paper we consider p_{ij} having equal denominators (the large common multiplication of the denominators of the probabilities) "n".

2. JPD-Coloring of the Regular Tilings

In this section we will investigate the coloring of r-monohedral tiling.

Let R^2 be equipped by r-monohedral tiling $\mathfrak{J}$, and let $V(\mathfrak{J})$ be the set of all vertices of the tiling. Here, we consider the probability values p_{ij} to represent the coloring of the set $V(\mathfrak{J})$ as in the following definition where

$$p_{ij} = P\left(X = x_i, Y = y_j\right) = f_{X,Y}\left(x_i, y_j\right) = \begin{cases} \text{nonzero value with equal denominators } n, \\ \qquad\qquad \text{if } \left(x_i, y_j\right) \in V(\mathfrak{J}), \\ \text{zero value,} \quad \text{if } \left(x_i, y_j\right) \notin V(\mathfrak{J}). \end{cases} \quad (1)$$

For each n, a family of a corresponding JPD is denoted by "F(JPD)".

Definition 4. A coloring of the tiling $\mathfrak{J}$ is a partition of $V(\mathfrak{J})$ into k color-classes such that

(i) each color k represents a probability value p_{ij},

(ii) the different colors appear on adjacent vertices,

(iii) for each prototile $T_s \in \mathfrak{J}$ there exists a corresponding JPD $\in F$(JPD).

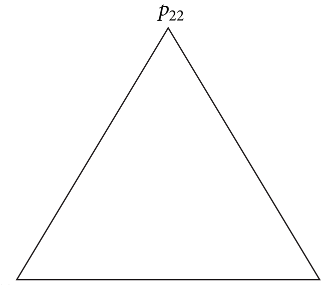


Figure 1: The used equilateral triangle with the corresponding JPD values.

Definition 5. The set of tiles colored by F(JPD) is called the mesh of tiling.

From the above definition the tiling $\mathfrak{J}$ can be colored by horizontal or vertical translation of the mesh.

Definition 6. The order $O(F(\text{JPD}))$ of F(JPD) is the number of JPDs which construct the mesh.

2.1. JPD-Coloring of the 3-Monohedral Tiling. Here, we consider the JPD $f_{X,Y}(x_i, y_j) = p_{ij}$, $i = 1, 2, 3$; $j = 1, 2$, with different nonzero values of p_{11}, p_{31}, p_{22} and zero values of p_{21}, p_{12}, p_{32}, where p_{ij} is with equal denominators n. The used equilateral triangle is illustrated in Figure 1.

Theorem 7. *If the plane is equipped by 3-monohedral tiling, then the number of colors "c" equals* 3 *where* $n \geq 6$. *If* $n < 6$, *then the tiling cannot be colored.*

Proof. Let R^2 be equipped by equilateral triangle tiling. We give the proof in two cases.

Case 1. If $n < 6$, then the tiling cannot be colored (i.e., the number of colors c equals 0) because we cannot find three different probability values (JPD) to color the three vertices of the mentioned equilateral triangle tiling (say, at $n = 5$ the probability values are $\{1/5, 2/5, 2/5\}$, at $n = 4$ the probability values are $\{1/4, 1/4, 2/4\}$, and at $n = 3$ the probability values are $\{1/3, 1/3, 1/3\}$).

Case 2. If $n \geq 6$, then for each n the number of colors c equals 3 and we can find three different probability values (JPD), which satisfied the condition (ii) in Definition 4, to color the three vertices of the mentioned equilateral triangle tiling. We can find the following:

(i) at $n = 6$ the different probability values are only $\{1/6, 2/6, 3/6\}$ (see Figure 2(a)),

(ii) at $n = 7$ the different probability values are only $\{1/7, 2/7, 4/7\}$,

(iii) at $n \geq 8$ there are more than one JPD with three different probability values (Figures 2(b) and 2(c)).

