Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 428042, 6 pages
http://dx.doi.org/10.1155/2015/428042

Research Article

Pontryagin's Maximum Principle for the Optimal Control Problems with Multipoint Boundary Conditions

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Received 5 August 2014; Revised 29 September 2014; Accepted 30 September 2014

Academic Editor: Allaberen Ashyralyev

Optimal control problem with multipoint boundary conditions is considered. Sufficient conditions for the existence and uniqueness of the solution of boundary value problem for every fixed admissible control are obtained. First order increment formula for the functional is derived. Pontryagin's maximum principle is proved by using the variations of admissible control.

1. Introduction

Boundary value problems appear in a large field of sciences to describe physical, biological, and chemical phenomena and several practically important problems lead to multipoint boundary value problems. Some examples are given in the area of elasticity and on the effects of soil settlement [1–5]. For boundary value problems with multipoint boundary conditions and comments on their importance, we refer the reader to the papers [6–11] and the references therein.

Pontryagin's maximum principle is the first order necessary optimality condition and occupies a special place in theory of optimal processes. Originally the maximum principle was proved for the Cauchy system of ordinary differential equations [12]. Later on this result was carried over the most complex objects described by the equations with a delay, integral equations, partial equations, stochastic equations, and so forth (see, e.g., [13, 14] and the references therein).

At present, there exists a great amount of work devoted to derivation of necessary optimality conditions of first and second orders for the systems with local conditions (see [12, 14–19] and the references therein).

Recently, the optimal control problems with nonlocal conditions are intensively investigated. In the papers [13, 20–24] the necessary optimality conditions for optimal control problems described by the systems of ordinary differential equations with nonlocal conditions were obtained. In these papers the nonlocal conditions contain two-point and integral boundary conditions.

It is known that the solution of problems of mechanics and control processes is reduced to multipoint boundary value problems. The constructive sufficient existence and uniqueness conditions and also the methods of numerical solution of such boundary value problems were studied in [6–9].

In the present paper, Pontryagin's maximum principle for optimal control problems for the ordinary differential equations with multipoint boundary conditions is proved. Since in optimal control problems with multipoint boundary conditions the solution of the associated system has discontinuities of the first kind of inner points, the direct applications of the solution methods of two-point boundary value problems to optimal control problems with multipoint boundary conditions are impossible.

The paper is organized as follows. First, we give the statement of the problem. Second, theorems on existence and uniqueness of the solution of problem (1)–(3) are established under some sufficient conditions on the nonlinear terms.

Third, the first order increment formula for the functional is presented and Pontryagin's maximum principle is provided.

2. Problem Statement

Let the controlled process on a fixed time interval $[0,T]$ be described by a system of differential equations

$$\frac{dx}{dt} = f(t,x,u) \tag{1}$$

with multipoint boundary conditions

$$\sum_{j=0}^{N} B_j x(t_j) = C, \tag{2}$$

where $x(t) \in R^n$; $f(t,x,u)$ is the given u dimensional vector-function; $C \in R^n$ is the given constant vector; $0 = t_0 < t_1 < \cdots < t_N = T$ are fixed points; $u(t)$ is the r dimensional and bounded vector of control actions with the values from the nonempty, bounded set U; that is,

$$u(t) \in U \subset R^r, \quad t \in [0,T]. \tag{3}$$

It is required to minimize the functional

$$J(u) = \varphi(x(0), x(T)) + \int_0^T F(t,x,u)\,dt \tag{4}$$

subject to (1)–(3).

Here we assumed that the functions $f(t,x,u)$, $F(t,x,u)$, and $\varphi(x,y)$ are continuous over the set of arguments and have bounded partial derivatives with respect to the arguments x and y. Under the solution of problem (1)–(3) that corresponds to the fixed admissible control $u(t)$ we take the function $x(t) : [0,T] \to R^n$ absolutely continuous on the interval $[0,T]$.

Denote by $C([0,T],R^n)$ a space of continuous functions on the interval $[0,T]$ with the values from R^n. Obviously, such a space is Banach with the norm

$$\|x\|_{C[0,T]} = \max_{[0,T]} |x(t)|, \tag{5}$$

where $|\cdot|$ is the norm R^n.

