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Research Article

Approximating Iterations for Nonexpansive and Maximal Monotone Operators

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We present two algorithms for finding a zero of the sum of two monotone operators and a fixed point of a nonexpansive operator in Hilbert spaces. We show that these two algorithms converge strongly to the minimum norm common element of the zero of the sum of two monotone operators and the fixed point of a nonexpansive operator.

1. Introduction

Throughout, we assume that $\mathscr{H}$ is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $\mathscr{C} \subset \mathscr{H}$ be a nonempty closed convex set.

Definition 1. An operator $\mathbb{S} : \mathscr{C} \to \mathscr{C}$ is said to be *nonexpansive* if

$$\|\mathbb{S}u - \mathbb{S}v\| \leq \|u - v\| \tag{1}$$

for all $u, v \in \mathscr{C}$.

We denote by $\mathrm{Fix}(\mathbb{S})$ the set of fixed points of $\mathbb{S}$.

Definition 2. An operator $\mathbb{A} : \mathscr{C} \to \mathscr{H}$ is said to be ξ*-inverse strong monotone* if

$$\langle \mathbb{A}u - \mathbb{A}v, u - v\rangle \geq \xi\|\mathbb{A}u - \mathbb{A}v\|^2 \tag{2}$$

for some $\xi > 0$ and for all $u, v \in \mathscr{C}$.

It is known that if $\mathbb{A}$ is ξ-inverse strong monotone, then $\mathbb{A}$ is $1/\xi$-lipschitz, that is,

$$\|\mathbb{A}u - \mathbb{A}v\| \leq \frac{1}{\xi}\|u - v\|, \tag{3}$$

for all $u, v \in \mathscr{C}$. Furthermore,

$$\begin{aligned}&\|(I - \delta\mathbb{A})u - (I - \delta\mathbb{A})v\|^2 \\ &\quad \leq \|u - v\|^2 + \delta(\delta - 2\xi)\|\mathbb{A}u - \mathbb{A}v\|^2, \quad \forall u, v \in \mathscr{C}.\end{aligned} \tag{4}$$

In particular, if $\delta \in (0, 2\xi)$, then $I - \delta\mathbb{A}$ is nonexpansive.

Let $\mathbb{B} : \mathscr{H} \to 2^{\mathscr{H}}$ be a set-valued operator. The effective domain of $\mathbb{B}$ is denoted by $\mathrm{dom}(\mathbb{B})$, that is, $\mathrm{dom}(\mathbb{B}) = \{x \in \mathscr{H} : \mathbb{B}x \neq \emptyset\}$.

Definition 3. A multivalued operator $\mathbb{B}$ is said to be a *monotone* on $\mathscr{H}$ if and only if

$$\langle x - y, u - v\rangle \geq 0 \tag{5}$$

for all $x, y \in \mathrm{dom}(\mathbb{B})$, $u \in \mathbb{B}x$, and $v \in \mathbb{B}y$.

A monotone operator $\mathbb{B}$ on $\mathscr{H}$ is said to be *maximal* if and only if its graph is not strictly contained in the graph of any other monotone operator on $\mathscr{H}$. We denote by $\mathbb{B}^{-1}0$ the set of the zero points of $\mathbb{B}$, that is, $\mathbb{B}^{-1}0 = \{x \in \mathscr{H} : 0 \in \mathbb{B}x\}$.

For $\lambda > 0$, we define a single-valued operator

$$J_\lambda^{\mathbb{B}} = (I + \lambda\mathbb{B})^{-1} : \mathscr{H} \longrightarrow \mathrm{dom}(\mathbb{B}), \tag{6}$$

which is called the resolvent of $\mathbb{B}$ for λ. It is known that the resolvent $J_\lambda^{\mathbb{B}}$ is firmly nonexpansive, that is,

$$\left\| J_\lambda^{\mathbb{B}} u - J_\lambda^{\mathbb{B}} v \right\|^2 \leq \left\langle J_\lambda^{\mathbb{B}} u - J_\lambda^{\mathbb{B}} v, u - v \right\rangle, \tag{7}$$

for all $u, v \in \mathscr{C}$ and $\mathbb{B}^{-1}0 = \operatorname{Fix}(J_\lambda^{\mathbb{B}})$ for all $\lambda > 0$.