□

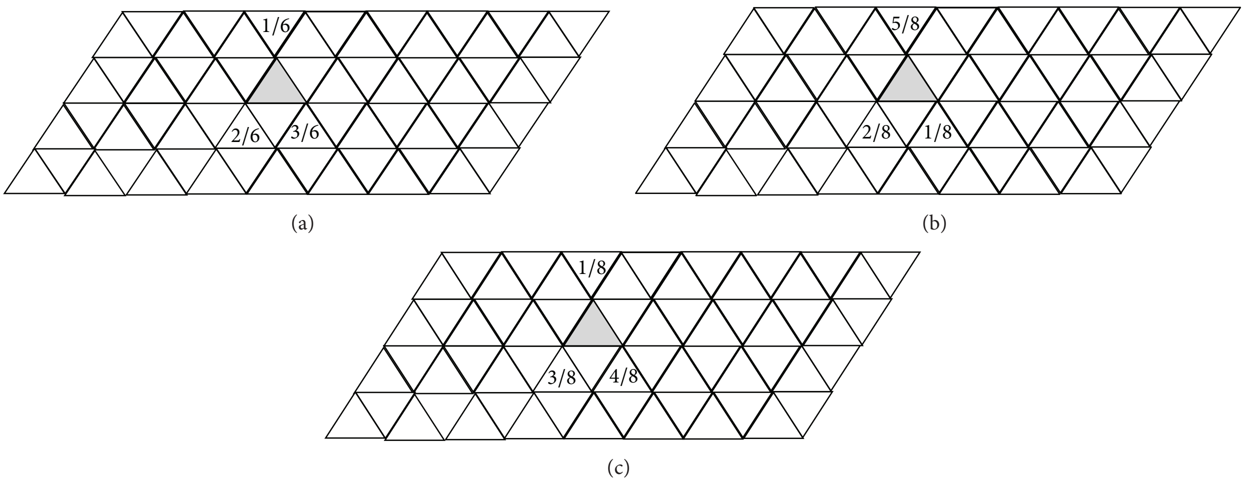


FIGURE 2: Some JPD values at $n = 6$ and $n = 8$ to color the three vertices of the equilateral triangle tiling.

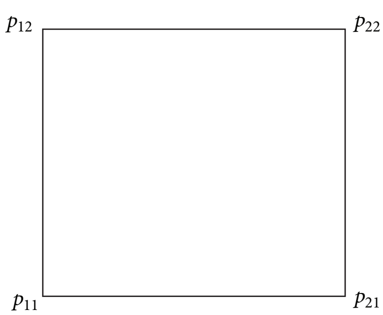


FIGURE 3: The used square with the corresponding JPD values.

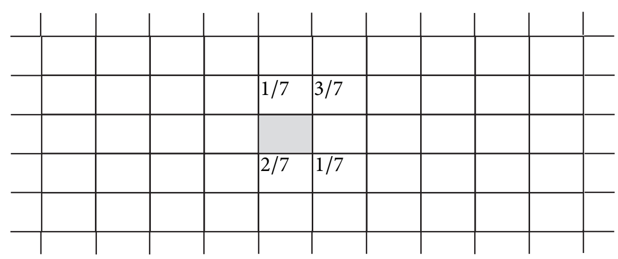


FIGURE 4: The JPD values at $n = 7$ to color the four vertices of the square triangle tiling.

Remark 8. $O(F(\text{JPD}))$ of the 3-monohedral tiling equals 1.

2.2. JPD-Coloring of the 4-Monohedral Tiling. Here, we consider the following JPD: $f_{X,Y}(x_i, y_j) = p_{ij}$, $i = 1, 2$; $j = 1, 2$, with the assumptions $p_{11} \neq p_{21}$, $p_{11} \neq p_{12}$, $p_{22} \neq p_{21}$, $p_{22} \neq p_{12}$ and where p_{ij} is with equal denominators n. The used square is illustrated in Figure 3.

Theorem 9. *If the plane is equipped by the 4-monohedral tiling, then the greatest number of colors "c" is given as follows:*

$$c = \begin{cases} n-4, & n \geq 6 \text{ and } n \text{ is even}, \\ 3, & n = 7, \\ n-5, & n \geq 9 \text{ and } n \text{ is odd}, \end{cases} \quad (2)$$

and if $n < 6$, then the tiling cannot be colored.

Proof. Let R^2 be equipped by square tiling. We give the proof in four cases.

Case 1. If $n < 6$, then the tiling cannot be colored (i.e., the greatest number of colors c equals 0) because we cannot find four probability values (JPD) to color the four vertices of the mentioned square tiling which satisfied (ii) in Definition 4 for coloring (say, at $n = 5$ the probability values are $\{1/5, 1/5, 2/5, 1/5\}$ and at the minimum value $n = 4$ the probability values are $\{1/4, 1/4, 1/4, 1/4\}$).

Case 2. For $n = 7$, take the corresponding probability values of two adjacent vertices 1/7 and 2/7. Then, the rest corresponding probability values must be 1/7 and 3/7 which satisfied the condition (ii) in Definition 4. So, the greatest number of colors "c" equals 3 and the different probability values are only $\{1/7, 2/7, 3/7\}$, see Figure 4.

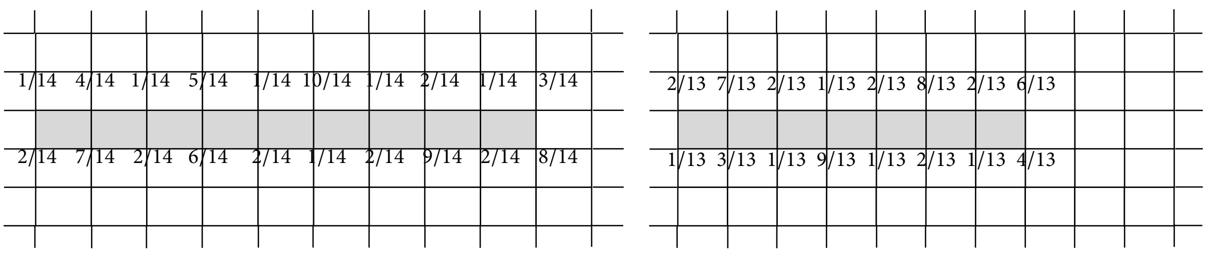
Case 3. For $n \geq 6$ and n is even, take the corresponding probability values of two adjacent vertices $1/n$ and $2/n$. Then, the rest corresponding probability value is $(n-3)/n$. This probability value must be distributed on the other two vertices as follows: $\{(n-4)/n, 1/n\}, \{(n-5)/n, 2/n\}, \ldots, \{(n-i)/n, (i-3)/n\}$, where i is an integer number and $4 \leq i \leq n-1$.