The admissible process $\{u(t), x(t,u)\}$, being the solution of problem (1)–(4), that is, delivering minimum to the functional (4) under restrictions (1)–(3), will be called an optimal process and $u(t)$ an optimal control.

3. Existence of Solutions of Boundary Value Problem (1)–(3)

Introduce the following conditions.

(À1) Let $\det B \neq 0$, where $B = \sum_{i=0}^{N} B_i$.

(À2) The function $f : [0,T] \times R^n \times R^n \to R^n$ is continuous and there exists a constant $K \geq 0$ such that

$$\begin{gathered} |f(t,x,u) - f(t,y,u)| \leq K|x-y|, \\ t \in [0,T], \quad x,y \in R^n, \quad u \in U. \end{gathered} \tag{6}$$

(À3) $L = KTM < 1$, where $M = \max_{0\leq t,s\leq T} \|M(t,s)\|$.

$M(t,s)$ is a piecewise matrix such that $t_{k-1} \leq s < t_k$, $(k = 1,2,\ldots,N)$:

$$M(t,s) = \begin{cases} B^{-1}\sum_{i=0}^{k-1} B_i, & \text{if } s < t, \\ -B^{-1}\sum_{i=k}^{N} B_i, & \text{if } t \leq s. \end{cases} \tag{7}$$

Theorem 1. *Let condition (À1) be fulfilled. The function $x(\cdot) \in C([0,T],R^n)$ is an absolute continuous solution of problem (1)–(3) if and only if*

$$x(t) = B^{-1}C + \int_0^T M(t,s) f(s,x(s),u(s))\,ds, \tag{8}$$

where the matrix function $M(t,s)$ is determined by equality (7).

Proof. Let the function $x = x(t)$ be a solution of (1). Then for $t \in (0,T)$ the formula is valid:

$$x(t) = x(0) + \int_0^t f(s,x(s),u(s))\,ds, \tag{9}$$

where $x(0)$ is an arbitrary constant vector. In order to determine $x(0)$ the required function defined by equality (9) satisfies condition (2):

$$\sum_{i=0}^{N} B_i x(0) = C - \sum_{i=0}^{N} B_i \int_0^{t_i} f(t,x(t),u(t))\,ds. \tag{10}$$

Since, according to condition (À1), $\det B \neq 0$, then it follows from equality (10) that

$$x(0) = B^{-1}C - B^{-1}\sum_{i=0}^{N} B_i \int_0^{t_i} f(s,x(s),u(s))\,ds, \tag{11}$$

which may be rewritten in the form

$$x(0) = B^{-1}C - B^{-1}\sum_{k=1}^{N}\left[\sum_{i=k}^{N} B_i \int_{t_{k-1}}^{t_k} f(s,x(s),u(s))\,ds\right]. \tag{12}$$

Now taking into account the value of $x(0)$ determined by equality (12) in (9) we get

$$\begin{aligned} x(t) = {} & B^{-1}C - B^{-1}\sum_{k=1}^{N}\left[\sum_{i=k}^{N} B_i \int_{t_{k-1}}^{t_k} f(s,x(s),u(s))\,ds\right] \\ & + \int_0^t f(s,x(s),u(s))\,ds. \end{aligned} \tag{13}$$

Obviously, for $t_{k-1} \leq t < t_k$, we can write equality (13) in equivalent form:

$$\begin{aligned} x(t) &= B^{-1}C + \int_0^t \left[E - B^{-1}\sum_{i=k}^{N} B_i\right] f(s,x(s),u(s))\,ds \\ &= \int_t^{t_k} \left(B^{-1}\sum_{i=k}^{N} B_i\right) f(s,x(s),u(s))\,ds. \end{aligned} \tag{14}$$

So

$$E - B^{-1}\sum_{i=k}^{N} B_i = B^{-1}\left(\sum_{i=0}^{N} B_i - \sum_{i=k}^{N} B_i\right) = B^{-1}\sum_{i=0}^{k-1} B_i \tag{15}$$

holds; then by using (7) we can rewrite equality (14) in the following equivalent form:

$$x(t) = B^{-1}C + \int_0^T M(t,s) f(s, x(s), u(s))\, ds. \tag{16}$$

Thus, it is shown that boundary value problem (1)–(3) may be rewritten in equivalent integral form (8). By direct calculation we can show that the solution of integral equation (8) is a solution of boundary value problem (1)–(3). □