In the present paper, we consider the variational inclusion of finding a zero $x \in \mathscr{H}$ of the sum of two monotone operators $\mathbb{A}$ and $\mathbb{B}$ such that

$$0 \in \mathbb{A}(x) + \mathbb{B}(x), \tag{8}$$

where $\mathbb{A} : \mathscr{H} \to \mathscr{H}$ is a single-valued operator and $\mathbb{B} : \mathscr{H} \to 2^{\mathscr{H}}$ is a set-valued operator. The set of solutions of problem (8) is denoted by $(\mathbb{A} + \mathbb{B})^{-1}(0)$.

Special Cases. (i) If $\mathscr{H} = \mathbb{R}^m$, then problem (8) becomes the generalized equation introduced by Robinson [1].

(ii) If $\mathbb{A} = 0$, then problem (8) becomes the inclusion problem introduced by Rockafellar [2].

It is known that (8) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, and game theory. Also various types of variational inclusions problems have been extended and generalized. For related work, please see [3–20].

Zhang et al. [21] introduced the following iterative algorithm for finding a common element of the set of solutions to the problem (8) and the set of fixed points of a nonexpansive operator:

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) \mathbb{S} J_\lambda^{\mathbb{B}} (x_n - \lambda \mathbb{A} x_n), \tag{9}$$

where $\mathbb{S} : \mathscr{C} \to \mathscr{C}$ is a nonexpansive operator. Under some mild conditions, they prove that the sequence $\{x_n\}$ converges strongly to $x^* \in \operatorname{Fix}(\mathbb{S}) \cap (\mathbb{A} + \mathbb{B})^{-1}(0)$.

Recently, Takahashi et al. [22] introduced another iterative algorithm for finding a zero of the sum of two monotone operators and a fixed point of a nonexpansive operator

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) \mathbb{S} \left(\alpha_n x_0 + (1 - \alpha_n) J_{\lambda_n}^{\mathbb{B}} (x_n - \lambda_n \mathbb{A} x_n) \right) \tag{10}$$

for all $n \geq 0$. Under some assumptions, they proved that the sequence $\{x_n\}$ converges strongly to a point of $\operatorname{Fix}(\mathbb{S}) \cap (\mathbb{A} + \mathbb{B})^{-1}(0)$.

Motivated and inspired by (9) and (10), in the present paper, we suggest two algorithms

$$x_t = J_\lambda^{\mathbb{B}} ((1 - t) \mathbb{S} x_t - \lambda \mathbb{A} \mathbb{S} x_t), \quad t \in (0, 1), \tag{11}$$

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) J_{\lambda_n}^{\mathbb{B}} ((1 - \alpha_n) \mathbb{S} x_n - \lambda_n \mathbb{A} \mathbb{S} x_n), \quad n \geq 0. \tag{12}$$

It is obvious that (12) is very different from (9) and (10). Furthermore, we prove that both (11) and (12) converge strongly to the minimum norm element in $\operatorname{Fix}(\mathbb{S}) \cap (\mathbb{A} + \mathbb{B})^{-1}0$. It should be pointed out that we do not use the metric projection in (11) and (12).

2. Lemmas

In this section, we collect several useful lemmas for our next section.

First, the following resolvent equality is well known.