This implies that the available total probability values to obtain the mesh are $\{1/n, 2/n, (n-4)/n, (n-5)/n, \ldots, (n-i)/n\}$, where $4 \leq i \leq n-1$. To avoid the repetition, the last two probability values $(n-3)/n, (n-1)/n$ are excluded. In this case we obtain the following.

(i) For $n = 6$, the available total probability values $\{1/6, 2/6, (6-4)/6, (6-5)/6\}$ are equivalent to $\{1/6, 2/6\}$, and the greatest number of colors c equals 2 (i.e., $n-4$).

Table 1: The relation between k, n, $O(F(\text{JPD}))$, and the $F(\text{JPD})$ of the square tiling at $n = 13$.

k	O(F(JPD))	Example of the corresponding F(JPD)
3	1	$\{\{5/13, 2/13, 1/13, 5/13\}\}$
4	3	$\{\{2/13, 1/13, 1/13, 9/13\}, \{1/13, 2/13, 9/13, 1/13\}, \{2/13, 8/13, 1/13, 2/13\}\}$
5	3	$\{\{2/13, 1/13, 1/13, 9/13\}, \{1/13, 2/13, 9/13, 1/13\}, \{2/13, 3/13, 1/13, 7/13\}\}$
6	5	$\{\{2/13, 1/13, 1/13, 9/13\}, \{1/13, 2/13, 9/13, 1/13\}, \{2/13, 8/13, 1/13, 2/13\}, \{8/13, 2/13, 2/13, 1/13\}, \{2/13, 3/13, 1/13, 7/13\}\}$
7	5	$\{\{2/13, 1/13, 1/13, 9/13\}, \{1/13, 2/13, 9/13, 1/13\}, \{2/13, 3/13, 1/13, 7/13\}, \{3/13, 2/13, 7/13, 1/13\}, \{2/13, 6/13, 1/13, 4/13\}\}$
8	7	$\{\{2/13, 1/13, 1/13, 9/13\}, \{1/13, 2/13, 9/13, 1/13\}, \{2/13, 8/13, 1/13, 2/13\}, \{8/13, 2/13, 2/13, 1/13\}, \{2/13, 3/13, 1/13, 7/13\}, \{3/13, 2/13, 7/13, 1/13\}, \{2/13, 4/13, 1/13, 6/13\}\}$



(a) (b)

Figure 5: Some JPD values at $n = 13$ and $n = 14$ to color the four vertices of the square tiling.

(ii) For $n \geq 8$ and n is even, the available total probability values (without repetition) are $\{1/n, 2/n, (n-4)/n, (n-5)/n, \ldots, (n-i)/n\}$, where $4 \leq i \leq n-3$. The greatest number of colors c equals $(n-4)$; for example, at $n = 8$, the available total probability values are $\{1/8, 2/8, 3/8, 4/8\}$, and so c equals 4 (i.e., $n-4$); at $n = 14$, the available total probability values are $\{1/14, 2/14, 3/14, 4/14, 5/14, 6/14, 7/14, 8/14, 9/14, 10/14\}$, and so the greatest number of colors c equals 10 (i.e., $n-4$); see Figure 5(a).

Case 4. For $n \geq 9$ and n is odd, the proof is similar to Case 3. Since n is odd, then in this case the rest corresponding probability value $(n-3)/n$ has even value of its numerator. Then, this probability value can be distributed on the other two vertices by two equal probability values "$((n-3)/2)/n$".

As in Case 3, the last two probability values $(n-3)/n$, $(n-1)/n$ and $((n-3)/2)/n$ are excluded. In this case we obtain that, for $n \geq 9$ and n is odd, the available total probability values are $\{1/n, 2/n, (n-4)/n, (n-5)/n, \ldots, (((n-3)/2)+1)/n, (((n-3)/2)-1)/n, \ldots, (n-i)/n\}$, where $4 \leq i \leq n-3$. The greatest number of colors c equals $(n-5)$: for example,

(i) at $n = 9$, the available total distinct probability values are $\{1/9, 2/9, 4/9, 5/9\}$ and the greatest number of colors c equals 4 (i.e., $n-5$);

(ii) at $n = 13$, the total distinct probability values are $\{1/13, 2/13, 3/13, 4/13, 6/13, 7/13, 8/13, 9/13\}$ and the greatest number of colors c equals 8 (i.e., $n-5$); see Figure 5(b).

