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*Research Article*

# Integer and Fractional General $T$-System and Its Application to Control Chaos and Synchronization

**Mihaela Neamţu, Anamaria Liţoiu, and Petru C. Strain**

*West University of Timisoara, 4 Bulevardul Vasile Pârvan, 300115 Timisoara, Romania*

Correspondence should be addressed to Mihaela Neamţu; mihaela.neamtu@e-uvt.ro

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We propose a three-dimensional autonomous nonlinear system, called the general $T$ system, which has potential applications in secure communications and the electronic circuit. For the general $T$ system with delayed feedback, regarding the delay as bifurcation parameter, we investigate the effect of the time delay on its dynamics. We determine conditions for the existence of the Hopf bifurcations and analyze their direction and stability. Also, the fractional order general $T$-system is proposed and analyzed. We provide some numerical simulations, where the chaos attractor and the dynamics of the Lyapunov coefficients are taken into consideration. The effectiveness of the chaotic control and synchronization on schemes for the new fractional order chaotic system are verified by numerical simulations.

## 1. Introduction

Lorenz found the first canonical chaotic attractor [1]. During the time, it has been proved that chaos can occur in simple three-dimensional autonomous systems with one, two, and three nonlinearities. Tigan and Opris [2] proposed and analyzed a new chaotic three-dimensional nonlinear system, called $T$ system, which is similar to the Lorenz system. Because $T$ system allows a larger possibility in choosing the parameters of the system, it can display more complex dynamics [3–10].

Recently, based on the study of integer order chaos, the fractional order Lorenz system [11] and the fractional order Liu system [12] were introduced. The system with fractional order still shows the chaotic behavior [13, 14].

The $T$ system is described by [2]

$$\begin{aligned}
\dot{x}(t) &= a\left(y(t) - x(t)\right), \\
\dot{y}(t) &= (b-a)\,x(t) - a x(t)\, z(t), \\
\dot{z}(t) &= -cz(t) + x(t)\, y(t),
\end{aligned} \tag{1}$$

which is chaotic when $a = 2.1$, $b = 30$, and $c = 0.6$ [2].

Li et al. [15] have proposed a new Lorenz-like chaotic system derived from (1). The nonlinear differential three-dimensional system is given by

$$\begin{aligned}
\dot{x}(t) &= a\left(y(t) - x(t)\right), \\
\dot{y}(t) &= abx(t) - ax(t)\, z(t), \\
\dot{z}(t) &= -cz(t) + x(t)\, y(t),
\end{aligned} \tag{2}$$

which is chaotic when $a = 5$, $b = 4$, and $c = 2$ [15].

Chaotic phenomena in electric circuits have been studied with great interest. The electronic circuit for (2) is designed and a chaotic attractor is implemented and verified [15].

Yang [16] proposed another new Lorenz-like system. The nonlinear differential three-dimensional system is

$$\begin{aligned}
\dot{x}(t) &= a\left(y(t) - x(t)\right), \\
\dot{y}(t) &= b_1 x(t) - b_2 x(t)\, z(t), \\
\dot{z}(t) &= -cz(t) + lx(t)^2 + ky(t)^2,
\end{aligned} \tag{3}$$

which is chaotic when $a = 10$, $b_1 = 40$, $b_2 = 1$, $c = 2.5$, $l = 2$, and $k = 2$ [16].

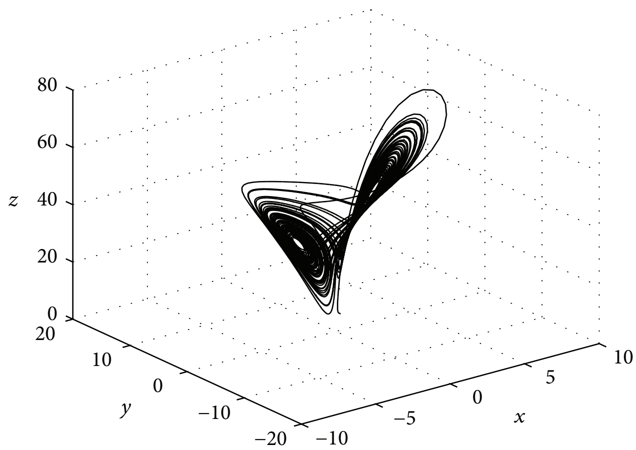


Figure 1: Phase portrait of the chaotic attractor for the general $T$ system (4) with $a = 10$, $b_1 = 40$, $b_2 = 1$, $c = 2.5$, $d = 3$, $l = 0.9$, and $k = 2$.

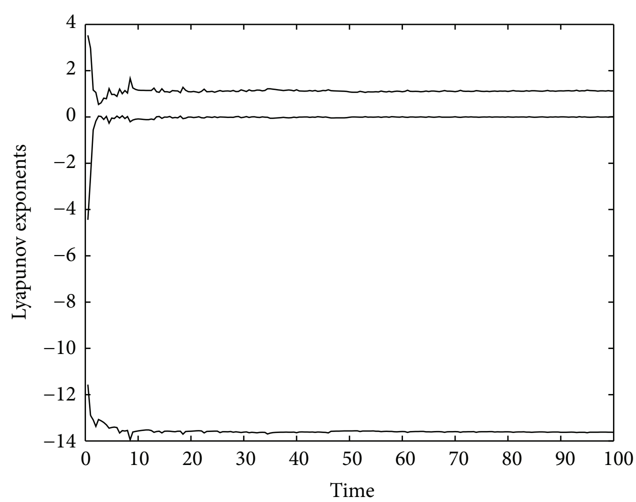


Figure 2: Dynamics of Lyapunov exponents of (4) with $a = 10$, $b_1 = 40$, $b_2 = 1$, $c = 2.5$, $d = 3$, $l = 0.9$, and $k = 2$.

Based on (1), (2), and (3), we propose a general $T$ system described by the following differential equations:

$$\begin{aligned} \dot{x}(t) &= a\left(y(t) - x(t)\right), \\ \dot{y}(t) &= b_1 x(t) - b_2 x(t) z(t), \\ \dot{z}(t) &= -cz(t) + dx(t) y(t) + lx(t)^2 + ky(t)^2, \end{aligned} \tag{4}$$

where $a, b_2, c, d, h, k$ are positive real parameters, $b_1$ is a real parameter, and $a \neq 0$.

An experimental electronic circuit for (4) can be described and implemented in a similar way as in [15, 17].

System (4) is chaotic when $a = 10$, $b_1 = 40$, $b_2 = 1$, $c = 2.5$, $d = 3$, $l = 0.9$, and $k = 2$ (see Figure 1).

The dynamics of Lyapunov exponents of system (4) is displayed in Figure 2.

For $b_1 = b - a$, $b_2 = a$, $d = 1$, $l = 0$, and $k = 0$, system (4) is given by (1). For $b_1 = ab$, $b_2 = a$, $d = 1$, $l = 0$, and $k = 0$, system (4) becomes (2). For $d = 0$, system (4) is (3).

Systems (1), (2), (3), and (4) are chaotic and using the method from [15] we can verify that they are not equivalent to the Lorenz, Chen, and Lü systems.

Time-delayed feedback is a powerful tool to control unstable periodic orbits or control unstable steady states [18]. Following the idea of Pyragas [19], as in [9, 18], we add a delayed force $k_1(y(t) - y(t-\tau))$ to the second equation of (4) and we obtain the delayed feedback control system:

$$\begin{aligned} \dot{x}(t) &= a\left(y(t) - x(t)\right), \\ \dot{y}(t) &= b_1 x(t) - b_2 x(t) z(t) + k_1\left(y(t) - y(t-\tau)\right), \\ \dot{z}(t) &= -cz(t) + dx(t) y(t) + lx(t)^2 + ky(t)^2, \end{aligned} \tag{5}$$

where $\tau$ is the time delay.

We assume that $\tau > 0$ and $k_1 \in \mathbb{R}$, which indicates the strength of the feedback [10].

The fractional-order general $T$ system can be described by

$$\begin{aligned} D^{q_1} x(t) &= a\left(y(t) - x(t)\right), \\ D^{q_2} y(t) &= b_1 x(t) - b_2 x(t) z(t), \\ D^{q_3} z(t) &= -cz(t) + dx(t) y(t) + lx(t)^2 + ky(t)^2, \end{aligned} \tag{6}$$

where ${}_aD_t^q$ is defined by [13]

$${}_aD_t^q = \begin{cases} \dfrac{d^q}{dt^q}, & \operatorname{Re}(q) > 0 \\ 1 & \operatorname{Re}(q) = 0 \\ \displaystyle\int_a^t (ds)^{-q} & \operatorname{Re}(q) < 0 \end{cases} \tag{7}$$

and $q$ is the derivative order that can be a complex number with $\operatorname{Re}(q)$ the real part of $q$. The numbers $a$ and $t$ are the limits of the operator. There are many definitions for general fractional derivative. The most frequently used ones are the Grunwald-Letnikov definition, the Riemann-Liouville, and the Caputo definitions.

As in [13], in this paper we use the Caputo definition for the fractional derivative.

In the present paper, we focus on (5) and (6). The aim is to provide a new investigation of the Hopf bifurcation and chaos control on the general $T$ system given by (5) and an analysis of the fractional general $T$-system as well.

For system (5), we consider $\tau$ as the bifurcation parameter. When it passes through some certain critical values, the equilibrium will lose its stability and Hopf bifurcation will occur. We study the direction of the Hopf bifurcation, as well as the stability and period of the bifurcating periodic solutions. Moreover, with different values for $k_1$ and $\tau$, we realize the chaos control.

The chaotic dynamics in the general $T$-system with fractional derivative is taken into account. Some properties are given. Then, synchronization problem of (6) is provided.