Lemma 4. *For $\lambda > 0$ and $\lambda^\dagger > 0$, one has*

$$J_\lambda^{\mathbb{B}} u = J_{\lambda^\dagger}^{\mathbb{B}} \left(\frac{\lambda^\dagger}{\lambda} u + \left(1 - \frac{\lambda^\dagger}{\lambda} \right) J_\lambda^{\mathbb{B}} u \right), \quad \forall u \in \mathscr{H}. \tag{13}$$

Lemma 5 (see [23]). *Let $\mathscr{C} \subset \mathscr{H}$ be a closed convex set. Let $\mathbb{S} : \mathscr{C} \to \mathscr{C}$ be a nonexpansive operator. Then* $\operatorname{Fix}(\mathbb{S})$ *is a closed convex subset of $\mathscr{C}$ and the operator $I - \mathbb{S}$ is demiclosed at* 0.

Lemma 6 (see [24]). *Let $\mathscr{X}$ be a Banach space. Let $\{u_n\} \subset \mathscr{X}$ and $\{v_n\} \subset \mathscr{X}$ be two bounded sequences. Let the sequence $\{\zeta_n\} \subset (0,1)$ satisfy $0 < \liminf_{n\to\infty} \zeta_n \leq \limsup_{n\to\infty} \zeta_n < 1$. Suppose $u_{n+1} = (1 - \zeta_n) v_n + \zeta_n u_n$ for all $n \geq 0$ and $\limsup_{n\to\infty} (\|v_{n+1} - v_n\| - \|u_{n+1} - u_n\|) \leq 0$. Then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.*

Lemma 7 (see [25]). *Let $\{\sigma_n\} \subset [0, \infty)$, $\{\gamma_n\} \subset (0,1)$, and $\{\delta_n\} \subset \mathbb{R}$ be three sequences satisfying*

$$\sigma_{n+1} \leq (1 - \gamma_n) \sigma_n + \delta_n \gamma_n. \tag{14}$$

If $\sum_{n=1}^\infty \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$ (or $\sum_{n=1}^\infty |\delta_n \gamma_n| < \infty$), then $\lim_{n\to\infty} \sigma_n = 0$.

3. Strong Convergence Results

Let $\mathscr{C} \subset \mathscr{H}$ be a nonempty closed convex set. Let $\mathbb{A} : \mathscr{C} \to \mathscr{H}$ be a ϱ-inverse strong monotone operator. Let $\mathbb{B}$ be a maximal monotone operator on $\mathscr{H}$ such that $\operatorname{dom}(\mathbb{B}) \subset \mathscr{C}$. Let $\mathbb{S} : \mathscr{C} \to \mathscr{C}$ be a nonexpansive operator.

Pick up a constant $\tau \in (0, 2\varrho)$. For any $t \in (0, (2\varrho - \tau)/2\varrho)$, we define an operator

$$\psi(x) = J_\tau^{\mathbb{B}} ((1 - t) \mathbb{S} - \tau \mathbb{A} \mathbb{S}) x, \tag{15}$$

for all $x \in \mathscr{C}$.

Since $J_\tau^{\mathbb{B}}$, $\mathbb{S}$, and $I - \tau\mathbb{A}/(1-t)$ are nonexpansive, we have

$$\begin{aligned} \|\psi(x) - \psi(y)\| &= \Big\| J_\tau^{\mathbb{B}} \left((1-t) \left(I - \frac{\tau}{1-t} \mathbb{A} \right) \mathbb{S} x \right) \\ &\quad - J_\tau^{\mathbb{B}} \left((1-t) \left(I - \frac{\tau}{1-t} \mathbb{A} \right) \mathbb{S} y \right) \Big\| \\ &\leq (1-t) \Big\| \left(I - \frac{\tau}{1-t} \mathbb{A} \right) \mathbb{S} x \\ &\quad - \left(I - \frac{\tau}{1-t} \mathbb{A} \right) \mathbb{S} y \Big\| \\ &\leq (1-t) \|x - y\|, \end{aligned} \tag{16}$$

for any $x, y \in \mathscr{C}$. Hence ψ is a contraction on $\mathscr{C}$. We use x_t to denote the unique fixed point of ψ in $\mathscr{C}$. Thus, $\{x_t\}$ satisfies the fixed point equation

$$x_t = J_\tau^{\mathbb{B}} ((1 - t) \mathbb{S} x_t - \tau \mathbb{A} \mathbb{S} x_t). \tag{17}$$

Next, we give the convergence analysis of (17).

