Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 278313, 7 pages
http://dx.doi.org/10.1155/2015/278313

Research Article

New Approach to Fractal Approximation of Vector-Functions

Konstantin Igudesman,[1] Marsel Davletbaev,[2] and Gleb Shabernev[3]

[1] *Geometry Department, Lobachevskii Institute of Mathematics and Mechanics, Kazan (Volga Region) Federal University, Kazan 420008, Russia*
[2] *Kazan (Volga Region) Federal University, IT-Lyceum of Kazan University, Kazan 420008, Russia*
[3] *Department of Autonomous Robotic Systems, High School of Information Technologies and Information Systems, Kazan (Volga Region) Federal University, Kazan 420008, Russia*

Correspondence should be addressed to Konstantin Igudesman; kigudesm@yandex.ru

Received 19 August 2014; Revised 24 January 2015; Accepted 8 February 2015

Academic Editor: Poom Kumam

This paper introduces new approach to approximation of continuous vector-functions and vector sequences by fractal interpolation vector-functions which are multidimensional generalization of fractal interpolation functions. Best values of fractal interpolation vector-functions parameters are found. We give schemes of approximation of some sets of data and consider examples of approximation of smooth curves with different conditions.

1. Introduction

It is well known that interpolation and approximation are an important tool for interpretation of some complicated data. But there are multitudes of interpolation methods using several families of functions: polynomial, exponential, rational, trigonometric, and splines to name a few. Still it should be noted that all these conventional nonrecursive methods produce interpolants that are differentiable a number of times except possibly at a finite set of points. But, in many situations, we deal with irregular forms, which can not be approximate with desired precision. Fractal approximation became a suitable tool for that purpose. This tool was developed and studied in [1–3].

We know that such curves as coastlines, price graphs, encephalograms, and many others are fractals since their Hausdorff-Besicovitch dimension is greater than unity. To approximate them, we use fractal interpolation curves [1] and their generalizations [4] instead of canonical smooth functions (polynomials and splines).

This paper is multidimensional generalization of [5]. In Section 2, we consider fractal interpolation vector-functions which depend on several matrices of parameters. Example of such functions is given. In Section 3, we set the optimization problem for approximation of vector-function from L_2 by fractal approximation vector-functions. We find best values of matrix parameters by means of matrix differential calculus. Section 4 illustrates some examples.

2. Fractal Interpolation Vector-Functions

Let $[a,b] \subset \mathbb{R}$ be a nonempty interval; let $1 < N \in \mathbb{N}$ and $\{(t_n, \mathbf{x}_n) \in [a,b] \times \mathbb{R}^M \mid a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b\}$ be the interpolation points. For all $n = \overline{1,N}$, consider affine transformation

$$A_n : \mathbb{R}^{M+1} \longrightarrow \mathbb{R}^{M+1},$$
$$A_n \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} := \begin{pmatrix} a_n & \mathbf{0} \\ \mathbf{c}_n & \mathbf{D}_n \end{pmatrix} \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} + \begin{pmatrix} e_n \\ \mathbf{f}_n \end{pmatrix}. \tag{1}$$

Henceforth, small bold letters denote columns (rows) of length M and big bold letters denote matrices of $M \times M$.

Require that for all n the following conditions hold true:

$$A_n(t_0, \mathbf{x}_0) = (t_{n-1}, \mathbf{x}_{n-1}), \qquad A_n(t_N, \mathbf{x}_N) = (t_n, \mathbf{x}_n). \tag{2}$$

Then,

$$\begin{aligned}
a_n t_0 + e_n &= t_{n-1},\\
a_n t_N + e_n &= t_n,\\
\mathbf{c}_n t_0 + \mathbf{D}_n \mathbf{x}_0 + \mathbf{f}_n &= \mathbf{x}_{n-1},\\
\mathbf{c}_n t_N + \mathbf{D}_n \mathbf{x}_N + \mathbf{f}_n &= \mathbf{x}_n.
\end{aligned} \tag{3}$$

Solving the system, we have

$$\begin{aligned}
a_n &= \frac{t_n - t_{n-1}}{b-a},\\
e_n &= \frac{bt_{n-1} - at_n}{b-a},\\
\mathbf{c}_n &= \frac{\mathbf{x}_n - \mathbf{x}_{n-1} - \mathbf{D}_n(\mathbf{x}_N - \mathbf{x}_0)}{b-a},\\
\mathbf{f}_n &= \frac{b\mathbf{x}_{n-1} - a\mathbf{x}_n - \mathbf{D}_n(b\mathbf{x}_0 - a\mathbf{x}_N)}{b-a},
\end{aligned} \tag{4}$$

where matrices $\{\mathbf{D}_n\}_{n=1}^N$ are considered as parameters.

Remark 1. Notice that $\sum_{n=1}^N a_n = 1$.

Also notice that for all n operator A_n takes straight segment between $(t_0, \mathbf{x}_0)$ and $(t_N, \mathbf{x}_N)$ to straight segment which connects points of interpolation $(t_{n-1}, \mathbf{x}_{n-1})$ and $(t_n, \mathbf{x}_n)$.