The paper is structured as follows. Section 2 provides the stability of the steady states when there is no time delay. For system (5), Section 3 analyzes the local stability, the existence of the Hopf bifurcation, and the direction and the stability of the Hopf bifurcation. The analysis of system (6) is presented in Section 4. The chaos control of (6) is given in Section 5 and the synchronization in Section 6. Finally, conclusions are drawn in Section 7.

## 2. Existence of Steady States: Stability Analysis for System (4)

The equilibrium of system (4) can be obtained by solving the following algebraic system:

$$\begin{aligned} a(y-x) &= 0, \\ b_1 x - b_2 xz &= 0, \\ -cz + dxy + lx^2 + ky^2 &= 0. \end{aligned} \tag{8}$$

Then, we have the following.

**Proposition 1.** *Consider the following:*

(i) *If $b_1 < 0$, then system (4) has only one real steady state $S_0(0,0,0)$.*

(ii) *If $b_1 > 0$, then system (4) has three real steady states: $S_0(0,0,0)$, $S_-(-x_0,-y_0,z_0)$, $S_+(x_0,y_0,z_0)$, where*

$$\begin{aligned} x_0 = y_0 &= \sqrt{\frac{b_1 c}{b_2(d+l+k)}}, \\ z_0 &= \frac{b_1}{b_2}. \end{aligned} \tag{9}$$

In order to analyze the local stability of the above steady states, the Jacobian matrix of (4) is given by

$$J(x,y,z) = \begin{pmatrix} -a & a & 0 \\ b_1 - b_2 z & 0 & -b_2 z \\ dy + 2lx & dx + 2ky & -c \end{pmatrix}. \tag{10}$$

If $b_1 < 0$, the characteristic equation of $J(S_0)$ is given by

$$(\lambda + c)\left(\lambda^2 + a\lambda - ab_1\right) = 0. \tag{11}$$

The eigenvalues of (11) are $\lambda_1 = -c$, and $\lambda_{2,3} = (1/2)(-a \pm \sqrt{a^2 + 4ab_1})$.

For $b_1 > 0$, both $S_-$ and $S_+$ have the same characteristic equation; that is

$$\lambda^3 + (a+c)\lambda^2 + \left(ac + \frac{b_1 c(d+2k)}{d+l+k}\right)\lambda + 2ab_1 c = 0. \tag{12}$$

The eigenvalues of (12), which are dependent on parameters $a, b_1, c, d, l, k$, can be obtained by the Cardano formula. Since $a, c, d, l, k$ are all positive real parameters, one can ensure that

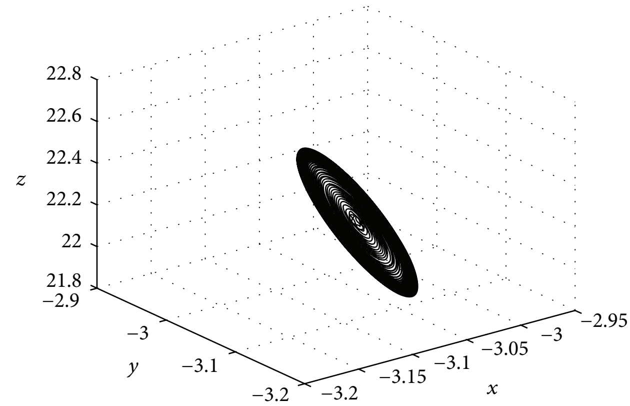


Figure 3: The steady state $S_-$ of system (4) is locally asymptotically stable when $b_1 < b_c$.

(12) has at least one eigenvalue with negative real part as $b_1 > 0$. The other two eigenvalues could be

(a) two negative real roots;

(b) two positive real roots;

(c) two complex-conjugate roots with negative real part;

(d) two complex-conjugate roots with positive real part.

Therefore, analyzing the characteristic equation of (4) and using the Routh-Hurwitz theorem, we obtain the following propositions.

**Proposition 2.** *Consider the following:*

(i) *If $b_1 < 0$, then the steady state $S_0(0,0,0)$ of system (4) is locally asymptotically stable.*

(ii) *If $b_1 > 0$, then the steady state $S_0(0,0,0)$ of system (4) is unstable.*

(iii) *If $0 < b_1 < b_c$ and $a(d+2l) - c(d+2k) > 0$, where*

$$b_c = \frac{a(a+c)(d+l+k)}{a(d+2l) - c(d+2k)}, \tag{13}$$

*then the steady states $S_-$ and $S_+$ of system (4) are locally asymptotically stable (see Figures 3 and 4).*

(iv) *If $b_1 > b_c$ and $a(d+2l) - c(d+2k) > 0$, then the steady states $S_-$ and $S_+$ of system (4) are unstable (see Figures 5 and 6).*

Using the Hopf bifurcation theorem [9, 16], we have the following.

**Proposition 3.** *If $b_1 > 0$ and $b_1 = b_c$, then for the steady state $S_+$ (or $S_-$) of system (4), the corresponding characteristic equation has three eigenvalues: one negative and one pair of purely imaginary conjugate roots, satisfying $\mathrm{Re}(d\lambda/db_1)|_{b_1=b_c} \neq 0$; that is, system (4) undergoes a Hopf bifurcation at the steady state $S_+$ (or $S_-$) (see Figures 7 and 8).*

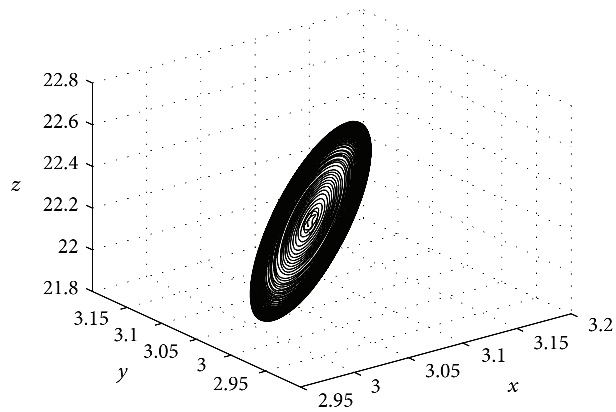


Figure 4: The steady state $S_+$ of system (4) is locally asymptotically stable when $b_1 < b_c$.

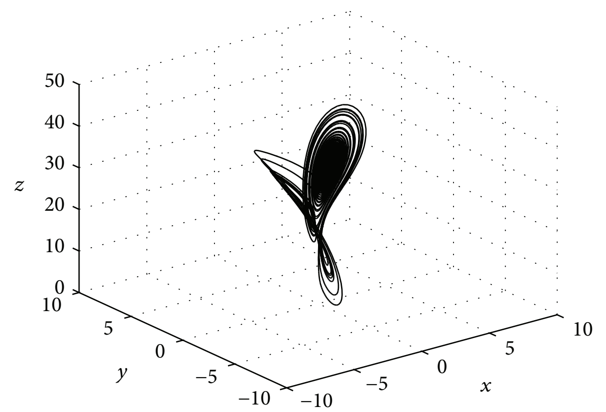


Figure 6: The steady state $S_+$ of system (4) is unstable when $b_1 > b_c$.

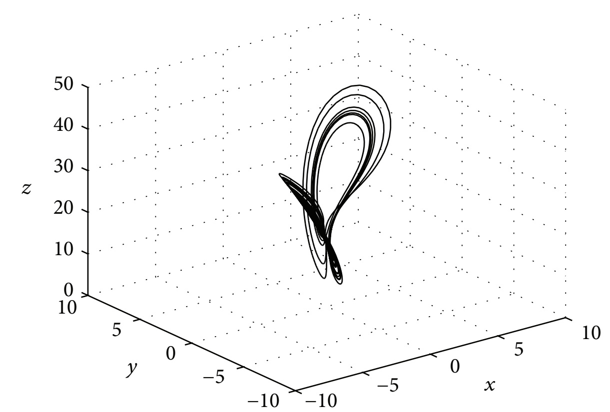


Figure 5: The steady state $S_-$ of system (4) is unstable when $b_1 > b_c$.

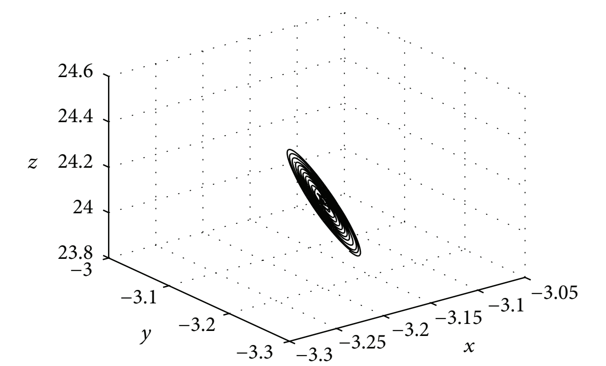


Figure 7: A Hopf bifurcation occurs at the steady state $S_-$ of system (4) when $b_1 = b_c$.

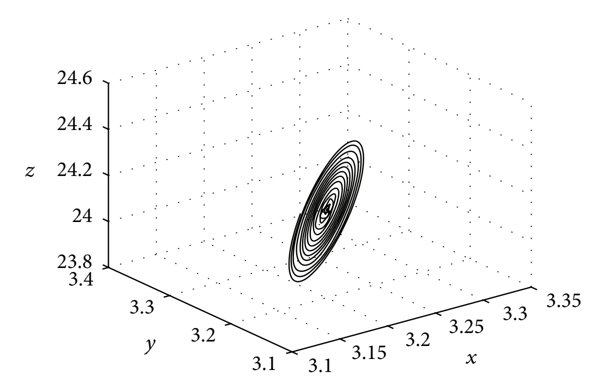


Figure 8: A Hopf bifurcation occurs at the steady state $S_+$ of system (4) when $b_1 = b_c$.