Theorem 2. *Let conditions (À1)–(À3) be fulfilled. Then for any $C \in R^n$ and any fixed admissible control, boundary value problem (1)–(3) has the unique solution satisfying the integral equation (8).*

Proof. Let $C \in R^n$ and $u(\cdot) \in U$ be fixed. Let us consider the mapping $P : C([0,T], R^n) \to C([0,T], R^n)$ determined according to the rule:

$$(Px)(t) = B^{-1}C + \int_0^T M(t,s) f(s, x(s), u(s))\, ds. \tag{17}$$

Obviously, the fixed points of the operator P are a solution of boundary value problem (1)-(2). Using the Banach method of contractive operators we show that the operator P determined by equality (17) has a fixed point. For any $v, \omega \in C([0,T], R^n)$ we have

$$\begin{aligned} &|(Pv)(t) - (P\omega)(t)| \\ &\le \int_0^T |M(t,s)| \cdot \left| f(s, v(s), u(s)) - f(s, \omega(s), u(s)) \right| ds \\ &\le KTM \|v(\cdot) - \omega(\cdot)\|_{C[0,T]} \end{aligned} \tag{18}$$

or

$$\|(Pv)(t) - (P\omega)(t)\|_{C[0,T]} \le KTM \|v(\cdot) - \omega(\cdot)\|_{C[0,T]}. \tag{19}$$

Here taking into account condition (À3) we get that the operator P has the unique fixed point in (17). This shows that integral equation (8) has the unique solution and therefore the equivalent boundary value problem (1)–(3) also has a unique solution. Theorem 2 is proved. □

4. Increment Formula for the Functional

The increment method is one of the simplest ones among the methods for proving the maximum principle. In order to obtain the necessary conditions for optimality, we will use the standard procedure (see, e.g., [16]).

Let $u = u(t)$ and $\tilde{u}(t) = u(t) + \Delta u(t)$, $t \in [0,T]$, be two admissible controls and let $x(t)$, $\tilde{x}(t) = x(t) + \Delta x(t)$, $t \in [0,T]$, be appropriate trajectories. Then, obviously, $\Delta x(t)$ is the solution of the following boundary value problem:

$$\begin{gathered} \Delta\dot{x}(t) = \Delta f(t,x,u), \quad t \in [0,T], \\ \sum_{i=0}^{N} B_i \Delta x(t_i) = 0. \end{gathered} \tag{20}$$

Here $\Delta f(t,x,u) = f(t,\tilde{x},\tilde{u}) - f(t,x,u)$ are the denotations of total increment of the function $f(t,x,u)$. We can write the increment of functional (4) in the form

$$\begin{aligned} \Delta J(u) = J(\overline{u}) - J(u) &= \Delta\varphi(x(0), x(T)) \\ &+ \int_0^T \Delta F(t,x,u)\, dt. \end{aligned} \tag{21}$$

Let $\psi(t) \in R^n$ be an arbitrary nontrivial vector-function and let $x \in R^n$ be a scalar vector. Then we can rewrite the increment of functional (4) in the form

$$\begin{aligned} \Delta J(u) = J(\tilde{u}) - J(u) &= \Delta\varphi(x(0), x(T)) \\ &+ \int_0^T F(t,x,u)\, dt \\ &+ \int_0^T \langle \psi(t), \Delta\dot{x}(t) - \Delta f(t,x,u) \rangle\, dt \\ &+ \left\langle \lambda, \sum_{i=0}^{N} B_i \Delta x(t_i) \right\rangle. \end{aligned} \tag{22}$$

After some standard operators usually used in deriving necessary optimality conditions of the first order, for the increments formula we get the following equality:

$$\begin{aligned} &\Delta J(u) \\ &= -\int_0^T \Delta_{\tilde{u}} H(t,\psi,x,u)\, dt \\ &\quad - \int_0^T \left\langle \Delta_{\tilde{u}} \frac{\partial H(t,\psi,x,u)}{\partial x}, \Delta x(t) \right\rangle dt \\ &\quad + \int_0^T \left\langle \dot{\psi}(t) + \frac{\partial H(t,\psi,x,u)}{\partial x}, \Delta x(t) \right\rangle dt \\ &\quad + \left\langle -\psi(0) + B_0'\lambda + \frac{\partial\varphi}{\partial x(0)}, \Delta x(0) \right\rangle \\ &\quad + \left\langle \psi(T) + B_N'\lambda + \frac{\partial\varphi}{\partial x(T)}, \Delta x(T) \right\rangle \\ &\quad + \left\langle \sum_{i=1}^{N-1} \left[-\psi(t_i+0) + \psi(t_i-0) + B_i'\lambda \right], \Delta x(t_i) \right\rangle + \eta_{\overline{u}}, \\ &\eta_{\overline{u}} = o_\varphi(\|\Delta x(0)\|, \|\Delta x(T)\|) - \int_0^T o_H(\Delta\|x(t)\|)\, dt, \end{aligned} \tag{23}$$

where $H(t,\psi,x,u) = \langle \psi(t), f(t,x,u) \rangle - F(t,x,u)$.

Now assume that the unknown vector-function $\psi(t) \in R^n$ and the λ scalar vector is a solution of the following boundary value problem:

$$\dot{\psi}(t) = -\frac{\partial H(t,\psi,x,u)}{\partial x}, \quad t \in [0,T], \quad t \neq t_i, \quad i = 1,2,\ldots,N-1, \tag{24}$$

$$\psi(0) = B_0'\lambda + \frac{\partial\varphi}{\partial x(0)}, \qquad \psi(T) = -B_N'\lambda - \frac{\partial\varphi}{\partial x(T)}, \tag{25}$$

$$\psi(t_i+0) - \psi(t_i-0) = B_i'\lambda, \quad i = 1,2,\ldots,N-1. \tag{26}$$

The difference-differential (24)–(26) boundary value problem is called an adjoint problem in parametric form since it conditions the unknown parameter λ. From the adjoint system (24)-(25) it is seen that the solution of this system at the points $t = t_i$, $(i = 1,2,\ldots,N-1)$, has the first order discontinuities. This is the essential peculiarity of multipoint boundary conditions.

From condition (À1) and from the system (24)-(25) we can exclude the unknown vector λ. Indeed, from equality (25) we have

$$\sum_{i=0}^{N} B_i'\lambda = \psi(0) - \frac{\partial\varphi}{\partial x(0)} - \psi(T) - \frac{\partial\varphi}{\partial x(T)} + \sum_{i=1}^{N-1}\left(\psi(t_i+0) - \psi(t_i-0)\right). \tag{27}$$

Hence

$$\lambda = \left(B'\right)^{-1}\left[\psi(0) - \frac{\partial\varphi}{\partial x(0)} - \psi(T) - \frac{\partial\varphi}{\partial x(T)} + \sum_{i=1}^{N-1}\left(\psi(t_i+0) - \psi(t_i-0)\right)\right]. \tag{28}$$

Now take into account the found value of λ expressed in (26) in equalities (25) and (26). Then equalities (25) and (26) take the form

$$\begin{aligned}
\psi(0) &= B_0'\left(B'\right)^{-1}\left[\psi(0) - \frac{\partial\varphi}{\partial x(0)} - \psi(T) - \frac{\partial\varphi}{\partial x(T)} + \sum_{i=1}^{N-1}\left(\psi(t_i+0) - \psi(t_i-0)\right)\right] + \frac{\partial\varphi}{\partial x(0)},\\
\psi(T) &= B_N'\left(B'\right)^{-1}\left[\psi(0) - \frac{\partial\varphi}{\partial x(0)} - \psi(T) - \frac{\partial\varphi}{\partial x(T)} + \sum_{i=1}^{N-1}\left(\psi(t_i+0) - \psi(t_i-0)\right)\right] - \frac{\partial\varphi}{\partial x(T)},\\
\psi(t_i+0) - \psi(t_i-0) &= B_i'\left(B'\right)^{-1} \times \left[\psi(0) - \frac{\partial\varphi}{\partial x(0)} - \psi(T) - \frac{\partial\varphi}{\partial x(T)} + \sum_{i=1}^{N-1}\left(\psi(t_i+0) - \psi(t_i-0)\right)\right].
\end{aligned} \tag{29}$$