Let $\mathscr{K}$ be a space of nonempty compact subsets of $\mathbb{R}^{M+1}$, with Hausdorff metric. Define the Hutchinson operator [6]

$$\Phi : \mathscr{K} \longrightarrow \mathscr{K}, \quad \Phi(E) = \bigcup_{n=1}^N A_n(E). \tag{5}$$

By the condition (2) Hutchinson operator Φ takes a graph of any continuous vector-function on segment $[a,b]$ to a graph of a continuous vector-function on the same segment. Thus, Φ can be treated as operator on the space of continuous vector-functions $(C[a,b])^M$.

For all $n = \overline{1,N}$, denote

$$\begin{aligned}
p_n &: [a,b] \longrightarrow [t_{n-1}, t_n], \quad p_n(t) := a_n t + e_n,\\
\mathbf{q}_n &: [a,b] \longrightarrow \mathbb{R}^M, \quad \mathbf{q}_n(t) := \mathbf{c}_n t + \mathbf{f}_n.
\end{aligned} \tag{6}$$

In (1), substitute $\mathbf{x}$ to vector-function $\mathbf{g}(t)$. We have that Φ acts on $(C[a,b])^M$ according to

$$\begin{aligned}
&(\Phi\mathbf{g})(t)\\
&= \sum_{n=1}^N \left(\left(\mathbf{q}_n \circ p_n^{-1}\right)(t) + \mathbf{D}_n \left(\mathbf{g} \circ p_n^{-1}\right)(t) \right) \chi_{[t_{n-1}, t_n]}(t).
\end{aligned} \tag{7}$$

Suppose that we consider all matrices $\mathbf{D}_n$ as linear operators on $\mathbb{R}^M$. Furthermore, they are contractive mappings; that is, constant $c \in [0,1)$ exists such that for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^M$ and $n = \overline{1,N}$ we have

$$\left|\mathbf{D}_n(\mathbf{v}) - \mathbf{D}_n(\mathbf{w})\right| \le c\,|\mathbf{v} - \mathbf{w}|. \tag{8}$$

Then, from (7), it follows that operator Φ is contraction with contraction coefficient c on Banach space $((C[a,b])^M, \|\cdot\|_\infty)$, where $\|\mathbf{g}(t) - \mathbf{h}(t)\|_\infty := \sup\{t \in [a,b] : |\mathbf{g}(t) - \mathbf{h}(t)|\}$. By the fixed-point theorem, there exists unique vector-function $\mathbf{g}^\star \in (C[a,b])^M$ such that $\Phi\mathbf{g}^\star = \mathbf{g}^\star$ and for all $\mathbf{g} \in (C[a,b])^M$ we have

$$\lim_{k\to\infty} \left\|\Phi^k(\mathbf{g}) - \mathbf{g}^\star\right\|_\infty = 0. \tag{9}$$

Function $\mathbf{g}^\star$ is called fractal interpolation vector-function. It is easy to notice that if $\mathbf{g} \in (C[a,b])^M$, $\mathbf{g}(t_0) = \mathbf{x}_0$, and $\mathbf{g}(t_N) = \mathbf{x}_N$, then $\Phi(\mathbf{g})$ passes through points of interpolation. In this case functions $\Phi^k(\mathbf{g})$ are called prefractal interpolation vector-functions of order k.

Example 2. Figure 1 shows fractal interpolation vector-function of plane. Here $t_0 = -1$, $t_1 = 0$, and $t_2 = 1$ and $x_0 = (1,-1)$, $x_1 = (0,0)$, and $x_2 = (1,1)$. Values of matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ are

$$\begin{pmatrix} -\frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}, \qquad \begin{pmatrix} -\frac{1}{4} & -\frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}. \tag{10}$$

3. Approximation

Henceforth, we assume that for all $n = \overline{1,N}$ linear operator $\mathbf{D}_n$ is contractive mapping with contraction coefficient $c \in [0,1)$. We approximate vector-function $\mathbf{g} \in (C[a,b])^M$ by fractal interpolation vector-function $\mathbf{g}^\star$ constructed on points of interpolation $\{(t_n, \mathbf{x}_n)\}_{n=0}^N$. Thus, we need to fit matrix parameters $\mathbf{D}_n$ to minimize the distance between $\mathbf{g}$ and $\mathbf{g}^\star$.