## 3. Local Stability and Hopf Bifurcation for System (5)

*3.1. Local Stability and the Existence of Hopf Bifurcation for System (5).* Consider system (5). When $\tau = 0$, it becomes (4). Since the time delay does not change, the delayed feedback system (5) has the same equilibria as system (4).

From Proposition 1, under the assumption $b_1 > 0$, system (5) also has three real steady states: $S_0(0, 0, 0)$, $S_-(-x_0, -y_0, z_0)$, and $S_+(x_0, y_0, z_0)$.

Now, we analyze the effect of the time delay on the stability of these steady states.

The linearization of system (5) at $S_0$ is

$$\begin{aligned} \dot{u}_1(t) &= a\left(u_2(t) - u_1(t)\right), \\ \dot{u}_2(t) &= b_1 u_1(t) + k_1\left(u_2(t) - u_2(t-\tau)\right), \\ \dot{u}_3(t) &= -cu_3(t). \end{aligned} \tag{14}$$

The characteristic equation of (14) is

$$(\lambda + c)\left(\lambda^2 + (a - k_1)\lambda - ak_1 - ab_1 + (\lambda + a)k_1 e^{-\lambda\tau}\right) = 0. \tag{15}$$

Equation (15) has a negative root $\lambda = -c$ for all $\tau > 0$; thus, we need to analyse the following equation:

$$\lambda^2 + (a - k_1)\lambda - ak_1 - ab_1 + (\lambda + a)k_1 e^{-\lambda\tau} = 0. \tag{16}$$

If there is no delay, (16) becomes

$$\lambda^2 + a\lambda - ab_1 = 0. \tag{17}$$

Let $i\omega$ $(\omega > 0)$ be a root of (16). It follows that

$$\begin{aligned} \omega^2 + ak_1 + ab_1 &= ak_1 \cos(\omega\tau) + \omega k_1 \sin(\omega\tau), \\ (a - k_1)\,\omega &= ak_1 \sin(\omega\tau) - \omega k_1 \cos(\omega\tau), \end{aligned} \tag{18}$$

which can be written as

$$\omega^4 + A\omega^2 + B = 0, \tag{19}$$

where

$$\begin{aligned} A &= 2ab_1 + a^2, \\ B &= 2a^2 b_1 (2k_1 + b_1). \end{aligned} \tag{20}$$

As in [9], we consider the following analysis:

(1) If the conditions

$$(H_1) \quad A > 0, \; B > 0 \tag{21}$$

hold, then (19) has no positive roots.

(2) If the condition

$$(H_2) \quad B < 0 \tag{22}$$

holds, then (19) has a unique positive root $\omega_+^2$. Using (19), we have

$$\omega_+ = \left( \frac{-A + \sqrt{A^2 - 4B}}{2} \right)^{1/2}. \tag{23}$$

(3) If the conditions

$$(H_3) \quad A < 0, \; B > 0, \; A^2 - 4B > 0 \tag{24}$$

hold, then (19) has two positive roots $\omega_\pm^2$. Thus, we have

$$\omega_\pm = \left( \frac{-A + \sqrt{A^2 - 4B}}{2} \right)^{1/2}. \tag{25}$$

A stability switch may occur, through the roots $\pm\omega_\pm i$, where $\omega_\pm$ are given by (25). Therefore, from (18), we have

$$\tau_n^\pm = \frac{1}{\omega_\pm} \arccos\left( \frac{k_1\omega_\pm^2 + a(ak_1 + ab_1)}{k_1(a^2 + \omega_\pm^2)} \right) + \frac{2n\pi}{\omega_\pm}, \tag{26}$$

where $n = 1, 2, \ldots$ at which (16) has a pair of purely imaginary roots $\pm\omega_\pm i$.

Consider

$$\lambda(\tau) = \mu(\tau) + i\omega(\tau) \tag{27}$$

the root of (16) so that $\mu(\tau_n^\pm) = 0$. Using (16) and considering $\lambda = \lambda(\tau)$, $(d\lambda(\tau)/d\tau)^{-1}$ is given by

$$\begin{aligned} \left( \frac{d\lambda(\tau)}{d\tau} \right)^{-1} &= \frac{(2\lambda + a - k_1)\, e^{\lambda\tau}}{\lambda(k_1\lambda + ak_1)} + \frac{k_1}{\lambda(k_1\lambda + ak_1)} \\ &\quad - \frac{\tau}{\lambda}. \end{aligned} \tag{28}$$

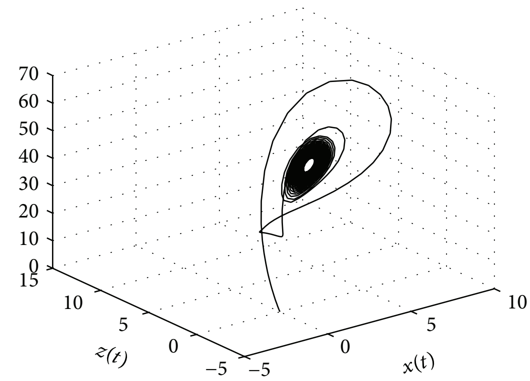


Figure 9: A Hopf bifurcation occurs at the steady state $S_0$ of system (5), when $k_1 = -20.3$, $\tau_1 = 0.415$.

From (18) and (25), we obtain

$$\operatorname{Re}\left( \frac{d\lambda(\tau)}{d\tau} \right)^{-1} \Bigg|_{\tau=\tau_n^\pm} = \frac{1}{k_1^2(\omega_\pm^2 + a^2)} \cdot \left( \pm\sqrt{A^2 - 4B} \right). \tag{29}$$

Thus, if $A^2 - 4B > 0$, we have

$$\begin{aligned} \operatorname{Re}\left( \frac{d\lambda(\tau)}{d\tau} \right)^{-1} \Bigg|_{\tau=\tau_n^+} &= \frac{1}{k_1^2(\omega_+^2 + a^2)} \sqrt{A^2 - 4B} > 0, \\ \operatorname{Re}\left( \frac{d\lambda(\tau)}{d\tau} \right)^{-1} \Bigg|_{\tau=\tau_n^-} &= -\frac{1}{k_1^2(\omega_-^2 + a^2)} \sqrt{A^2 - 4B} < 0. \end{aligned} \tag{30}$$

From the above findings we have the following.

**Theorem 4.** *Let $\tau_n^\pm$ $(n = 0, 1, 2, \ldots)$ be defined by (26) and $\tau_0 = \min\{\tau_0^+, \tau_0^-\}$.*

(i) *If $(H_1)$ holds, (16) and (17) have the same number of roots with positive real part for all $\tau > 0$.*

(ii) *If either $(H_2)$ or $(H_3)$ holds, when $\tau \in [0, \tau_0)$, (16) and (17) have the same number of roots with positive real part. Moreover, if the transversality conditions (30) hold, then a Hopf bifurcation occurs at the steady state $S_0$ in $\tau = \tau_n^\pm$ (see Figure 9).*

From the symmetry of $S_+$ and $S_-$, it is sufficient to analyze the stability of $S_+$. By the linear transformation

$$\begin{aligned} x_1(t) &= x(t) - x_0, \\ y_1(t) &= y(t) - y_0, \\ z_1(t) &= z(t) - z_0, \end{aligned} \tag{31}$$

system (5) becomes

$$\begin{aligned} \dot{x}_1(t) &= a(y_1(t) - x_1(t)), \\ \dot{y}_1(t) &= -b_2 x_0 z_1(t) - b_2 x_1(t) z_1(t) \\ &\quad + k_1(y_1(t) - y_1(t - \tau)), \\ \dot{z}_1(t) &= (d + 2l)\, x_0 x_1(t) + (d + 2k)\, y_0 y_1(t) - c z_1(t) \\ &\quad + d x_1(t) y_1(t) + l x_1(t)^2 + k y_1(t)^2. \end{aligned} \tag{32}$$

The characteristic equation of system (32) in (0, 0, 0) is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + \left(c_2\lambda^2 + c_1\lambda + c_0\right)e^{-\lambda\tau} = 0, \tag{33}$$

where

$$\begin{aligned} a_2 &= a + c - k_1, \\ a_1 &= ac - (a+c)\,k_1 + \frac{b_1 c\,(d+2k)}{d+l+k}, \\ a_0 &= -ack_1 + 2ab_1c, \\ c_2 &= k_1, \\ c_1 &= k_1\,(a+c), \\ c_0 &= k_1 ac. \end{aligned} \tag{34}$$

If there is no delay, (33) becomes

$$\lambda^3 + \left(a_2 + c_2\right)\lambda^2 + \left(a_1 + c_1\right)\lambda + a_0 + c_0 = 0. \tag{35}$$

Let $i\omega$ $(\omega > 0)$ be a root of (33). Then, $\omega$ satisfies the equation

$$\begin{aligned} &\omega^6 + \left(a_2^2 - c_2^2 - 2a_1\right)\omega^4 \\ &\quad + \left(a_1^2 - 2a_0a_2 - c_1^2 + 2c_0c_2\right)\omega^2 + a_0^2 - c_0^2 = 0. \end{aligned} \tag{36}$$

If $z = \omega^2$, then (36) becomes

$$h(z) := z^3 + pz^2 + qz + r = 0, \tag{37}$$

where

$$\begin{aligned} p &= a_2^2 - c_2^2 - 2a_1, \\ q &= a_1^2 - 2a_0a_2 - c_1^2 + 2c_0c_2, \\ r &= a_0^2 - c_0^2. \end{aligned} \tag{38}$$

Therefore, applying the findings from [20] we obtain the following.