Taking into account equalities (24) and (25) in (23), we get the final form for the increment of the functional

$$\Delta J(u) = -\int_0^T \Delta_{\tilde{u}} H(t,\psi,x,u)\,dt - \int_0^T \left\langle \Delta_{\tilde{u}} \frac{\partial H(t,\psi,x,u)}{\partial x}, \Delta x(t) \right\rangle dt + \eta_{\tilde{u}}. \tag{30}$$

5. Pontryagin's Maximum Principle

At different proofs of the maximum principle, the needle-shaped variation plays one of the main parts. We choose the "perturbed" control $\tilde{u}(t)$ in the special way:

$$\tilde{u}(t) = \begin{cases} u(t), & t \notin [\theta, \theta+\varepsilon), \\ v, & t \in [\theta, \theta+\varepsilon), \end{cases} \tag{31}$$

where the parameters of the needle-shaped variation satisfy the following conditions. $\theta \in [0,T]$ is a regular point of the control $u(t)$, $\varepsilon > 0$, $\theta + \varepsilon < T$, $v \in U$. For any θ, ε, v satisfying the enumerated conditions, the control $\overline{u}(t)$ is admissible.

The traditional form of necessary optimality conditions will follow from increments formula (30) if we show that on the needle-shaped variation $\tilde{u}(t) = u_\varepsilon(t)$ the increment of phase states $\Delta_\varepsilon x(t)$ is of order ε. This will follow from conditions (À1)–(À3) and boundary value problem (20):

$$\Delta x(t) = \int_0^T M(t,s)\left[f(s,x+\Delta x,\tilde{u}) - f(s,x,\tilde{u})\right] ds + \int_0^T M(t,s)\,\Delta_{\tilde{u}} f(s,x,u)\,ds. \tag{32}$$

From (32) we get

$$\|\Delta x(t)\| \le (1-L)^{-1} M \int_0^T \left\|\Delta_{\tilde{u}} f(t,x,u)\right\| dt. \tag{33}$$

If in inequality we take $\tilde{u}(t) = u_\varepsilon(t)$, we have

$$\|\Delta_\varepsilon x(t)\| \le \tilde{L}\varepsilon, \quad t \in [0,T], \ \tilde{L} = \text{const} > 0. \tag{34}$$

Estimation (33) shows that, for $\tilde{u}(t) = u_\varepsilon(t)$,

$$\begin{aligned}\int_\theta^{\theta+\varepsilon} \left\langle \Delta_v \frac{\partial H(t,\psi,x,u)}{\partial x}, \Delta_\varepsilon x(t) \right\rangle dt \\ + \eta_{u_\varepsilon}(\|\Delta_\varepsilon x(t)\|) = o(\varepsilon),\end{aligned} \tag{35}$$

where $\Delta_\varepsilon x(t) = x(t,u_\varepsilon) - x(t,u) \sim \varepsilon$.

Use the increment formula (30) and the property of the needle-shaped variation. Then

$$\Delta J(u) = -\int_\theta^{\theta+\varepsilon} \Delta_v H(t,\psi,x,u)\, dt + o(\varepsilon). \tag{36}$$

Since the point $t = \theta$ is a regular point of the control $u = u(t)$, from the Taylor formula it follows that

$$\begin{aligned}\Delta J(u) = -\Delta_v H(\theta,\psi(\theta),x(\theta),u(\theta))\,\varepsilon + o(\varepsilon), \\ v \in U, \quad \theta \in [0,T).\end{aligned} \tag{37}$$

Pontryagin's maximum principle follows from formula (37).

Theorem 3 (maximum principle). *Let the admissible process $u^0(t)$, $x^0(t,u^0)$ be optimal in problem (1)–(4) and let $\psi^0(t)$ be a solution of conjugated problem (24)-(25) calculated on optimal process. Then for all $t \in [0,T]$ the following equality is fulfilled:*

$$\max_{v\in U} H(t,\psi^0(t),x^0(t),v) = H(t,\psi^0(t),x^0(t),u^0(t)). \tag{38}$$

Corollary 4. *If in the optimal control problem the function f is linear with respect to (x,u) and the functions φ, F are convex with respect to $x(0)$, $x(T)$ and $x(t)$, respectively, then the maximum principle is necessary and sufficient for optimality. This fact follows from increment formula (30). Indeed, in this case,*

$$\begin{aligned}\Delta J(u) = -\int_0^T \Delta_{\bar{u}} H(t,\psi,x,u)\, dt + o_\varphi(\|\Delta x(0)\|, \|\Delta x(T)\|) \\ + \int_0^T o_F(\|\Delta x(t)\|)\, dt.\end{aligned} \tag{39}$$

Since the functions φ and F are convex, then $o_\varphi \ge 0$, $o_F \ge 0$.