Theorem 8. *Assume that* $\mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}0 \neq \emptyset$. *Then* $\{x_t\}$ *defined by* (17) *converges strongly, as* $t \to 0+$, *to the minimum norm element in* $\mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}(0)$.

Proof. Choose any $z \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}(0)$. It is obvious that $z = \mathbb{S}z = J_\tau^{\mathbb{B}}(z - \tau\mathbb{A}z)$ for all $\tau > 0$. So, we have

$$z = \mathbb{S}z = J_\tau^{\mathbb{B}}(z - \tau\mathbb{A}z) = J_\tau^{\mathbb{B}}\left(tz + (1-t)\left(I - \frac{\tau}{1-t}\mathbb{A}\right)\mathbb{S}z\right) \tag{18}$$

for all $t \in (0,1)$.

From (17), we have

$$\begin{aligned}
\|x_t - z\| &= \left\|J_\tau^{\mathbb{B}}\left((1-t)\left(I - \frac{\tau}{1-t}\mathbb{A}\right)\mathbb{S}x_t\right) - z\right\| \\
&= \left\|J_\tau^{\mathbb{B}}\left((1-t)\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right)\right. \\
&\qquad \left. - J_\tau^{\mathbb{B}}\left(tz + (1-t)\left(\mathbb{S}z - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right)\right\| \\
&\le \left\|(1-t)\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right. \\
&\qquad \left. - \left(tz + (1-t)\left(\mathbb{S}z - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right)\right\| \\
&= \left\|(1-t)\left(\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.\right. \\
&\qquad \left.\left. - \left(\mathbb{S}z - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right) - tz\right\| \\
&\le (1-t)\left\|\left(I - \frac{\tau}{1-t}\mathbb{A}\right)\mathbb{S}x_t - \left(I - \frac{\tau}{1-t}\mathbb{A}\right)\mathbb{S}z\right\| \\
&\quad + t\|z\| \\
&\le (1-t)\|x_t - z\| + t\|z\|.
\end{aligned} \tag{19}$$

Hence, we get

$$\|x_t - z\| \le \|z\|. \tag{20}$$

Thus, $\{x_t\}$ is bounded.

By (4) and (19), we derive

$$\begin{aligned}
\|x_t - z\|^2 &\le \left\|(1-t)\left(\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.\right. \\
&\qquad \left.\left. - \left(\mathbb{S}z - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right) + t(-z)\right\|^2 \\
&\le (1-t)\left\|\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right. \\
&\qquad \left. - \left(\mathbb{S}z - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right\|^2 + t\|z\|^2 \\
&= (1-t)\left\|(\mathbb{S}x_t - \mathbb{S}z) - \frac{\tau}{1-t}(\mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z)\right\|^2 \\
&\quad + t\|z\|^2 \\
&= (1-t)\left(\|\mathbb{S}x_t - \mathbb{S}z\|^2 - \frac{2\tau}{1-t}\right. \\
&\qquad \times \langle \mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z, \mathbb{S}x_t - \mathbb{S}z\rangle \\
&\qquad \left. + \frac{\tau^2}{(1-t)^2}\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z\|^2\right) + t\|z\|^2 \\
&\le (1-t)\left(\|\mathbb{S}x_t - \mathbb{S}z\|^2 - \frac{2\varrho\tau}{1-t}\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z\|^2\right. \\
&\qquad \left. + \frac{\tau^2}{(1-t)^2}\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z\|^2\right) + t\|z\|^2 \\
&= (1-t)\left(\|\mathbb{S}x_t - \mathbb{S}z\|^2 + \frac{\tau}{(1-t)^2}(\tau - 2(1-t)\varrho)\right. \\
&\qquad \left. \times \|\mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z\|^2\right) + t\|z\|^2 \\
&\le (1-t)\|x_t - z\|^2 + \frac{\tau}{1-t}(\tau - 2(1-t)\varrho) \\
&\quad \times \|\mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z\|^2 + t\|z\|^2.
\end{aligned} \tag{21}$$