**Proposition 5.** *For the polynomial equation (37), we have the following:*

(i) *If $r < 0$, then (37) has at least one positive root.*

(ii) *If $r \geq 0$ and $\Delta = p^2 - 3q \leq 0$, then (37) has no positive roots.*

(iii) *If $r \geq 0$ and $\Delta > 0$, then (37) has positive roots if and only if $z_1^* = (1/2)(-p + \sqrt{\Delta}) > 0$ and $h(z_1^*) \leq 0$.*

Suppose that (37) has positive roots. Without loss of generality, we assume that it has three positive roots, defined by $z_1, z_2$, and $z_3$. Then, (36) has three positive roots $\omega_h = \sqrt{z_h}$, $h = 1, 2, 3$.

By direct computation, we have the following.

**Theorem 6.** *For the simple pairs of conjugate purely imaginary roots of (36), $\pm i\omega_h$, $h = 1, 2, 3$, we have*

$$\begin{aligned} &\tau_h^{(j)} \\ &= \frac{1}{\omega_h}\left[\arccos\frac{c_1\omega_h^2\left(\omega_h^2 - a_1\right) - \left(a_2\omega_h^2 - a_0\right)\left(c_2\omega_h^2 - c_0\right)}{\left(c_2\omega_h^2 - c_0\right)^2 + c_1^2\omega_h^2} + 2j\pi\right]. \end{aligned} \tag{39}$$

*We have $h = 1, 2, 3$, $j = 0, 1, 2, \ldots$, with*

$$\begin{aligned} \tau_0 &:= \tau_{h_0}^{(0)} = \min\left\{\tau_1^{(0)}, \tau_2^{(0)}, \tau_3^{(0)}\right\}, \\ \omega_0 &= \omega_{h_0}. \end{aligned} \tag{40}$$

For (33), using Proposition 5 and [21] to (33), we have the following.

**Proposition 7** (see [10]). *For (33) we have the following:*

(i) *If $r \geq 0$ and $\Delta = p^2 - 3q \leq 0$, then all roots with positive real part of (33) have the same sum to those of the polynomial equation (35) for all $\tau \geq 0$.*

(ii) *If either $r < 0$ or $r \geq 0$, $\Delta = p^2 - 3q > 0$, $z_1^* = (1/3)(-p + \sqrt{\Delta}) > 0$, and $h(z_1^*) \leq 0$, then all roots with positive real parts of (33) have the same sum to those of the polynomial equation (35) for $\tau \in [0, \tau_0)$.*

Let us denote the root of (33) by $\lambda(\tau) = \mu(\tau) + i\omega(\tau)$ with $\mu(\tau_h^{(j)}) = 0$, $\omega(\tau_h^{(j)}) = \omega_h$.

**Proposition 8.** *If $z_h = \omega_h^2$ and $h'(z_h) \neq 0$, then the transversality condition*

$$\operatorname{Re}\left(\frac{d\lambda(\tau)}{d\tau}\right)\bigg|_{\tau=\tau_h^{(j)}} \neq 0 \tag{41}$$

*is satisfied and $\operatorname{Re}(d\lambda(\tau)/d\tau)|_{\tau=\tau_h^{(j)}}$ and $h'(z_h)$ have the same signs.*

Applying Propositions 7 and 8 to (33), we have the following theorems.

**Theorem 9.** *Let $\tau_h^{(j)}$ and $\tau_0$ be defined by (39) and (40). Suppose that condition*

$$a\,(d + 2l) - c\,(d + 2k) > 0 \tag{42}$$

*holds.*

(1) *If $0 < b_1 < b_c$, then we have the following:*

(i) *If $k_1 \leq b_1$ and $\Delta \leq 0$, then (33) has all the roots with negative real part for all $\tau \geq 0$ and the steady state $S_+$ (or $S_-$) of system (5) is locally asymptotically stable.*

(ii) *If $k_1 > b_1$ or $k_1 \leq b_1$ and $\Delta > 0$, $z_1^* > 0$ and $h(z_1^*) \leq 0$, then (33) has all the roots with negative real part for all $\tau \in [0, \tau_0)$ and the steady state $S_+$ (or $S_-$) of system (5) is locally asymptotically stable.*

(iii) *If the conditions of (ii) hold and $h'(z_h) \neq 0$, then a Hopf bifurcation occurs at the steady state $S_+$ (or $S_-$) for $\tau = \tau_h^{(j)}$.*

(2) *If $b_1 > b_c$, then we have the following:*

(i) *If $k_1 \le b_1$ and $\Delta \le 0$, then (33) has two roots with positive real part for all $\tau \ge 0$ and steady state $S_+$ (or $S_-$) of system (5) is unstable.*

(ii) *If $k_1 > b_1$ or $k_1 \le b_1$ and $\Delta > 0$, $z_1^* > 0$ and $h(z_1^*) \le 0$, then (33) has two roots with positive real part for all $\tau \in [0, \tau_0)$ and the steady state $S_+$ (or $S_-$) of system (5) is unstable.*

(iii) *If the conditions of (ii) hold and $h'(z_h^*) \neq 0$, then a Hopf bifurcation occurs at the steady state $S_+$ (or $S_-$) for $\tau = \tau_h^{(j)}$.*

*3.2. Direction and Stability of the Hopf Bifurcation.* In the previous section, for system (5), we have obtained conditions for the Hopf bifurcations to occur for a sequence of values of $\tau$. Using the techniques from normal form theory and center manifold theory introduced by [22], we determine the direction, the stability, and the periodicity of the bifurcating solutions. Let $(x_*, y_*, z_*)$ be the steady state of (5), where system (5) undergoes Hopf bifurcations at $\tau = \tau_h$ and $\pm i\omega_h$ are the corresponding pure imaginary roots of the characteristic equation.

For convenience, let $x_1(t) = x(t) - x_*$, $x_2(t) = y(t) - y_*$, and $x_3(t) = z(t) - z_*$, $\overline{x}_i(t) = x_i(t\tau)$, $\tau = \tau_h + \mu$ and dropping the bars for simplification of notation, system (5) can be written as a FDE (functional differential equation) in $C = C([-1,0], \mathbb{R}^3)$ as follows:

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{43}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T \in \mathbb{R}^3$ and $L_\mu : C \to \mathbb{R}$, $f : \mathbb{R} \times C \to \mathbb{R}$ are given, restrictively, by

$$\begin{aligned} L_\mu(\phi) &= (\tau_h + \mu) \\ &\cdot \begin{pmatrix} -a & a & 0 \\ 0 & k_1 & -b_2 x_* \\ (d+2l) y_* & (d+2k) x_* & -c \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} \\ &+ (\tau_h + \mu) \\ &\cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & -k_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -b_2 \phi_1(0) \phi_3(0) \\ d\phi_1(0)\phi_2(0) + l\phi_1(0)^2 + k\phi_2(0)^2 \end{pmatrix}. \end{aligned} \tag{44}$$

From the previous section, if $\mu = 0$, system (5) undergoes Hopf bifurcations at $(x_*, y_*, z_*)$. By Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$ so that

$$L_\mu(\phi) = \int_{-1}^{0} d\eta(\theta, 0)\, \phi(\theta), \quad \phi \in C[-1, 0]. \tag{45}$$

We can choose

$$\begin{aligned} &\eta(\theta, \mu) \\ &= (\tau_h + \mu) \begin{pmatrix} -a & a & 0 \\ 0 & k_1 & -b_2 x_* \\ (d+2l) y_* & (d+2k) x_* & -c \end{pmatrix} \delta(\theta) \\ &- (\tau_h + \mu) \begin{pmatrix} 0 & 0 & 0 \\ 0 & -k_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(\theta + 1), \end{aligned} \tag{46}$$

where $\delta$ is the Dirac delta function.

For $\phi \in C([-1, 0], (\mathbb{R}^3)^*)$ we define

$$\begin{aligned} A(\mu)\phi &= \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0) \\ \displaystyle\int_{-1}^{0} d\eta(\mu, s)\, \phi(s), & \theta = 0, \end{cases} \\ R(\mu)\phi &= \begin{cases} 0, & \theta \in [-1, 0) \\ f(\mu, \phi(\theta)), & \theta = 0. \end{cases} \end{aligned} \tag{47}$$

Then, system (43) is equivalent to the abstract differential equation:

$$\dot{u}_t = A(\mu) u_t + R(\mu) u_t, \tag{48}$$

where $u_t(\theta) = u(t + \theta)$, for $\theta \in [-1, 0]$.

For $\Psi \in C^1([0, 1], \mathbb{R}^3)$, we define

$$A^* \Psi(s) = \begin{cases} -\dfrac{d\Psi(s)}{ds}, & s \in (0, 1] \\ \displaystyle\int_{-1}^{0} d\eta^T(t, \theta)\, \Psi(-t), & s = 0 \end{cases} \tag{49}$$

and the bilinear form

$$\begin{aligned} &\langle \Psi(s), \phi(\theta) \rangle \\ &= \overline{\Psi}(0)\phi(0) - \int_{\theta=-1}^{0} \int_{\varepsilon=0}^{\theta} \overline{\Psi}(\varepsilon - \theta)\, d\eta(\theta)\, \phi(\varepsilon)\, d\varepsilon, \end{aligned} \tag{50}$$

where $\eta(\theta) = \eta(\theta, 0)$.

Then, $A = A(0)$ and $A^* = A^*(0)$ are adjoining operators. From Section 2, $\pm\omega_h\tau_h$ are eigenvalues of $A$; thus, they are also eigenvalues of $A^*$.