□

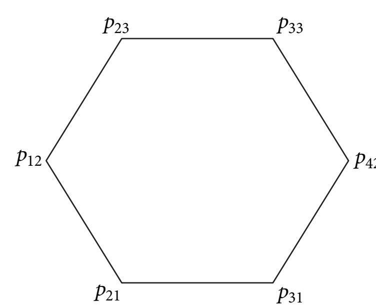


Figure 6: The used regular hexagon with the corresponding JPD values.

There are a relation between $k, n, O(F(\text{JPD}))$ and the corresponding $F(\text{JPD})$. Tables 1 and 2 show this relation:

(i) for $n = 13$, see Table 1;

(ii) for $n = 14$, see Table 2.

Corollary 10. *The smallest number of colors for square tiling is 2 if n is even and 3 if n is odd.*

2.3. JPD-Coloring of the 6-Monohedral Tiling. Here, we consider the JPD: $f_{X,Y}(x_i, y_j) = p_{ij}$, $i = 1, 2, 3, 4$; $j = 1, 2, 3$, with the assumptions $p_{21} \neq p_{31}$, $p_{31} \neq p_{42}$, $p_{42} \neq p_{33}$, $p_{33} \neq p_{23}$, $p_{23} \neq p_{12}$, $p_{12} \neq p_{21}$ (where p_{ij} is with equal denominators n) and zero values of p_{11}, p_{41}, p_{22} p_{32}, p_{13}, p_{43}. The used regular hexagon is illustrated in Figure 6.

Table 2: The relation between k, n, $O(F(\text{JPD}))$, and the $F(\text{JPD})$ of the square tiling at $n = 14$.

k	$O(F(\text{JPD}))$	Example of the corresponding $F(\text{JPD})$
2		{{4/14, 3/14, 3/14, 4/14}}
3	1	{{8/14, 1/14, 1/14, 4/14}}
4		{{2/14, 7/14, 1/14, 4/14}}
5	3	{{3/14, 1/14, 1/14, 9/14}, {1/14, 3/14, 9/14, 1/14}, {3/14, 4/14, 1/14, 6/14}}
6	4	{{6/14, 1/14, 1/14, 6/14}, {1/14, 5/14, 6/14, 2/14}, {5/14, 1/14, 2/14, 6/14}, {1/14, 4/14, 6/14, 3/14}}
7	5	{{2/14, 7/14, 1/14, 4/14}, {7/14, 2/14, 4/14, 1/14}, {2/14, 6/14, 1/14, 5/14}, {6/14, 2/14, 5/14, 1/14}, {2/14, 1/14, 1/14, 10/14}}
8		{{2/14, 7/14, 1/14, 4/14}, {7/14, 2/14, 4/14, 1/14}, {2/14, 6/14, 1/14, 5/14}, {6/14, 2/14, 5/14, 1/14}, {2/14, 8/14, 1/14, 3/14}}
9	7	{{2/14, 7/14, 1/14, 4/14}, {7/14, 2/14, 4/14, 1/14}, {2/14, 6/14, 1/14, 5/14}, {6/14, 2/14, 5/14, 1/14}, {2/14, 1/14, 1/14, 10/14}, {1/14, 2/14, 1/14, 10/14}, {2/14, 8/14, 1/14, 3/14}}
10	9	{{2/14, 7/14, 1/14, 4/14}, {7/14, 2/14, 4/14, 1/14}, {2/14, 6/14, 1/14, 5/14}, {6/14, 2/14, 5/14, 1/14}, {2/14, 1/14, 1/14, 10/14}, {1/14, 2/14, 10/14, 1/14}, {2/14, 9/14, 1/14, 2/14}, {9/14, 2/14, 2/14, 1/14}, {2/14, 8/14, 1/14, 3/14}}