So,

$$\begin{aligned}
&\frac{\tau}{1-t}(2(1-t)\varrho - \tau)\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}z\|^2 \\
&\quad \le t\|z\|^2 - t\|x_t - z\|^2 \longrightarrow 0.
\end{aligned} \tag{22}$$

Since $2(1-t)\varrho - \tau > 0$ for all $t \in (0, 1 - \tau/2\varrho)$, we obtain

$$\lim_{t \to 0+}\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}z\| = 0. \tag{23}$$

Using the firm nonexpansivity of $J_\tau^{\mathbb{B}}$, we have

$$\begin{aligned}
\|x_t - z\|^2 &= \left\|J_\tau^{\mathbb{B}}((1-t)\mathbb{S}x_t - \tau\mathbb{A}\mathbb{S}x_t) - z\right\|^2 \\
&= \left\|J_\tau^{\mathbb{B}}((1-t)\mathbb{S}x_t - \tau\mathbb{A}\mathbb{S}x_t) - J_\tau^{\mathbb{B}}(z - \tau\mathbb{A}z)\right\|^2 \\
&\le \langle (1-t)\mathbb{S}x_t - \tau\mathbb{A}\mathbb{S}x_t - (z - \tau\mathbb{A}z), x_t - z\rangle \\
&= \frac{1}{2}\left(\|(1-t)\mathbb{S}x_t - \tau\mathbb{A}\mathbb{S}x_t - (z - \tau\mathbb{A}z)\|^2\right. \\
&\qquad + \|x_t - z\|^2 \\
&\qquad \left. - \|(1-t)\mathbb{S}x_t - \tau(\mathbb{A}\mathbb{S}x_t - \tau\mathbb{A}z) - x_t\|^2\right).
\end{aligned} \tag{24}$$

Note that

$$\begin{aligned}
&\|(1-t)\mathbb{S}x_t - \tau\mathbb{A}\mathbb{S}x_t - (z - \tau\mathbb{A}z)\|^2 \\
&\quad = \left\|(1-t)\left(\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.\right. \\
&\qquad \left.\left. - \left(\mathbb{S}z - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right) + t(-z)\right\|^2
\end{aligned}$$

$$\leq (1-t)\left\|\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.$$
$$\left. - \left(\mathbb{S}z - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right\|^2 + t\|z\|^2$$
$$\leq (1-t)\left\|x_t - z\right\|^2 + t\|z\|^2. \tag{25}$$

Thus,

$$\left\|x_t - z\right\|^2 \leq \frac{1}{2}\left((1-t)\left\|x_t - z\right\|^2 + t\|z\|^2 + \left\|x_t - z\right\|^2\right.$$
$$\left. - \left\|(1-t)\mathbb{S}x_t - \tau\left(\mathbb{A}\mathbb{S}x_t - \mathbb{A}z\right) - x_t\right\|^2\right). \tag{26}$$

It follows that

$$\left\|x_t - z\right\|^2 \leq (1-t)\left\|x_t - z\right\|^2 + t\|z\|^2$$
$$- \left\|(1-t)\mathbb{S}x_t - x_t - \tau\left(\mathbb{A}\mathbb{S}x_t - \mathbb{A}z\right)\right\|^2$$
$$= (1-t)\left\|x_t - z\right\|^2 + t\|z\|^2 - \left\|(1-t)\mathbb{S}x_t - x_t\right\|^2$$
$$+ 2\tau\left\langle (1-t)\mathbb{S}x_t - x_t, \mathbb{A}\mathbb{S}x_t - \mathbb{A}z\right\rangle$$
$$- \tau^2\left\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}z\right\|^2$$
$$\leq (1-t)\left\|x_t - z\right\|^2 + t\|z\|^2 - \left\|(1-t)\mathbb{S}x_t - x_t\right\|^2$$
$$+ 2\tau\left\|(1-t)\mathbb{S}x_t - x_t\right\|\left\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}z\right\|. \tag{27}$$