We use methods that have been developed for fractal image compression [7]. Denote Banach space of square integrated vector-functions on segment as $(L_2^M[a,b], \|\cdot\|_2)$, where norm $\|\cdot\|_2$ defines

$$\|\mathbf{g}\|_2 = \sqrt{\int_a^b |\mathbf{g}(t)|^2\, \mathrm{d}t}. \tag{11}$$

Then from (7) and (8) and Remark 1 it follows that for all $\mathbf{g}, \mathbf{h} \in L_2^M[a,b]$

$$\begin{aligned}
&\|\Phi\mathbf{g} - \Phi\mathbf{h}\|_2^2\\
&= \int_a^b |\Phi\mathbf{g} - \Phi\mathbf{h}|^2\, \mathrm{d}t\\
&= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left|\mathbf{D}_n \circ (\mathbf{g} - \mathbf{h}) \circ u_n^{-1}(t)\right|^2 \mathrm{d}t\\
&= \sum_{n=1}^N a_n \int_a^b \left|\mathbf{D}_n \circ (\mathbf{g} - \mathbf{h})(t)\right|^2 \mathrm{d}t\\
&\le \sum_{n=1}^N a_n c^2 \int_a^b \left|(\mathbf{g} - \mathbf{h})(t)\right|^2 \mathrm{d}t = c^2 \|\mathbf{g} - \mathbf{h}\|_2^2.
\end{aligned} \tag{12}$$

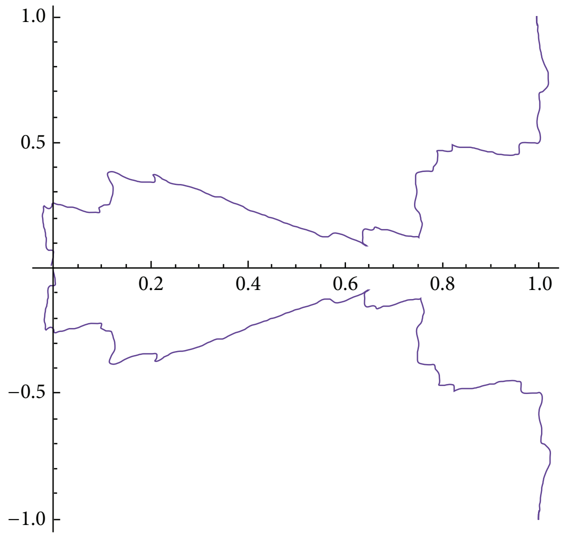


Figure 1: Fractal interpolation vector-function $\mathbf{g}^{\star}$.

Thus, $\Phi : L_2^M[a,b] \to L_2^M[a,b]$ is a contractive operator and $\mathbf{g}^{\star}$ is its fixed point.

Instead of minimizing $\|\mathbf{g} - \mathbf{g}^{\star}\|_2$ we minimize $\|\mathbf{g} - \Phi\mathbf{g}\|_2$ that makes the problem of optimization much easier. The collage theorem provides validity of such approach [8].

Theorem 3. *Let (X, d) be complete metric space and $T : X \to X$ is contractive mapping with contraction coefficient $c \in [0, 1)$ and fixed point $x^{\star}$. Then*

$$d(x, x^{\star}) \le \frac{d(x, T(x))}{1 - c} \tag{13}$$

for all $x \in X$.

Considering (4) and (6), rewrite (7)

$$(\Phi\mathbf{g})(t) = \sum_{n=1}^{N} \left(\mathbf{u}_n(t) + \mathbf{D}_n(\mathbf{g} \circ w_n(t) - \mathbf{v}_n(t))\right) \chi_{[t_{n-1}, t_n]}(t), \tag{14}$$

where

$$\begin{aligned} \mathbf{u}_n(t) &= \frac{(\mathbf{x}_n - \mathbf{x}_{n-1})\,t + (t_n \mathbf{x}_{n-1} - t_{n-1}\mathbf{x}_n)}{t_n - t_{n-1}}, \\ \mathbf{v}_n(t) &= \frac{(\mathbf{x}_N - \mathbf{x}_0)\,t + (t_n \mathbf{x}_0 - t_{n-1}\mathbf{x}_N)}{t_n - t_{n-1}}, \\ w_n(t) &= \frac{(b - a)\,t + (t_n a - t_{n-1} b)}{t_n - t_{n-1}}. \end{aligned} \tag{15}$$

Thus, we minimize the functional

$$\begin{aligned} &\|\mathbf{g} - \Phi\mathbf{g}\|_2^2 \\ &\quad = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left|\mathbf{g}(t) - \mathbf{u}_n(t) - \mathbf{D}_n(\mathbf{g} \circ w_n(t) - \mathbf{v}_n(t))\right|^2 \mathrm{d}t. \end{aligned} \tag{16}$$

Lemma 4. *Let $\mathbf{f}, \mathbf{h} \in L_2^M[a,b]$ be square integrated vector-functions. Suppose that matrix $\int_a^b \mathbf{h}\mathbf{h}^T \mathrm{d}t$ is nondegenerated. Matrix integration is implied to be componentwise. Then, the functional*

$$\Psi : \mathbb{R}^{M\times M} \longrightarrow \mathbb{R}, \quad \Psi(\mathbf{X}) = \int_a^b |\mathbf{f} - \mathbf{X}\mathbf{h}|^2 \,\mathrm{d}t \tag{17}$$

reaches its minimum in $\mathbf{X} = \int_a^b \mathbf{f}\mathbf{h}^T \mathrm{d}t \, (\int_a^b \mathbf{h}\mathbf{h}^T \mathrm{d}t)^{-1}$.