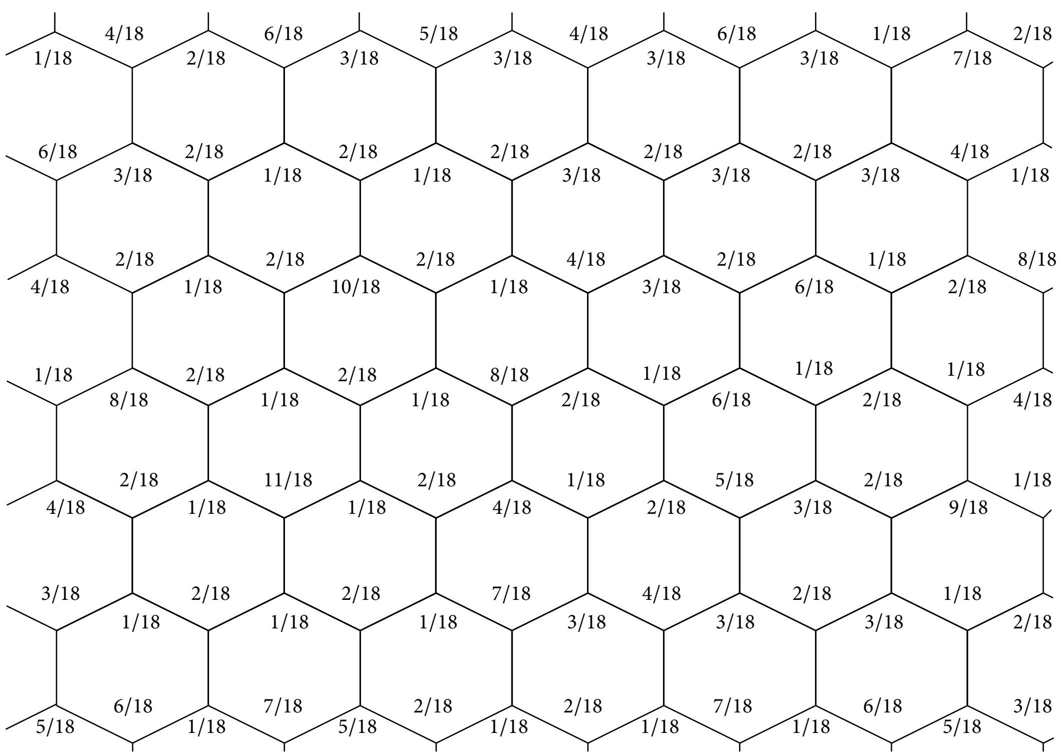


Figure 7: The JPD values at $n = 18$ to color the six vertices of the regular hexagon tiling.

Theorem 11. *If the plane is equipped by 6-monohedral tiling, then the greatest number of colors "c" is given as $c = n-7$ where $n \geq 9$. If $n < 9$, then the tiling cannot be colored.*

Proof. Let R^2 be equipped by hexagon tiling. The proof can be given as follows.

Case 1. If $n < 9$, then the tiling cannot be colored (i.e., the greatest number of colors c equals 0) because we cannot find six probability values (JPD), which satisfied the condition (ii) in Definition 4, to color the six vertices of the mentioned hexagon tiling (say, at $n = 8$ the available probability values are $\{1/8, 2/8, 1/8, 2/8, 1/8, 1/8\}$, at $n = 7$ the available probability values are $\{1/7, 2/7, 1/7, 1/7, 1/7, 1/7\}$, and at the smallest value $n = 6$ the available probability values are $\{1/6, 1/6, 1/6, 1/6, 1/6, 1/6\}$).

Case 2. For $n \geq 9$, take the corresponding probability value of a vertex $(n-i)/n$ where $i \geq 7$ because it is impossible to take the value of i less than 7. Then, the rest corresponding probability value is i/n. This probability value must be distributed on the other five vertices under consideration of