By direct computation, we obtain that

$$q(\theta) = q_0 e^{i\theta\omega_h\tau_h}, \tag{51}$$

with

$$\begin{aligned} q_0 &= (1, \alpha, \beta)^T \\ &= \left(1, \frac{b_2 + i\omega_h}{b_2}, \frac{b_2(x_* + y_*) + i\omega_h x_*}{b_2(c + i\omega_h)}\right) \end{aligned} \tag{52}$$

the eigenvector of $A$ corresponding to $i\omega_h\tau_h$ and

$$q^*(s) = D q_0^* e^{is\omega_h\tau_h} \tag{53}$$

with

$$q_0^* = (1, \alpha^*, \beta^*)^T = \left(1, -\frac{(a-i\omega_h)(c-i\omega_h)}{a(d+2l)(d+2k)x_* y_*}, \frac{a-i\omega_h}{(d+2l)y_*}\right) \quad (54)$$

the eigenvector of $A^*$ corresponding to $-i\omega_h\tau_h$, where

$$D = \frac{1}{1+\overline{\alpha}\alpha^* + \overline{\beta}\beta^* - k_1\tau_h\overline{\alpha}\alpha^* e^{i\omega_h\tau_h}}. \quad (55)$$

Using the same notations as in [22], we compute the coordinates to describe the center manifold $C_0$ at $\mu = 0$.

Let $u_t$ be the solution of (43) when $\mu = 0$ and define

$$z(t) = \langle q^*, u_t\rangle, \quad W(t,\theta) = u_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}. \quad (56)$$

On the center manifold $C_0$ we have

$$W(t,\theta) = W(z(t), \overline{z}(\theta), \theta), \quad (57)$$

where

$$W(z,\overline{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots \quad (58)$$

and $z$ and $\overline{z}$ are local coordinates for center manifold $C_0$ in the direction of $q^*$ and $\overline{q}^*$. Note that $W$ is real if $u_t$ is real. We consider only real solutions. For the solution $u_t \in C_0$ of (43), since $\mu = 0$, we have

$$\begin{aligned}\dot{z}(t) &= \langle q^*, \dot{u}_t\rangle = \langle q^*, A(\mu)u_t + R(\mu)u_t\rangle \\ &= i\tau_h\omega_h z \\ &\quad + \overline{q}^*(0) f(0, W(z,\overline{z},0) + 2\mathrm{Re}\{zq(0)\}) \\ &= i\tau_h\omega_h z + \overline{q}^*(0) f_0(z,\overline{z}).\end{aligned} \quad (59)$$

We rewrite this equation as

$$\dot{z}(t) = i\tau_h\omega_h z(t) + g(z(t), \overline{z}(t)), \quad (60)$$

where

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{01}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \quad (61)$$

Note that

$$u_t(\theta) = (u_{1t}(\theta), u_{2t}(\theta), u_{3t}(\theta)) = W(t,\theta) + zq(\theta) + \overline{z}\overline{q}(\theta) \quad (62)$$

and $q(\theta) = (1-\alpha,\beta)^T e^{i\theta\omega_h\tau_h}$.

We have

$$\begin{aligned}u_{1t}(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots, \\ u_{2t}(0) &= \alpha z + \overline{\alpha}\,\overline{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots, \\ u_{3t}(0) &= \beta z + \overline{\beta}\overline{z} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\overline{z} + W_{02}^{(3)}(0)\frac{\overline{z}^2}{2} + \cdots.\end{aligned} \quad (63)$$

Using (61), we obtain

$$\begin{aligned}g_{20} &= 2\overline{D}\tau_h\left(d\overline{\beta}^*\alpha - b_2\overline{\alpha}^*\beta + \overline{\beta}^* l + \overline{\beta}^* k\alpha^2\right), \\ g_{11} &= \overline{D}\tau_h\left(-b_2\alpha^*(\beta+\overline{\beta}) + \beta^*(d(\alpha+\overline{\alpha}) + 2l + 2k\alpha\overline{\alpha})\right), \\ g_{02} &= \overline{D}\tau_h\left(d\overline{\beta}^*\overline{\alpha} - b_2\overline{\alpha}^*\overline{\beta} + \overline{\beta}^* l + \beta^* k\overline{\alpha}^2\right), \\ g_{21} &= -b_2\overline{D}\tau_h\overline{\alpha}^*\left(2W_{20}^{(3)}(0) + W_{20}^{(3)}(0) + 2\beta W_{11}^{(1)}(0) + \overline{\beta}W_{20}^{(1)}(0)\right) + \overline{D}\tau_h\overline{\beta}^*\left[d\alpha W_{11}^{(2)}(0) + dW_{20}^{(2)}(0) \right. \\ &\quad + 2\alpha dW_{11}^{(1)}(0) + d\overline{\alpha}W_{20}^{(2)}(0) \\ &\quad + l\left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right) \\ &\quad \left. + k\left(2\alpha W_{11}^{(2)}(0) + \overline{\alpha}W_{02}^{(2)}(0)\right)\right].\end{aligned} \quad (64)$$

Since there are $W_{20}(0)$ and $W_{11}(0)$ in $g_{21}$, we need to compute them.

From (43) and (56), we have

$$\begin{aligned}\dot{W} &= \dot{x}_t - \dot{z}q - \dot{\overline{z}}\overline{q} \\ &= \begin{cases} AW - 2\mathrm{Re}\{\overline{q}^*(0) f_0 q(\theta)\}, & \theta \in [-1,0) \\ AW - 2\mathrm{Re}\{\overline{q}^*(0) f_0 q(0)\} + f_0, & \theta = 0 \end{cases} \\ &= AW + H(z,\overline{z},\theta),\end{aligned} \quad (65)$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \quad (66)$$

Expending the above series and comparing the corresponding coefficients, we obtain

$$(A - 2i\tau_h\omega_h)W_{20}(\theta) = -H_{20}(\theta), \quad AW_{11}(\theta) = -H_{11}(\theta). \quad (67)$$

From (65), we know that, for $\theta \in [-1, 0)$,

$$\begin{aligned} H(z, \bar{z}, \theta) &= -\bar{q}^*(0) f_0 q(\theta) - q^*(0) \bar{f}_0 \bar{q}(0) \\ &= -g q(\theta) + \bar{q} q(\theta). \end{aligned} \tag{68}$$

Comparing the coefficients with (66), we obtain

$$\begin{aligned} H_{20}(\theta) &= -g_{20} q(\theta) - \bar{g}_{02} \bar{q}(\theta), \\ H_{11}(\theta) &= -g_{11} q(\theta) - \bar{g}_{11} \bar{q}(\theta). \end{aligned} \tag{69}$$

From (67) and (68) and the definition of $A$, it follows that

$$\dot{W}_{20}(\theta) = 2i\tau_h \omega_h W_{20}(\theta) + g_{20} q(\theta) + \bar{g}_{02} \bar{q}(\theta). \tag{70}$$

Notice that $q(\theta) = (1, \alpha, \beta)^T e^{i\theta\omega_h\tau_h}$ and

$$\begin{aligned} W_{20}(\theta) = {} & \frac{i g_{20}}{\omega_h \tau_h} q(0) e^{i\theta\omega_h\tau_h} + \frac{i\bar{g}_{02}}{3\omega_h\tau_h} \bar{q}(0) \bar{e}^{i\theta\omega_h\tau_h} \\ & + E_1 e^{2i\theta\omega_h\tau_h}, \end{aligned} \tag{71}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T \in \mathbb{R}^3$ is a constant vector.

In a similar way, we can obtain

$$W_{11}(\theta) = -\frac{i g_{11}}{\omega_h \tau_h} q(0) e^{i\theta\omega_h\tau_h} + \frac{i\bar{g}_{11}}{\omega_h\tau_h} \bar{q}(0) \bar{e}^{i\theta\omega_h\tau_h} + E_2, \tag{72}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in \mathbb{R}^3$ is a constant vector.

From the definition of $A$ and (67), we obtain

$$\int_{-1}^{0} d\eta(\theta) W_{20}(\theta) = 2i\omega_h\tau_h W_{20}(0) - H_{20}(0), \tag{73}$$

$$\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0), \tag{74}$$

where $\eta(\theta) = \eta(\theta, 0)$.