Proof. To prove it, we use matrix differential calculus [9].

Consider

$$\begin{aligned} \mathrm{d}\Psi(\mathbf{X}, \mathbf{U}) &= \mathrm{d}\left(\int_a^b (\mathbf{f} - \mathbf{X}\mathbf{h})^T (\mathbf{f} - \mathbf{X}\mathbf{h})\,\mathrm{d}t\right)\mathbf{U} \\ &= \mathrm{d}\left(\int_a^b \left(\mathbf{f}^T\mathbf{f} - \mathbf{h}^T\mathbf{X}^T\mathbf{f} - \mathbf{f}^T\mathbf{X}\mathbf{h} + \mathbf{h}^T\mathbf{X}^T\mathbf{X}\mathbf{h}\right)\mathrm{d}t\right)\mathbf{U} \\ &= \int_a^b \left(-\mathbf{h}^T\mathbf{U}^T\mathbf{f} - \mathbf{f}^T\mathbf{U}\mathbf{h} + \mathbf{h}^T\mathbf{U}^T\mathbf{X}\mathbf{h} + \mathbf{h}^T\mathbf{X}^T\mathbf{U}\mathbf{h}\right)\mathrm{d}t \\ &= 2\int_a^b \left(-\mathbf{h}^T\mathbf{U}^T\mathbf{f} + \mathbf{h}^T\mathbf{U}^T\mathbf{X}\mathbf{h}\right)\mathrm{d}t. \end{aligned} \tag{18}$$

Necessary condition of existence of functional Ψ extremum is $\mathrm{d}\Psi(\mathbf{X}, \mathbf{U}) = 0$ for all $\mathbf{U} \in \mathbb{R}^{M\times M}$. Since there is U-linearity of functional $\mathrm{d}\Psi(\mathbf{X}, \mathbf{U})$, it is sufficient to prove $\mathrm{d}\Psi(\mathbf{X}, \mathbf{U}) = 0$ only for matrices $\mathbf{U}$ that consist of $M^2 - 1$ zeros and one unity. Therefore, we have M^2 expressions for finding coefficients of matrix $\mathbf{X}$. In matrix form these expressions are as follows:

$$\int_a^b \mathbf{f}\mathbf{h}^T \mathrm{d}t = \int_a^b \mathbf{X}\mathbf{h}\mathbf{h}^T \mathrm{d}t, \tag{19}$$

from which

$$\mathbf{X} = \int_a^b \mathbf{f}\mathbf{h}^T \mathrm{d}t \left(\int_a^b \mathbf{h}\mathbf{h}^T \mathrm{d}t\right)^{-1}. \tag{20}$$

Hence,

$$\mathrm{d}^2\Psi(\mathbf{X}, \mathbf{U}) = 2\int_a^b \mathbf{h}^T\mathbf{U}^T\mathbf{U}\mathbf{h}\,\mathrm{d}t = 2\int_a^b |\mathbf{U}\mathbf{h}|^2 \,\mathrm{d}t \ge 0, \tag{21}$$

and then functional Ψ is convex one. Thus, the value $\mathbf{X}$ is absolute minimum of Ψ. □

From Lemma 4, it follows that functional (16) reaches minimum when

$$\begin{aligned} \mathbf{D}_n = &\int_{t_{n-1}}^{t_n} (\mathbf{g}(t) - \mathbf{u}_n(t))(\mathbf{g} \circ w_n(t) - \mathbf{v}_n(t))^T \,\mathrm{d}t \\ &\cdot \left(\int_{t_{n-1}}^{t_n} (\mathbf{g} \circ w_n(t) - \mathbf{v}_n(t))(\mathbf{g} \circ w_n(t) - \mathbf{v}_n(t))^T \,\mathrm{d}t\right)^{-1}. \end{aligned} \tag{22}$$

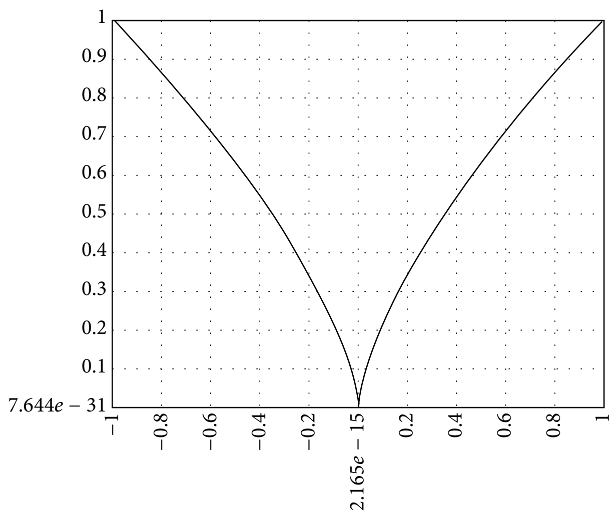


Figure 2: Vector-function $\mathbf{g}(t) = (t^2, t^3)$ and fractal interpolation vector-function $\mathbf{g}^\star$ completely identical.