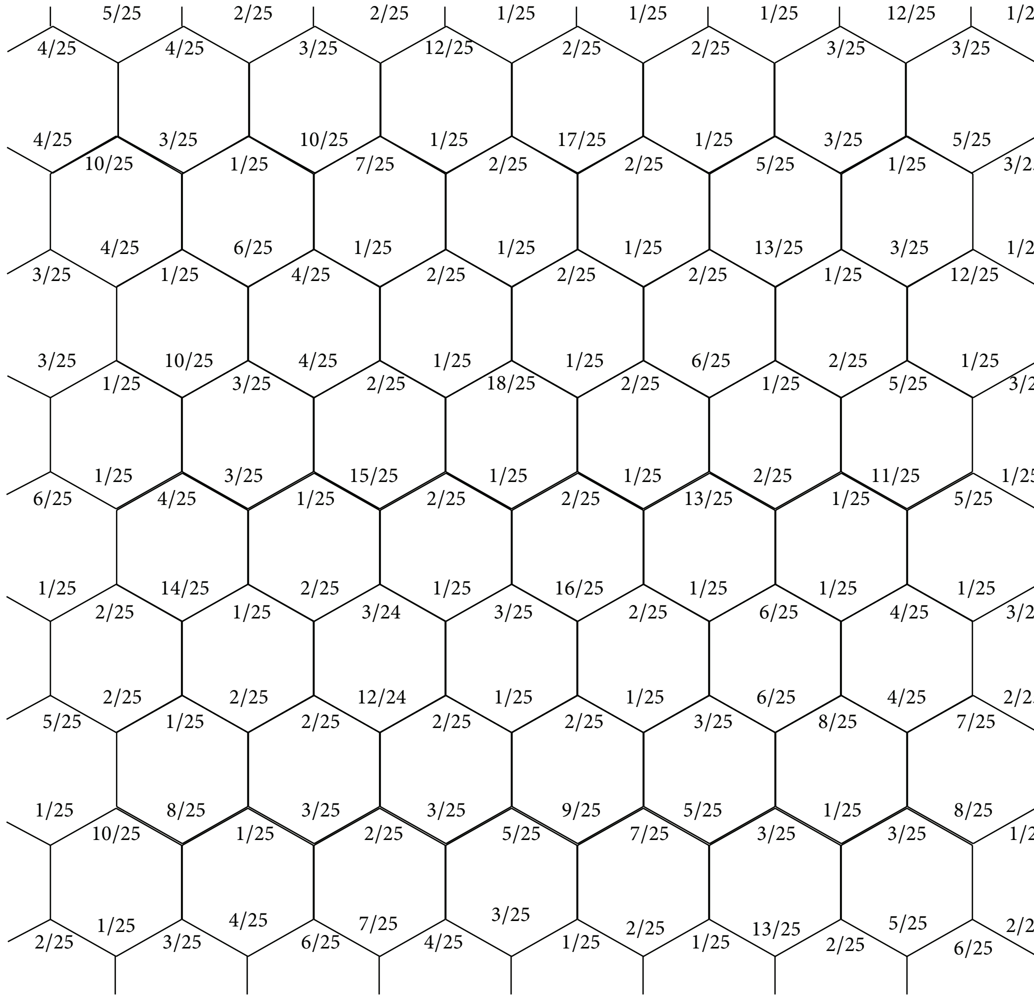


FIGURE 8: The JPD values at $n = 25$ to color the six vertices of the regular hexagon tiling.

the conditions in Definition 4 for an integer number $i \geq 7$. This implies that the available total probability values to obtain the mesh under consideration of the conditions in Definition 4 are $\{1/n, 2/n, 3/n, \ldots, (n-i-1)/n\}$, where $7 \leq i \leq n-2$. Avoiding the repetition, we obtain the following:

(i) for $n = 9$, the total probability values are {1/9, 2/9} and c equals 2,

(ii) for $n = 18$, the total probability values are {1/18, 2/18, 3/18, 4/18, 5/18, 6/18, 7/18, 8/18, 9/18, 10/18, 11/18} and c equals 11 (see Figure 7),

(iii) for $n = 25$, the total probability values are {1/25, 2/25, 3/25, 4/25, 5/25, 6/25, 7/25, 8/25, 9/25, 10/25, 11/25, 12/25, 13/25, 14/25, 15/25, 16/25, 17/25, 18/25} and c equals 18 (see Figure 8).

In this case, the greatest number of colors c equals $n - 7$. □

Corollary 12. *The relation between k and n can be shown in Table 3.*

TABLE 3: The relation between k and n of the regular hexagon tiling.

k	n
2	$6r + 3, r \geq 1$
3	$6r + 3, 6r + 4, 6r + 5, r \geq 1$
	$6r, 6r + 1, 6r + 2, r \geq 2$
4	$6r, 6r + 1, 6r + 2, 6r + 3, 6r + 4, 6r + 5, r \geq 2$
5	$6r + 5, r \geq 2$
	$6r, 6r + 1, 6r + 2, 6r + 3, 6r + 4, r \geq 3$
6	$6r + 3, 6r + 4, 6r + 5, r \geq 3$
	$6r, 6r + 1, 6r + 2, r \geq 4$

Appendix

As a MAPLE programming guide see [10].

Determination of the Probability Values by Using the MAPLE Program

Case A.1 (JPD-coloring of the 3-monohedral tiling). See Box 1.

Case A.2 (JPD-coloring of the 4-monohedral tiling). See Box 2.

```
> restart: with(linalg):
Probability_Values:=proc(n::posint,i::posint,j::posint)
local A,b,T,r,v,C; global PV;
A[T]:= matrix([[1,1,1]]): b:= vector([1]): C:=sum(V[w],w=1..2):
linsolve(A[T],b,‘r′): r: linsolve(A[T],b,‘r′,v):
PV[T[n,i,j]]:=subs(v[1]=i/n,v[2]=j/n,v[3]=1-C,%):
end proc:
n:=3:i:=1:j:=1: PV[T[n,i,j]]:=Probability_Values(n,i,j);
n:=4:i:=1:j:=2: PV[T[n,i,j]]:=Probability_Values(n,i,j);
n:=5:i:=1:j:=2: PV[T[n,i,j]]:=Probability_Values(n,i,j);
n:=6:i:=1:j:=2: PV[T[n,i,j]]:=Probability_Values(n,i,j);
n:=7:i:=1:j:=2: PV[T[n,i,j]]:=Probability_Values(n,i,j);
n:=8:i:=1:j:=2: PV[T[n,i,j]]:=Probability_Values(n,i,j);
n:=8:i:=1:j:=3: PV[T[n,i,j]]:=Probability_Values(n,i,j);
```

$$\mathrm{PV}_{T_{3,1,1}} := \left[\frac{1}{3},\frac{1}{3},\frac{1}{3}\right], \mathrm{PV}_{T_{4,1,2}} := \left[\frac{1}{2},\frac{1}{4},\frac{1}{4}\right], \mathrm{PV}_{T_{5,1,2}} := \left[\frac{2}{5},\frac{1}{5},\frac{2}{5}\right], \mathrm{PV}_{T_{6,1,2}} := \left[\frac{1}{2},\frac{1}{6},\frac{1}{3}\right],$$
$$\mathrm{PV}_{T_{7,1,2}} := \left[\frac{2}{7},\frac{4}{7},\frac{1}{7}\right], \mathrm{PV}_{T_{8,1,2}} := \left[\frac{5}{8},\frac{1}{8},\frac{1}{4}\right], \mathrm{PV}_{T_{8,1,3}} := \left[\frac{1}{2},\frac{1}{8},\frac{3}{8}\right]$$