By (65), we have

$$H_{20}(0) = -g_{20} q(0) - \bar{g}_{02} \bar{q}(0) + 2\tau_h \begin{pmatrix} 0 \\ -\beta \\ \alpha \end{pmatrix}, \tag{75}$$

$$H_{11}(0) = -g_{11} q(0) - \bar{g}_{11} \bar{q}(0) + 2\tau_h \begin{pmatrix} 0 \\ -\mathrm{Re}(\beta) \\ \mathrm{Re}(\alpha) \end{pmatrix}. \tag{76}$$

Substituting (71) and (75) into (73), we obtain

$$\left(2i\omega_h\tau_h I - \int_{-1}^{0} e^{2i\theta\omega_h\tau_h} d\eta(\theta)\right) E_1 = 2\tau_h \begin{pmatrix} 0 \\ -\beta \\ \alpha \end{pmatrix} \tag{77}$$

which leads to

$$\begin{aligned} & \begin{pmatrix} 2i\omega_h + a & -a & 0 \\ 0 & 2i\omega_h - k_1 + k_1 e^{-2\omega_h\tau_h} & b_2 x_* \\ -(d+2l) y_* & -(d+2k) x_* & 2i\omega_h + c \end{pmatrix} E_1 \\ & = 2 \begin{pmatrix} 0 \\ -\beta \\ \alpha \end{pmatrix}, \\ & E_1^{(1)} = \frac{-2a(c\beta + 2i\omega_h\beta + b_2\alpha x_*)}{\Delta_1}, \\ & E_1^{(2)} = -\frac{\beta(2i\omega_h + c) + \alpha b_2 x_*}{\Delta_1}, \\ & E_1^{(3)} \\ & = \frac{\alpha(2i\omega_h + a)(2i\omega_h - k_1 + k_1 e^{-2\omega_h\tau_h})}{\Delta_1} \\ & \quad - \frac{\beta a(d+2l) y_* - \beta(d+2k) x_* (2i\omega_h + a)}{\Delta_1}, \\ & \Delta_1 \\ & = \det \begin{pmatrix} 2i\omega_h + a & -a & 0 \\ 0 & 2i\omega_h - k_1 + k_1 e^{-2\omega_h\tau_h} & b_2 x_* \\ -(d+2l) y_* & -(d+2k) x_* & 2i\omega_h + c \end{pmatrix}. \end{aligned} \tag{78}$$

In a similar way, substituting (72) and (76) into (74), we obtain

$$\begin{aligned} & \begin{pmatrix} a & -a & 0 \\ 0 & 0 & b_2 x_* \\ -(d+2l) y_* & -(d+2k) x_* & c \end{pmatrix} E_2 \\ & \quad = 2 \begin{pmatrix} 0 \\ -\mathrm{Re}(\beta) \\ \mathrm{Re}(\alpha) \end{pmatrix}, \\ & E_2^{(1)} = E_1^{(2)} = \frac{-2ab_2 x_* \mathrm{Re}(\alpha) - 2ac\mathrm{Re}(\beta)}{\Delta_2}, \\ & E_2^{(3)} = \frac{-2a\mathrm{Re}(\beta)((d+2k) x_* + (d+2l) y_*)}{\Delta_2}, \\ & \Delta_2 = ab_2 x_* ((d+2k) x_* + (d+2l) y_*). \end{aligned} \tag{79}$$

Thus, we can determine $W_{20}(0)$ and $W_{11}(0)$ from (71) and (72). Furthermore, we can determine $g_{21}$. Therefore, each $g_{ij}$

in (64) is determined by the parameters and delay in (43). Thus, we can compute the following values:

$$\begin{aligned} C_1(0) &= \frac{i}{2\tau_h\omega_h}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}(C_1(0))}{\mathrm{Re}(\lambda'(\tau_h))}, \\ \beta_2 &= 2\mathrm{Re}(C_1(0)), \\ T_2 &= -\frac{\mathrm{Im}(C_1(0)) + \mu_2 \mathrm{Im}(\lambda'(\tau_h))}{\tau_h\omega_h}. \end{aligned} \tag{80}$$

The above quantities characterize the bifurcating periodic solutions in the center manifold at the critical value $\tau_h$ [22, 23]:

(i) $\mu = 0$ is the Hopf bifurcation of system (43).

(ii) $\mu_2$ determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exists for $\tau > \tau_h$ ($\tau < \tau_h$).

(iii) $\beta_2$ determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$).

(iv) $T_2$ determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

## 4. Analysis of System (6)

Let $q_1 = q_2 = q_3 = q \in (0, 1)$. In this case, the fractional order system is commesurate-order [24].

**Proposition 10.** *The initial value problem of the commensurate order system (6) can be rewritten as follows:*

$$\begin{aligned} D^q X(t) &= AX(t) + x(t)BX(t) + y(t)CX(t), \\ X(0) &= (x_0, y_0, z_0)^T, \end{aligned} \tag{81}$$

*where* $0 < t \le T$, $X(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$,

$$A = \begin{pmatrix} -a & a & 0 \\ b_1 & 0 & 0 \\ 0 & 0 & -c \end{pmatrix},$$
$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -b_2 \\ l & d & 0 \end{pmatrix}, \tag{82}$$
$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k & 0 \end{pmatrix}.$$

*Then, for constant* $T = \min\{T^*, (\epsilon\Gamma(q+1)/\|f\|_\infty)^{1/q}\}$, $T^* > 0$, *fractional order system (6) has a unique solution, where* $\Gamma(\cdot)$ *is the Gamma function.*

*Proof.* Consider the function $f(X(t)) = AX(t) + x(t)BX(t) + y(t)CX(t)$, which is continuous and bounded on the interval $X \in [0, T^*] \times [X_0 - \epsilon, X_0 + \epsilon]$, for any $T^*$, $\epsilon > 0$. For $X(t) = (x(t), y(t), z(t))^T$ and $Y(t) = (x_1(t), y_1(t), z_1(t))^T$, where $Y(t) \in [0, T^*] \times [X_0 - \epsilon, X_0 + \epsilon]$, we can obtain

$$|f(X(t)) - f(Y(t))| \le L(|X(t) - Y(t)|), \tag{83}$$

where $L = \|A\| + (\|B\| + \|C\|)(2|X_0| + 2\epsilon) > 0$.

Therefore, the fractional-order general $T$ system meets the Lipschitz condition. Then, according to the existence and uniqueness theorem of the fractional-order system [25], the initial value problem of the commensurate order system (81) has a unique solution in the interval $T = \min\{T^*, (\epsilon\Gamma(q+1)/\|f\|_\infty)^{1/q}\}$.

System (6) is dissipative for $a + c > 0$, because

$$\nabla V(x, y, z) = -a - c, \tag{84}$$

where $V(x, y, z) = (1/2)(x^2+y^2+z^2)$. Meanwhile, it is convergent in an exponential rate

$$\frac{dV(X(t))}{dt} = e^{-(a+c)}. \tag{85}$$

That is to say, an initial volume $V_0$ is the volume element at time $t$ contraction of volume element $V_0 e^{-(a+c)t}$. When $t \to \infty$ all the trajectory of systems will eventually be restricted in a volume element for zero point sets, and incremental dynamic behavior of it will be fixed in an attractor. □

In order to determine the stability conditions for the steady states $S_-$ and $S_+$, we first consider the integer-order case. Based on Proposition 2, $S_-$ and $S_+$ are locally asymptotically stable if and only if $0 < b_1 < b_c$, where

$$b_c = \frac{a(a+c)(d+l+k)}{a(d+2l) - c(d+2k)} \tag{86}$$

and $a(d+2l) - c(d+2k) > 0$. It is known that the fractional-order system is at least as stable as their integer-order counterpart, so we have the following conclusion.

**Proposition 11.** *The steady states* $S_-$ *and* $S_+$ *are stable with* $q \in (0, 1)$ *for* $b_1 < b_c$.

When $b_1 > b_c$, the steady states $S_-$ and $S_+$ become unstable in an integer-order system but may be stable in the fractional-order case.

## 5. Chaos Control of System (6)

In this section, we want to control the chaos for the fractional-order general $T$ system (6) to the steady state denoted by $S$ via feedback control.

An $n$-dimensional fractional-order system can be described as [13]

$$D^q X(t) = f(X(t)), \tag{87}$$

where $X(t) \in \mathbb{R}^n$, $q \in (0,1)$.

The system with controller is given by [13]

$$D^q X(t) = f(X(t)) - K(X(t) - S), \tag{88}$$

where $K = \operatorname{diag}(k_1, k_2, \ldots, k_n)$ is the matrix of positive feedback gains and $S$ is the steady state of system (87).

The controlled fractional-order general $T$ system (6) is

$$\begin{aligned} D^{q_1} x(t) &= a(y(t) - x(t)) - u_1(t), \\ D^{q_2} y(t) &= b_1 x(t) - b_2 x(t) z(t) - u_2(t), \\ D^{q_3} z(t) &= -cz(t) + dx(t) y(t) + lx(t)^2 + ky(t)^2 \\ &\quad - u_3(t), \end{aligned} \tag{89}$$

where $u_i(t)$, $i = 1, 2, 3$, are the external control inputs. The control law of single state variables feedback has the following form [13]:

$$\begin{aligned} u_1(t) &= k_1(x(t) - x_{01}), \\ u_2(t) &= k_2(y(t) - y_{02}), \\ u_3(t) &= k_3(z(t) - z_{03}), \end{aligned} \tag{90}$$

where $K = \operatorname{diag}(k_1, k_2, k_3)$ is the matrix of positive feedback gains and $S = (x_{01}, y_{02}, z_{03})^T$ is the steady state of system (6).