Example 5. Let us approximate vector-function $\mathbf{g}(t) = (t^2, t^3)$ on segment $[-1, 1]$ by the fractal interpolation vector-function constructed on values of $\mathbf{g}(t)$ in points $t_0 = -1$, $t_1 = 0$, and $t_2 = 1$ and $x_0 = (1, -1)$, $x_1 = (0, 0)$, and $x_2 = (1, 1)$ (see Figure 2). Then,

$$
\begin{gathered}
a_1 = a_2 = \frac{1}{2}, \\
e_1 = -\frac{1}{2}, \qquad e_2 = \frac{1}{2}, \\
\mathbf{u}_1 = (-t, t), \qquad \mathbf{u}_2 = (t, t), \\
\mathbf{v}_1 = (1, 1 + 2t), \qquad \mathbf{v}_2 = (1, -1 + 2t), \\
w_1 = 1 + 2t, \qquad w_2 = -1 + 2t^2.
\end{gathered} \tag{23}
$$

Calculate $\mathbf{D}_1$, $\mathbf{D}_2$ according to formula (22) as follows:

$$
\mathbf{D}_1 = \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & \frac{1}{8} \end{pmatrix} \qquad \mathbf{D}_2 = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{3}{8} & \frac{1}{8} \end{pmatrix}. \tag{24}
$$

Apply affine transformations from (1) to vector $\{t, t^2, t^3\}$

$$
\begin{aligned}
A_1 \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{3}{8} & -\frac{3}{8} & \frac{1}{8} \end{pmatrix} \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{4} \\ -\frac{1}{8} \end{pmatrix} \\
&= \begin{pmatrix} \frac{t}{2} - \frac{1}{2} \\ \frac{t^2}{4} - \frac{t}{2} + \frac{1}{4} \\ \frac{t^3}{8} - \frac{3t^2}{8} + \frac{3t}{8} - \frac{1}{8} \end{pmatrix} \\
&= \begin{pmatrix} \frac{t-1}{2} \\ \left(\frac{t-1}{2}\right)^2 \\ \left(\frac{t-1}{2}\right)^3 \end{pmatrix}, \\
A_2 \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{8} \end{pmatrix} \\
&= \begin{pmatrix} \frac{t}{2} + \frac{1}{2} \\ \frac{t^2}{4} + \frac{t}{2} + \frac{1}{4} \\ \frac{t^3}{8} + \frac{3t^2}{8} + \frac{3t}{8} + \frac{1}{8} \end{pmatrix} \\
&= \begin{pmatrix} \frac{t+1}{2} \\ \left(\frac{t+1}{2}\right)^2 \\ \left(\frac{t+1}{2}\right)^3 \end{pmatrix}.
\end{aligned} \tag{25}
$$

Thus, $\Phi(\mathbf{g}) = \mathbf{g}$ and $\mathbf{g} = \mathbf{g}^\star$.

4. Discretization and Results

In this section, we approximate discrete data $Z = \{(z_m, \mathbf{w}_m)\}_{k=0}^{K}$, $a = z_0 < z_1 < \cdots < z_K = b$ by fractal interpolation vector-function $\mathbf{g}^\star$ constructed on points of interpolation $X = \{(t_i, \mathbf{x}_i)\}_{i=0}^{N}$, $a = t_0 < t_1 < \cdots < t_N = b$, $N \ll K$. Assume that $X \subset Z$. We fit matrix parameters $\mathbf{D}_n$ to minimize functional

$$
\sum_{k=0}^{K} \left| \mathbf{w}_k - \mathbf{g}^\star (z_k) \right|^2. \tag{26}
$$

It is necessary to use results of previous section. Approximate Z by constant piecewise vector-function $\mathbf{g} : [a, b] \to \mathbb{R}^M$. More precisely $\mathbf{g}(z) = \mathbf{w}_k$, where $(z_k, \mathbf{w}_k) \in Z$ z_k is the nearest approximation neighbor of z. By substituting integrals in (22) to discretization points sums we obtain

$$
\begin{aligned}
&\mathbf{D}_n \\
&\quad = \left(\sum_{z_k \in [t_{n-1}, t_n]} \left(\mathbf{g}(z_k) - \mathbf{u}_n(z_k) \right) \left(\mathbf{g} \circ w_n(z_k) - \mathbf{v}_n(z_k) \right)^T \right) \\
&\qquad \cdot \left(\sum_{z_k \in [t_{n-1}, t_n]} \left(\mathbf{g} \circ w_n(z_k) - \mathbf{v}_n(z_k) \right)
\end{aligned}
$$