Box 1

```
> restart: with(linalg):
Probability_Values:=proc(n::posint,i::posint,j::posint,k::posint)
local A,b,F,r,v,C; global PV;
A[F]:=matrix([[1,1,1,1]]): b:=vector([1]): C:=sum(V[w],w=1..3):
linsolve(A[F],b,‘r′): r: linsolve(A[F],b,‘r′,v):
PV[S[n,i,j,k]]:=subs(v[1]=i/n,v[2]=j/n,v[3]=k/n,v[4]=1-C,%):
end proc:
n:=4: i:=1:j:=1:k:=1: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=5: i:=1:j:=2:k:=1: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=6: i:=1:j:=2:k:=1: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=7: i:=1:j:=2:k:=1: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=8: i:=1:j:=2:k:=1: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=8: i:=1:j:=2:k:=3: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=8: i:=1:j:=3:k:=1: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=9: i:=1:j:=2:k:=1: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=9: i:=1:j:=4:k:=3: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=13:i:=1:j:=2:k:=1: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=13:i:=1:j:=3:k:=4: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=13:i:=1:j:=1:k:=6: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=13:i:=1:j:=1:k:=7: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=13:i:=1:j:=1:k:=8: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=14:i:=1:j:=1:k:=2: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=14:i:=2:j:=3:k:=4: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=14:i:=1:j:=5:k:=6: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=14:i:=1:j:=2:k:=7: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=14:i:=1:j:=2:k:=8: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
n:=14:i:=2:j:=2:k:=9: PV[F[n,i,j,k]]:=Probability_Values(n,i,j,k);
```

$$\mathrm{PV}_{F_{4,1,1,1}} := \left[\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right], \mathrm{PV}_{F_{5,1,2,1}} := \left[\frac{1}{5},\frac{1}{5},\frac{2}{5},\frac{1}{5}\right], \mathrm{PV}_{F_{6,1,2,1}} := \left[\frac{1}{6},\frac{1}{3},\frac{1}{6},\frac{1}{3}\right], \mathrm{PV}_{F_{7,1,2,1}} := \left[\frac{3}{7},\frac{1}{7},\frac{2}{7},\frac{1}{7}\right],$$
$$\mathrm{PV}_{F_{8,1,2,1}} := \left[\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{1}{8}\right], \mathrm{PV}_{F_{8,1,2,3}} := \left[\frac{1}{4},\frac{1}{8},\frac{1}{4},\frac{3}{8}\right], \mathrm{PV}_{F_{8,1,3,1}} := \left[\frac{3}{8},\frac{1}{8},\frac{3}{8},\frac{1}{8}\right], \mathrm{PV}_{F_{9,1,2,1}} := \left[\frac{5}{9},\frac{1}{9},\frac{2}{9},\frac{1}{9}\right],$$
$$\mathrm{PV}_{F_{9,1,4,3}} := \left[\frac{1}{3},\frac{4}{9},\frac{1}{9},\frac{1}{9}\right], \mathrm{PV}_{F_{13,1,2,1}} := \left[\frac{9}{13},\frac{1}{13},\frac{1}{13},\frac{2}{13}\right], \mathrm{PV}_{F_{13,1,3,4}} := \left[\frac{5}{13},\frac{4}{13},\frac{3}{13},\frac{1}{13}\right], \mathrm{PV}_{F_{13,1,1,6}} := \left[\frac{1}{13},\frac{6}{13},\frac{5}{13},\frac{1}{13}\right],$$
$$\mathrm{PV}_{F_{13,1,1,7}} := \left[\frac{1}{13},\frac{7}{13},\frac{4}{13},\frac{1}{13}\right], \mathrm{PV}_{F_{13,1,1,8}} := \left[\frac{8}{13},\frac{1}{13},\frac{1}{13},\frac{3}{13}\right], \mathrm{PV}_{F_{14,1,1,2}} := \left[\frac{1}{7},\frac{5}{7},\frac{1}{14},\frac{1}{14}\right], \mathrm{PV}_{F_{14,2,3,4}} := \left[\frac{1}{7},\frac{3}{14},\frac{2}{7},\frac{5}{14}\right],$$
$$\mathrm{PV}_{F_{14,1,5,6}} := \left[\frac{1}{14},\frac{5}{14},\frac{3}{7},\frac{1}{7}\right], \mathrm{PV}_{F_{14,1,2,7}} := \left[\frac{2}{7},\frac{1}{14},\frac{1}{7},\frac{1}{4}\right], \mathrm{PV}_{F_{14,1,2,8}} := \left[\frac{1}{14},\frac{1}{7},\frac{4}{7},\frac{3}{14}\right], \mathrm{PV}_{F_{14,2,2,9}} := \left[\frac{1}{14},\frac{1}{7},\frac{1}{7},\frac{9}{14}\right]$$