The characteristic equation of the controlled system (89) evaluated at the steady state $S$ is

$$(\lambda + a + k_1)(\lambda + k_2)(\lambda + c + k_3) = 0. \tag{91}$$

The fractional-order Routh-Hurwitz conditions lead to

$$\begin{aligned} a + k_1 &> 0, \\ k_2 &> 0, \\ c + k_3 &> 0. \end{aligned} \tag{92}$$

## 6. Synchronization of System (6)

The nonlinear control method is used to design control in order to make the drive system (6) and response system state synchronization [14]. Two identical systems are introduced, one is the drive system and the other system added a nonlinear control to be the response system. Corresponding to (6), the drive system is

$$\begin{aligned} D^{q_1} x_1(t) &= a(y_1(t) - x_1(t)), \\ D^{q_2} y_1(t) &= b_1 x_1(t) - b_2 x_1(t) z_1(t), \\ D^{q_3} z_1(t) &= -cz_1(t) + dx_1(t) y_1(t) + lx_1(t)^2 \\ &\quad + ky_1(t)^2, \end{aligned} \tag{93}$$

and the response system is

$$\begin{aligned} D^{q_1} x_2(t) &= a(y_2(t) - x_2(t)) + v_1(t), \\ D^{q_2} y_2(t) &= b_1 x_2(t) - b_2 x_2(t) z_2(t) + v_2(t), \\ D^{q_3} z_2(t) &= -cz_2(t) + dx_2(t) y_2(t) + lx_2(t)^2 \\ &\quad + ky_2(t)^2 + v_3(t), \end{aligned} \tag{94}$$

where $V(t) = (v_1(t), v_2(t), v_3(t))^T$ is the nonlinear synchronization controller. The drive system and the response system achieve synchronization under the driver of $V(t)$. From (93) and (94), the error system is obtained:

$$\begin{aligned} D^{q_1} e_1(t) &= a(e_2(t) - e_1(t)) + v_1(t), \\ D^{q_2} e_2(t) &= b_1 e_1(t) - b_2 x_2(t) e_3 - b_2 x_1(t) e_1 + v_2(t), \\ D^{q_3} e_3(t) &= -ce_3(t) + dx_2(t) e_2(t) + dy_1(t) e_1(t) \\ &\quad + l(x_1(t) + x_2(t)) e_1(t) \\ &\quad + k(y_1(t) + y_2(t)) e_2(t) + v_3(t), \end{aligned} \tag{95}$$

where $e_1(t) = x_2(t) - x_1(t)$, $e_2(t) = y_2(t) - y_1(t)$, $e_3 = z_2(t) - z_1(t)$. We have to find the proper control function $v_i(t)$, $i = 1, 2, 3$, so that the response system (94) globally synchronizes with drive system (93); that is, $\lim_{t \to +\infty} \|e(t)\| = 0$, where $e(t) = (e_1(t), e_2(t), e_3(t))^T$. According to [14], we propose the following control law for system (94):

$$\begin{aligned} v_1(t) &= ae_1(t) - k_1 e_1(t), \\ v_2(t) &= -b_1 e_1(t) - ae_1(t) + b_2 z_1(t) e_1(t) \\ &\quad + b_2 x_2(t) e_3(t) - k_2 e_2(t), \\ v_3(t) &= ce_3(t) - dy_1(t) e_1(t) \\ &\quad - le_1(t)(x_1(t) + x_2(t)) - dx_2(t) e_2(t) \\ &\quad - k(y_2(t) + y_1(t)) e_2(t), \end{aligned} \tag{96}$$

where $k_1$, $k_2$, and $k_3$ are the control parameters.

**Proposition 12** (see [14]). *For any initial conditions, if $k_i > 0$, $i = 1, 2, 3$ then the drive system and response system will synchronize.*

## 7. Numerical Simulations

Now, we illustrate the findings from the previous sections. We have proved that at some critical values of the delay, a family of periodic solutions bifurcate from the steady states of system (5) and the stability of the steady states may be changed.

The numerical simulations indicate that when the delay passes through certain critical values, chaotic oscillation is converted into a stable steady state or a periodic orbit.

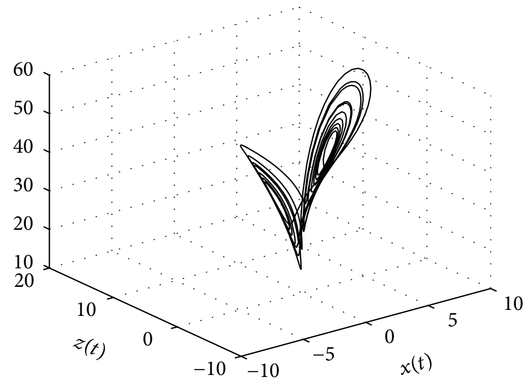


Figure 10: For system (97), there is chaos when $k_1 = -3$, $\tau = 0.01$.

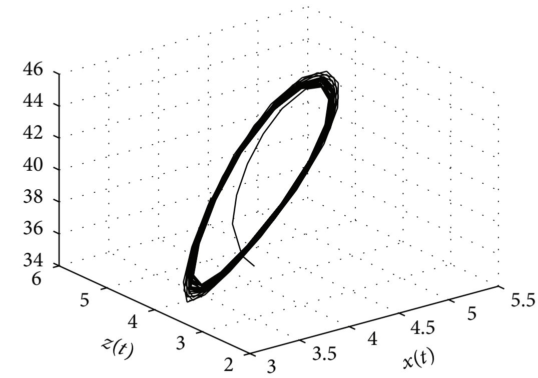


Figure 11: For system (97), there is chaos when $k_1 = -3$, $\tau = 0.051$.

We consider the delayed feedback control system (5) in the following particular form:

$$\begin{aligned} \dot{x}(t) &= 10\left(y(t) - x(t)\right), \\ \dot{y}(t) &= 40x(t) - x(t)z(t) + k_1\left(y(t) - y(t-\tau)\right), \\ \dot{z}(t) &= -2.5z(t) + 3x(t)y(t) + 0.9x(t)^2 + 2y(t)^2, \end{aligned} \tag{97}$$

which has three steady states $S_0 = (0, 0, 0)$, $S_- = (-4.11, -4.11, 40)$, and $S_+ = (4.11, 4.11, 40)$. When $\tau = 0$ and $k_1 = 0$, system (97) is chaotic.

The characteristic equation of system (97) is given by

$$\begin{aligned} \lambda^3 &+ (12.5 - k_1)\lambda^2 + (143.644 - 12.5k_1)\lambda + 2000 \\ &- 10k_1 + k_1\left(\lambda^2 + 12.5\lambda + 25\right)e^{-\lambda\tau} = 0. \end{aligned} \tag{98}$$

If there is no delay, (98) has a negative root and a pair of complex roots with positive real part.

Using Theorem 9, we have the following:

(i) For $5.0413 < k_1 < 28.403$, $r = 10^5(40 - k_1) > 0$, $\Delta = 1.052 \cdot 10^5 - 3101.694k_1 < 0$, (98) has two roots with positive real part for all $\tau \geq 0$, and the steady state $S_+$ (or $S_-$) of system (97) is unstable.

(ii) For $k_1 < 5.0413$ chaos can occur.

In what follows, we consider $k_1 = -3$. In this case: $\omega_1 = 11.58361053$, $\omega_2 = 13.32108470$,

$$\begin{aligned} \tau_1^j &= 0.473158 + \frac{2j}{\omega_1}, \\ \tau_2^j &= 0.51716 + \frac{2j}{\omega_2}, \\ j &= 0, 1, 2, \ldots. \end{aligned} \tag{99}$$

Then, $\tau_1^0 < \tau_2^0 < \tau_2^1 < \tau_1^1$.

(i) For $\tau \in [0, \tau_2^0) \cup (\tau_2^1, \infty)$, the steady states $S_+$ and $S_-$ of system (97) are unstable.

(ii) For $\tau \in [\tau_2^0, \tau_2^1)$, the steady states $S_+$ and $S_-$ of system (97) are locally asymptotically stable.

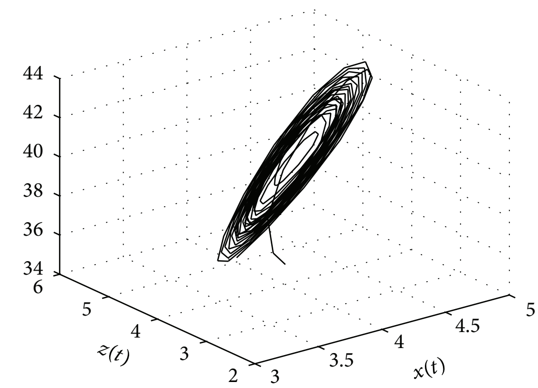


Figure 12: For system (97), an unstable periodic solution bifurcates from the steady state $S_+$ when $k_1 = -3$, $\tau = 1.5$.

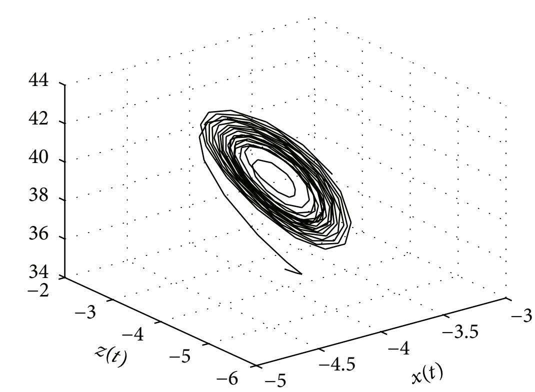


Figure 13: For system (97), an unstable periodic solution bifurcates from the steady state $S_-$ when $k_1 = -3$, $\tau = 1.5$.

(iii) For $\tau = \tau_h^j$, for $h = 1, 2$, $j = 0, 1, 2, \ldots$, system (97) undergos a Hopf bifurcation at the steady states $S_+$ and $S_-$.

For values of $\tau$, which satisfy the above conditions, we obtain the dynamical behaviors in Figures 10, 11, 12, and 13.

For the numerical simulation of the fractional differential equations (FDE) (6), with $a = 10$, $b_2 = 1$, $c = 2.5$, $d = 1$, $l = 2$, $k = 2$, and $q_1 = q_2 = q_3 = 0.99$, we use the method from [26, 27]. We obtain Figures 14, 15, and 16.

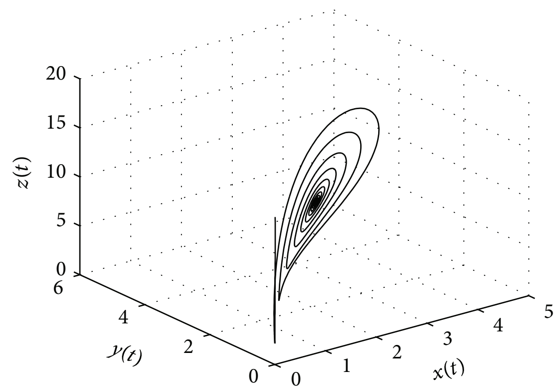


Figure 14: Asymptotic stable solution for the fractional general $T$ system (6) when $b_1 = 10$.