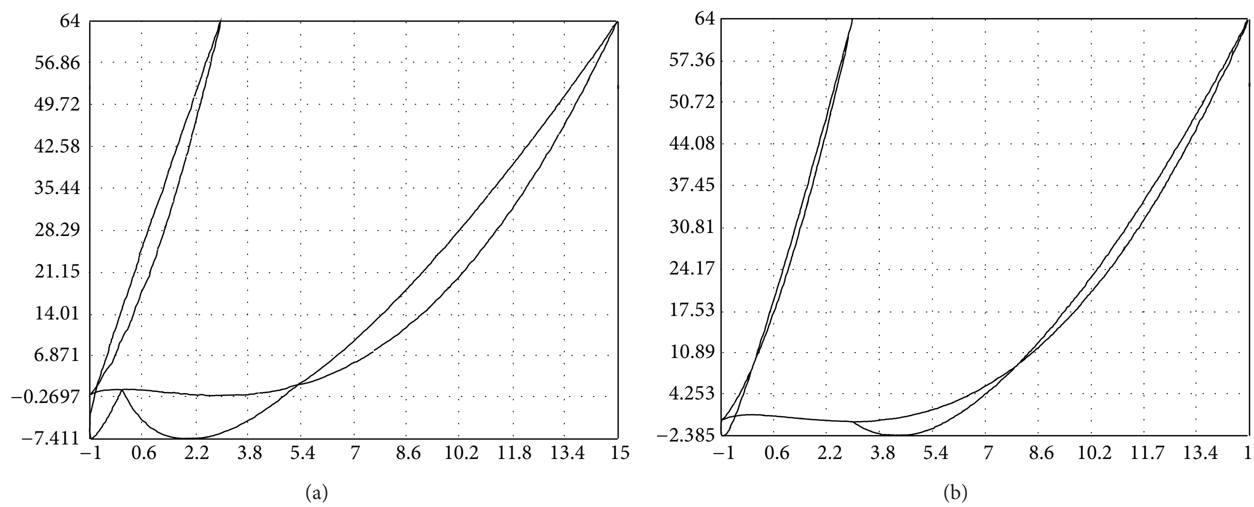


Figure 3: Approximation of vector-function $\mathbf{g}(t) = (t(t-2), (t-1)^2(t+1)^2)$ by fractal interpolation function $\mathbf{g}^*$ with three (a) and four (b) points of interpolation correspondingly.

$$\left. \cdot \left(\mathbf{g} \circ w_n(z_k) - \mathbf{v}_n(z_k)\right)^T \right)^{-1}, \qquad n = \overline{1, N}. \tag{27}$$

It is sufficient to apply (1) for constructing fractal interpolation vector-function after we find $\mathbf{D}_n$.

Consider several examples of approximation of discrete data.

Example 6. Let us approximate vector-function $\mathbf{g}(t) = (t(t-2), (t-1)^2(t+1)^2)$, where $t \in [-3, 3]$. Figure 3 shows the results. Here, we have two pictures; the first one illustrates initial vector-function and its approximation with 3 points and the second one with 4 points, where two functions are nearly identical.

In this case affine transformations (1) have the following form:

$$A_1 \begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ -0.8842 & 0.0943 & -0.1045 \\ -0.1038 & -0.0530 & 0.2287 \end{pmatrix} \begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0.7320 \\ 5.3989 \end{pmatrix},$$

$$A_2 \begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ 3.4602 & 0.7065 & -1.5549 \\ -0.4554 & 0.1504 & -0.2847 \end{pmatrix} \begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} 0.75 \\ 10.8875 \\ 8.1844 \end{pmatrix}. \tag{28}$$

Remark 7. Vectors $\mathbf{c}_n$ in matrices of affine transformations (1) equal $\mathbf{0}$ (like in previous example). It means that fractal interpolation vector-function can be treated as attractor of classical affine IFS in $\mathbb{R}^M$.

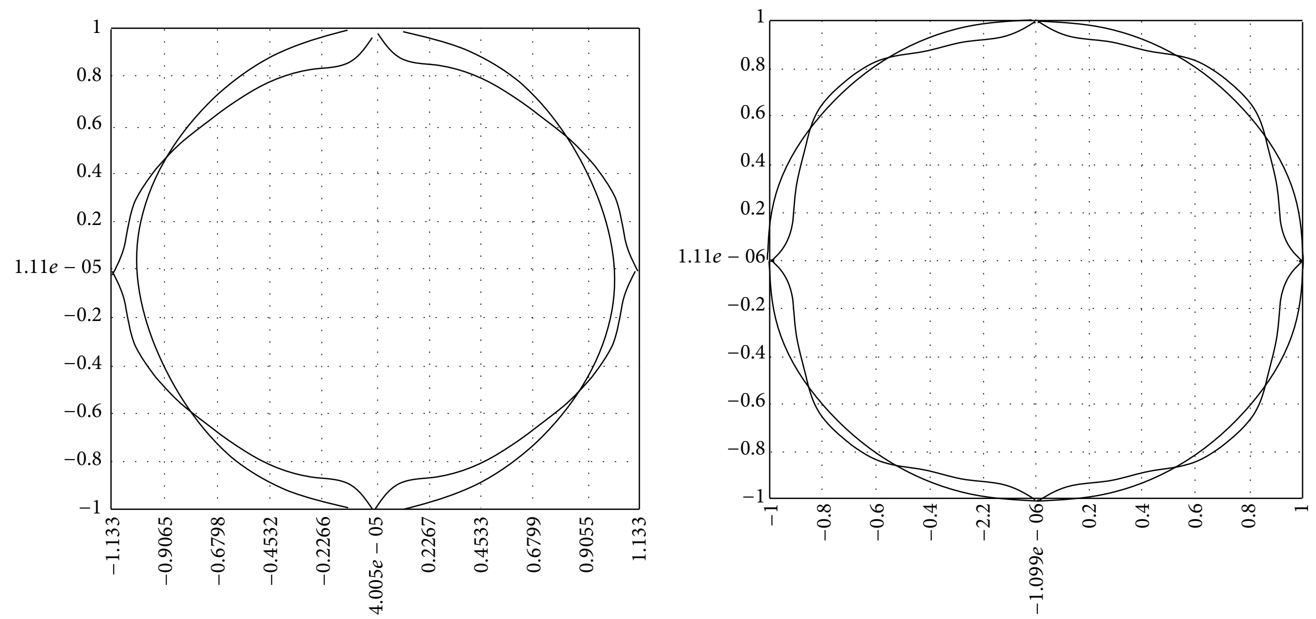
Example 8. Next example is devoted to a circle $\mathbf{g}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Figure 4 shows the results. Here we also have two pictures; the first one illustrates initial vector-function and its approximation with 3 points and the second one with 5 points.