Box 2

```
>restart: with(linalg):
Probability_Values:=proc(n::posint,i::posint,j::posint,k::posint,l::posint,m::posint)
local A,b,S,r,v,C; global PV;
A[S]:=matrix([[1,1,1,1,1,1]]): b:=vector([1]): C:=sum(v[w],w=1..5):
linsolve(A[S],b,'r'): r: linsolve(A[S],b,'r',v):
PV[S[n,i,j,k,l,m]]:=subs(v[1]=i/n,v[2]=j/n,v[3]=k/n,v[4]=l/n,v[5]=m/n,v[6]=1-C,%):
end proc:
n:=6: i:=1:j:=1:k:=1:l:=1:m:=1: PV[S[n,i,j,k,l,m]]:=Probability_Values(n,i,j,k,l,m);
n:=7: i:=1:j:=2:k:=1:l:=1:m:=1: PV[S[n,i,j,k,l,m]]:=Probability_Values(n,i,j,k,l,m);
n:=8: i:=1:j:=2:k:=1:l:=2:m:=1: PV[S[n,i,j,k,l,m]]:=Probability_Values(n,i,j,k,l,m);
n:=9: i:=1:j:=2:k:=1:l:=2:m:=1: PV[S[n,i,j,k,l,m]]:=Probability_Values(n,i,j,k,l,m);
n:=18:i:=1:j:=2:k:=3:l:=4:m:=6: PV[S[n,i,j,k,l,m]]:=Probability_Values(n,i,j,k,l,m);
n:=25:i:=1:j:=2:k:=3:l:=4:m:=5: PV[S[n,i,j,k,l,m]]:=Probability_Values(n,i,j,k,l,m);
n:=25:i:=1:j:=2:k:=3:l:=4:m:=5: PV[S[n,i,j,k,l,m]]:=Probability_Values(n,i,j,k,l,m);
```

$$\mathrm{PV}_{S_{6,1,1,1,1,1}} := \left[\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right], \mathrm{PV}_{S_{7,1,2,1,1,1}} := \left[\frac{1}{7},\frac{1}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7}\right], \mathrm{PV}_{S_{8,1,2,1,2,1}} := \left[\frac{1}{4},\frac{1}{8},\frac{1}{4},\frac{1}{8},\frac{1}{8},\frac{1}{8}\right],$$
$$\mathrm{PV}_{S_{9,1,2,1,2,1}} := \left[\frac{2}{9},\frac{1}{9},\frac{2}{9},\frac{1}{9},\frac{2}{9},\frac{1}{9}\right], \mathrm{PV}_{S_{18,1,2,3,4,6}} := \left[\frac{1}{9},\frac{1}{18},\frac{1}{9},\frac{1}{6},\frac{2}{9},\frac{1}{3}\right], \mathrm{PV}_{S_{25,1,2,3,4,5}} := \left[\frac{1}{5},\frac{4}{25},\frac{3}{25},\frac{2}{25},\frac{1}{25},\frac{2}{25}\right],$$
$$\mathrm{PV}_{S_{25,1,2,3,4,5}} := \left[\frac{1}{5},\frac{4}{25},\frac{2}{5},\frac{3}{25},\frac{2}{25},\frac{1}{25}\right]$$

Box 3

Case A.3 (JPD-coloring of the 6-monohedral tiling). See Box 3.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] B. Grünbaum and G. C. Shephard, "Tiling by regular polygons," *Mathematics Magazine*, vol. 50, no. 5, pp. 227–247, 1977.

[2] B. Grünbaum and G. C. Shephard, "Perfect colorings of transitive tilings and patterns in the plane," *Discrete Mathematics*, vol. 20, no. 3, pp. 235–247, 1977.

[3] S. Eigen, J. Navarro, and V. S. Prasad, "An aperiodic tiling using a dynamical system and Beatty sequences," in *Dynamics, Ergodic Theory, and Geometry*, vol. 54 of *Mathematical Sciences Research Institute*, pp. 223–241, Cambridge University Press, 2007.

[4] C. Mann, L. Asaro, J. Hyde, M. Jensen, and T. Schroeder, "Uniform edge-c-colorings of the Archimedean tilings," *Discrete Mathematics*, vol. 338, no. 1, pp. 10–22, 2015.

[5] R. Santos and R. Felix, "Perfect precise colorings of plane regular tilings," *Zeitschrift für Kristallographie*, vol. 226, no. 9, pp. 726–730, 2011.

[6] B. Grünbaum, P. Mani-Levitska, and G. C. Shephard, "Tiling three-dimensional space with polyhedral tiles of a given isomorphism type," *Journal of the London Mathematical Society*, vol. 29, no. 1, pp. 181–191, 1984.

[7] M. E. Basher, "σ-Coloring of the monohedral tiling," *International Journal of Mathematical Combinatorics*, vol. 2, pp. 46–52, 2009.

[8] J. E. Freund, I. Miller, and M. Miller, *Mathematical Statistics with Applications*, Prentice Hall PTR, 7th edition, 2003.

[9] D. C. Montgomery and G. C. Runger, *Applied Statistics and Probability for Engineers*, John Wiley & Sons, New York, NY, USA, 2nd edition, 1999.

[10] L. Bernardin, P. Chin, P. DeMarco et al., *Maple Programming Guide*, Maplesoft, 2011.