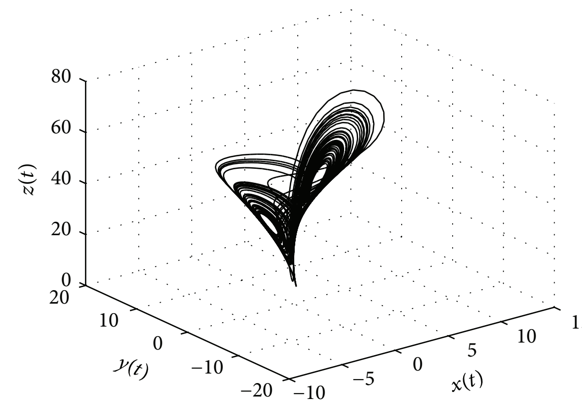


Figure 16: Chaos occurs for the fractional general $T$ system (6) when $b_1 = 40$.

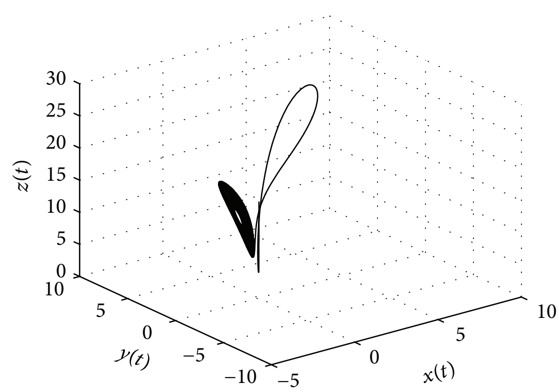


Figure 15: Periodic solution for the fractional general $T$ system (6) when $b_1 = 16.6$.

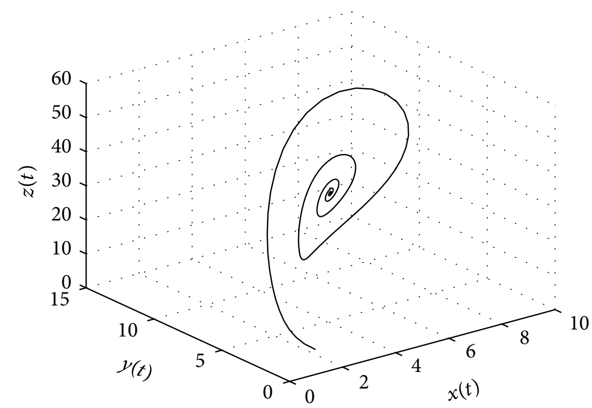


Figure 17: Asymptotic stable solution for the controlled fractional general $T$ system (89), when $b_1 = 10$.

For the controlled fractional order general $T$-system (89), using $K = \text{diag}(5, 5, 1)$ the matrix of positive feedback gains, we obtain Figure 17.

## 8. Conclusion

In the present paper we introduce a generalized $T$-system where the time delay is present. The linear stability is analyzed by using the Routh Hurwitz criterion. The existence of the Hopf bifurcation is studied. Then, the direction and the stability of the bifurcating periodic solutions are determined by using the normal form theory and the center manifold theorem. Chaotic behavior is also taken into account. The numerical simulations show that when the delay passes through certain critical values, chaotic oscillation is converted into a stable steady state or a periodic orbit.

Furthermore, the fractional-order general $T$ system has been proposed. The dynamics, chaos control as well as synchronization have been investigated.

The present study will be continued for the system which describes the financial risk [28]. Also, as in [29], the fractional-order chaotic complex system will be taken into consideration.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.


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## References


[1] E. N. Lorenz, "Deterministic non-periodic fows," *Journal of the Atmospheric Sciences*, vol. 20, pp. 130–141, 1963.

[2] G. Tigan and D. Opris, "Analysis of a 3D chaotic system," *Chaos, Solitons and Fractals*, vol. 36, no. 5, pp. 1315–1319, 2008.

[3] A. Algaba, F. Fernandez-Sanchez, and M. Merino, "Existence of heteroclinic and homoclinic orbits in two different chaotic dynamical systems," *Applied Mathematics and Computation*, vol. 218, pp. 11859–11870, 2012.

[4] A. Algaba, F. Fernández-Sánchez, M. Merino, and A. J. Rodríguez-Luis, "Chen's attractor exists if Lorenz repulsor

exists: the Chen system is a special case of the Lorenz system," *Chaos*, vol. 23, no. 3, Article ID 033108, 2013.

[5] B. Jiang, X. Han, and Q. Bi, "Hopf bifurcation analysis in the *T* system," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 1, pp. 522–527, 2010.

[6] G. Tigan and D. Constantinescu, "Heteroclinic orbits in the *T* and the Lü systems," *Chaos, Solitons & Fractals*, vol. 42, no. 1, pp. 20–23, 2009.

[7] R. A. van Gorder and S. R. Choudhury, "Classification of chaotic regimes in the T system by use of competitive modes," *International Journal of Bifurcation and Chaos*, vol. 20, no. 11, pp. 3785–3793, 2010.

[8] R. A. van Gorder and S. R. Choudhury, "Analytical Hopf bifurcation and stability analysis of T system," *Communications in Theoretical Physics*, vol. 55, no. 4, pp. 609–616, 2011.

[9] Y. Chen and Z.-Y. Yan, "Chaos control in a new three-dimensional chaotic T system," *Communications in Theoretical Physics*, vol. 49, no. 4, pp. 951–954, 2008.

[10] R. Zhang, "Bifurcation analysis for T system with delayed feedback and its application to control of chaos," *Nonlinear Dynamics*, vol. 72, no. 3, pp. 629–641, 2013.

[11] S. Bhalekar, "Chaos control and synchronization in fractional-order lorenz-like system," *International Journal of Differential Equations*, vol. 2012, Article ID 623234, 16 pages, 2012.

[12] S. Bhalekar and V. Daftardar-Gejji, "Fractional ordered Liu system with time-delay," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 8, pp. 2178–2191, 2010.

[13] X. Liu, L. Hong, and L. Yang, "Fractional-order complex T system: bifurcations, chaos control, and synchronization," *Nonlinear Dynamics*, vol. 75, no. 3, pp. 589–602, 2014.

[14] S. Ma, J. Zheng, and Y. Li, "Chaos control and synchronization of a new fractional order chaotic system," *Journal of Information & Computational Science*, vol. 11, no. 10, pp. 3469–3479, 2014.

[15] X.-F. Li, Y.-D. Chu, J.-G. Zhang, and Y.-X. Chang, "Nonlinear dynamics and circuit implementation for a new Lorenz-like attractor," *Chaos, Solitons & Fractals*, vol. 41, no. 5, pp. 2360–2370, 2009.

[16] G. Yang, "Hopf birurcation of Lorenz-like system about parameter h," *Modern Applied Science*, vol. 4, no. 1, pp. 91–95, 2010.

[17] I. Pehlivan and Y. Uyaroğlu, "A new chaotic attractor from general Lorenz system family and its electronic experimental implementation," *Turkish Journal of Electrical Engineering & Computer Sciences*, vol. 18, no. 2, pp. 171–184, 2010.

[18] M. Xiao and J. Cao, "Bifurcation analysis and chaos control for Lü system with delayed feedback," *International Journal of Bifurcation and Chaos*, vol. 17, no. 12, pp. 4309–4322, 2007.

[19] K. Pyragas, "Continuous control of chaos by self-controlling feedback," *Physics Letters A*, vol. 170, no. 6, pp. 421–428, 1992.

[20] Y. Song and J. Wei, "Bifurcation analysis for Chen's system with delayed feedback and its application to control of chaos," *Chaos, Solitons & Fractals*, vol. 22, no. 1, pp. 75–91, 2004.

[21] S. Ruan and J. Wei, "On the zeros of transcendental functions with applications to stability of delay differential equations with two delays," *Dynamics of Continuous, Discrete and Impulsive Systems*, vol. 10, pp. 863–874, 2003.

[22] B. D. Hassard, N. D. Kazarinoff, and Y. Wan, *Theory and Application of Hopf Bifurcation*, Cambridge University Press, Cambridge, UK, 1981.

[23] S. Wiggins, *Introduction to Applied Nonlinear Dynamical, Systems and Chaos*, Springer, New York, NY, USA, 2003.

[24] M. S. Tavazoei and M. Haeri, "Limitations of frequency domain approximation for detecting chaos in fractional order systems," *Nonlinear Analysis: Theory, Methods& Applications*, vol. 69, no. 4, pp. 1299–1320, 2008.

[25] K. Diethelm and N. J. Ford, "Analysis of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 265, no. 2, pp. 229–248, 2002.

[26] K. Diethelm, N. J. Ford, and A. D. Freed, "A predictor-corrector approach for the numerical solution of fractional differential equations," *Nonlinear Dynamics*, vol. 29, no. 1–4, pp. 3–22, 2002.

[27] I. Petras, *Fractional Order Nonlinear Systems: Modeling, Analysis and Simulation*, Nonlinear Physical Science, Springer, New York, NY, USA, 2011.

[28] Z. Xiao-Dan, L. Xiang-Dong, Z. Yuan, and L. Cheng, "Chaotic dynamic behavior analysis and control for a financial risk system," *Chinese Physics B*, vol. 22, no. 3, Article ID 030509, 2013.

[29] C. Jiang, S. Liu, and C. Luo, "A new fractional-order chaotic complex system and its antisynchronization," *Abstract and Applied Analysis*, vol. 2014, Article ID 326354, 12 pages, 2014.