In this case affine transformations (1) have the following form:

$$A_1 \begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ -0.3180 & 0.0006 & 0.2128 \\ -0.0013 & -0.5686 & -0.0038 \end{pmatrix} \begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0.9993 \\ 0.5686 \end{pmatrix},$$

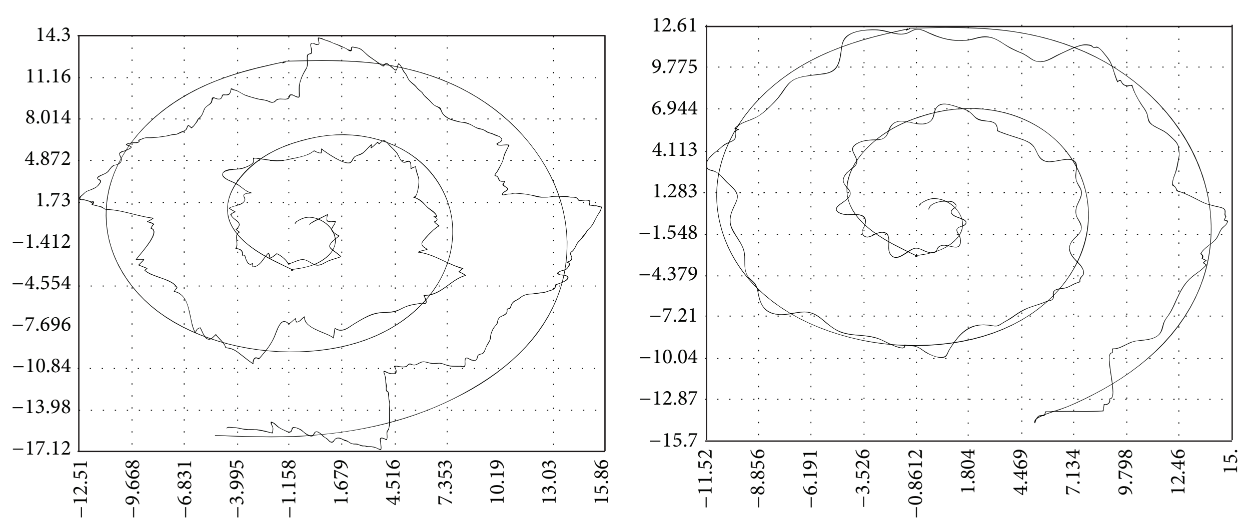
$$A_2 \begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0.3181 & -0.0053 & -0.2128 \\ 0.0040 & 0.5686 & 0.0020 \end{pmatrix} \begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} 3.151 \\ -0.9945 \\ -0.5770 \end{pmatrix}. \tag{29}$$

Example 9. Spiral of Archimedes $\mathbf{g}(t) = (t \cos t, t \sin t)$, $t \in [0, 5\pi]$, where the scheme is equal to the examples above, but here we use far more points of interpolation, as illustrated in Figure 5.



(a) (b)

Figure 4: Approximation of vector-function $\mathbf{g}(t) = (\cos t, \sin t)$ by fractal interpolation function $\mathbf{g}^{\star}$ with three (a) and five (b) points of interpolation correspondingly.



(a) (b)

Figure 5: Approximation of vector-function $\mathbf{g}(t) = (t\cos t, t\sin t)$, $t \in [0, 5\pi]$, by fractal interpolation function $\mathbf{g}^{\star}$ with twelve (a) and seventeen (b) points of interpolation correspondingly.