Other Optimality Conditions. In this item we suppose that the function $f(t,x,u)$ is differentiable and the set U is convex. Then from Theorem 3 we get the following theorem.

Theorem 5 (differential principle of maximum). *Let the process $(u^0(t), x^0(t,u^0))$, $t \in [0,T]$, be optimal in problem (1)–(4) and let $\psi^0(t)$ be an appropriate solution of adjoint problem (24)–(26). Then,*

$$\begin{aligned}\left\langle \frac{\partial H(t,\psi^0(t),x^0(t),u^0(t))}{\partial u}, u^0(t) \right\rangle \\ = \max_{v\in U} \left\langle \frac{\partial H(t,\psi^0(t),x^0(t),u^0(t))}{\partial u}, v \right\rangle.\end{aligned} \tag{40}$$

Proof. Suppose the contrary. Let there exist $\theta \in [0,T]$, $v \in U$, $\alpha > 0$, such that

$$\begin{aligned}\left\langle \frac{\partial H(\theta,\psi^0(\theta),x^0(\theta),u^0(\theta))}{\partial u}, u^0(\theta) \right\rangle \\ = \left\langle \frac{\partial H(\theta,\psi^0(\theta),x^0(\theta),u^0(\theta))}{\partial u}, v \right\rangle - \alpha.\end{aligned} \tag{41}$$

Let $0 \le \varepsilon \le 1$. Construct the vector

$$v(\varepsilon) = u^0(\theta) + \varepsilon(v - u^0(\theta)). \tag{42}$$

Since the set U is convex, then $v(\varepsilon) \in U$. From maximum principle (38) and equality (41) it follows that the inequality

$$\begin{aligned}0 &\ge \Delta_{v(u)} H(\theta,\psi^0(\theta),x^0(\theta),u^0(\theta)) \\ &= \left\langle \frac{\partial H(\theta,\psi^0(\theta),x^0(\theta),u^0(\theta))}{\partial u}, v(\varepsilon) - u^0(\theta) \right\rangle \\ &\quad + o(\|v(\varepsilon) - u^0(\theta)\|) = \alpha\varepsilon + o(\varepsilon)\end{aligned} \tag{43}$$

is valid for all rather small ε, which is impossible, since by supposition $\alpha > 0$. Indeed, the right part of (43) for rather small $\varepsilon > 0$ is positive. The theorem is proved. □

Note that condition (40) for verification is simpler than condition (38) by virtue of linearity of the right hand side of (43). However, assumptions on convexity of U and differentiability of the function $f(t,x,u)$ with respect to u contract the application of condition (41).

Note that there exist optimal control problems for which condition (40) is valid, and the maximum principle gives no information. This determines the value of the differential principle of maximum.

The following theorem follows from the maximum principle.

Theorem 6 (stationary state principle). *Let $U \subset R^n$ be an open set. Then at each $t \in [0,T]$ the optimal control delivers the stationary value to the function $H(t,\psi,x,u)$; that is,*

$$\frac{\partial H(t,\psi^0(t),x^0(t),u^0(t))}{\partial u} = 0. \tag{44}$$

6. Discussion of the Obtained Results

In this paper different necessary optimality conditions of first order were obtained for optimal control problems with multipoint boundary conditions. This problem is rather general and contains different special cases.

(i) The first case is the Cauchy problem (in this case $N = 0$ and B_0 is a unit matrix).

(ii) The second case is the problem with two-point boundary conditions (in this case $N = 1$).

(iii) Each equation of (1) has its initial condition; that is, dimension of the vector x equals $N+1$ and $B_i = (B^i_{jk})$ $(i = 0, 1, \ldots, N;\ j, k = 1, 2, \ldots, N+1)$ and

$$B^i_{jk} = \begin{cases} 1, & j = i+1, k = k(i), \\ 0. \end{cases} \tag{45}$$

Here $(k(0), k(1), \ldots, k(N))$ is some permutation $(1, 2, \ldots, N+1)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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