Example 10. Figure 6 shows approximation of vector-function $\mathbf{g}(t) = (\cos(1.5t), \sin(t))$, $t \in [0, 12\pi]$, by fractal interpolation vector-function with sixteen points of interpolation.

Example 11. The example illustrates approximation of graph of Weierstrass function $\omega(x) = \sum_{n=0}^{\infty}(1/2)^n \cos(2\pi 4^n x)$ (Figure 7) by fractal interpolation vector-function.

This example is taken from [10], where fractal approximation is used for approximate calculation of box dimension of fractal curves.

5. Conclusion

In this paper, we have introduced new effective method of approximation of continuous vector-functions and vector

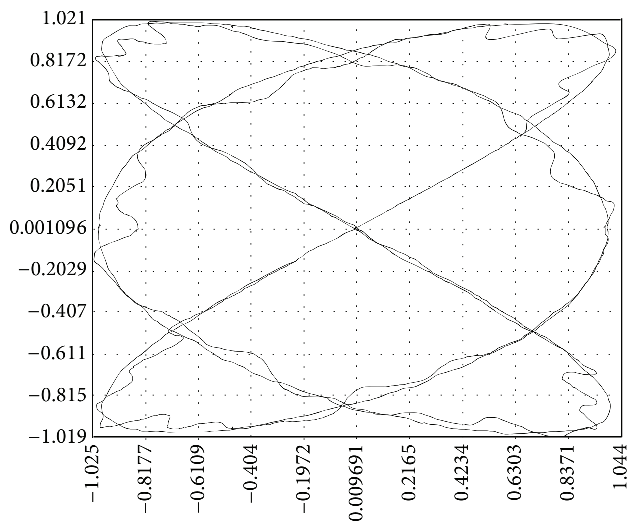


Figure 6: Approximation of vector-function $\mathbf{g}(t) = (\cos(1.5t), \sin(t))$ by fractal interpolation function $\mathbf{g}^*$.

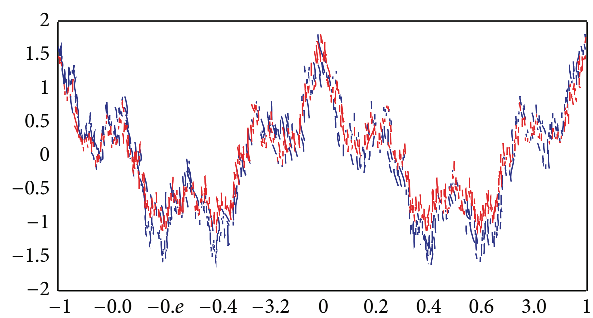


Figure 7: Weierstrass function (blue one) and approximating vector-function (red one).

sequences by fractal interpolation vector-functions, which are affine transformations with matrix parameters. Parameter fitting was a crucial part of approximation process. We have found appropriate parameter values of fractal interpolation vector-functions and illustrate it with several examples of different types of discrete data.

We assume that fractal approximation is highly promising computational tool for different types of data and it can be used in many ways, even in interdisciplinary fields, with a quite high precision that allows us to apply fractal approximation methods to a wide variety of curves, smooth and nonsmooth alike.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The work was performed according to the Russian Government Program of Competitive Growth of Kazan Federal University. The authors greatly acknowledge the anonymous reviewer for carefully reading the paper and providing constructive comments that have led to an improved paper.

References

[1] M. F. Barnsley, “Fractal functions and interpolation,” *Constructive Approximation*, vol. 2, no. 4, pp. 303–329, 1986.

[2] M. F. Barnsley, J. Elton, D. Hardin, and P. Massopust, “Hidden variable fractal interpolation functions,” *SIAM Journal on Mathematical Analysis*, vol. 20, no. 5, pp. 1218–1242, 1989.

[3] M. F. Barnsley and A. N. Harrington, “The calculus of fractal interpolation functions,” *Journal of Approximation Theory*, vol. 57, no. 1, pp. 14–34, 1989.

[4] P. Massopust, *Interpolation and Approximation with Splines and Fractals*, Oxford University Press, Oxford, UK, 2010.

[5] K. Igudesman and G. Shabernev, “Novel method of fractal approximation,” *Lobachevskii Journal of Mathematics*, vol. 34, no. 2, pp. 125–132, 2013.

[6] J. E. Hutchinson, “Fractals and self-similarity,” *Indiana University Mathematics Journal*, vol. 30, no. 5, pp. 713–747, 1981.

[7] M. F. Barnsley and L. P. Hurd, *Fractal Image Compression*, A. K. Peters, Wellesley, Mass, USA, 1993.

[8] M. Barnsley, *Fractals Everywhere*, Academic Press, Boston, Mass, USA, 1988.

[9] J. R. Magnus and H. Neudecker, “Matrix differential calculus with applications in statistics and econometrics,” *Applied Mathematical Sciences*, vol. 8, no. 144, pp. 7175–7181, 2014.

[10] K. Igudesman, R. Lavrenov, and V. Klassen, *Novel Approach to Calculation of Box Dimension of Fractal Functions*, Wiley, 3rd edition, 2007.