Hence,

$$\left\|(1-t)\mathbb{S}x_t - x_t\right\|^2$$
$$\leq t\|z\|^2 + 2\tau\left\|(1-t)\mathbb{S}x_t - x_t\right\|\left\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}z\right\|. \tag{28}$$

This together with (23) implies that

$$\lim_{t\to 0+}\left\|(1-t)\mathbb{S}x_t - x_t\right\| = 0. \tag{29}$$

So,

$$\lim_{t\to 0+}\left\|x_t - \mathbb{S}x_t\right\| = 0. \tag{30}$$

By (19), we have

$$\left\|x_t - z\right\|^2 \leq \left\|(1-t)\left(\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.\right.$$
$$\left.\left. - \left(z - \frac{\tau}{1-t}\mathbb{A}z\right)\right) + t(-z)\right\|^2$$
$$= (1-t)^2\left\|\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.$$
$$\left. - \left(z - \frac{\tau}{1-t}\mathbb{A}z\right)\right\|^2$$
$$+ 2t(1-t)\left\langle -z, \left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.$$
$$\left. - \left(z - \frac{\tau}{1-t}\mathbb{A}z\right)\right\rangle + t^2\|z\|^2$$
$$\leq (1-t)^2\left\|x_t - z\right\|^2 + 2t(1-t)$$
$$\times\left\langle -z, \mathbb{S}x_t - \frac{\tau}{1-t}\left(\mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z\right) - z\right\rangle$$
$$+ t^2\|z\|^2. \tag{31}$$

It follows that

$$\left\|x_t - z\right\|^2 \leq \left\langle -z, \mathbb{S}x_t - \frac{\tau}{1-t}\left(\mathbb{A}\mathbb{S}x_t - \mathbb{A}z\right) - z\right\rangle$$
$$+ \frac{t}{2}\left(\|z\|^2 + \left\|x_t - z\right\|^2\right)$$
$$+ t\|z\|\left\|\mathbb{S}x_t - \frac{\tau}{1-t}\left(\mathbb{A}\mathbb{S}x_t - \mathbb{A}z\right) - z\right\|$$
$$\leq \left\langle -z, \mathbb{S}x_t - z\right\rangle + \left(t + \left\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}z\right\|\right)M, \tag{32}$$

where M is some constant such that

$$\sup_{t\in(0,(2\varrho-\tau)/2\varrho)}\left\{\frac{1}{2}\left(\|z\|^2 + \left\|x_t - z\right\|^2\right),\right.$$
$$\left.\|z\|\left\|\mathbb{S}x_t - \frac{\tau}{1-t}\left(\mathbb{A}\mathbb{S}x_t - \mathbb{A}z\right) - z\right\|\right\} \leq M. \tag{33}$$

Now we show that $\{x_t\}$ is relatively norm-compact as $t \to 0+$. Assume $\{t_n\} \subset (0, (2\varrho-\tau)/2\varrho)$ such that $t_n \to 0+$ as $n \to \infty$. Put $x_n := x_{t_n}$. From (32), we have

$$\left\|x_n - z\right\|^2 \leq \left\langle -z, \mathbb{S}x_n - z\right\rangle + \left(t_n + \left\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\|\right)M. \tag{34}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_j} \rightharpoonup \tilde{x} \in C$. Hence, $x_{n_j} - (\tau/(1-t_{n_j}))(\mathbb{A}\mathbb{S}x_{n_j} - \mathbb{A}z) \rightharpoonup \tilde{x}$ because of $\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\| \to 0$ by (23). From (30), we have

$$\lim_{n\to\infty}\left\|x_n - \mathbb{S}x_n\right\| = 0. \tag{35}$$

By Lemma 5 and (35), we deduce $\tilde{x} \in \mathrm{Fix}(\mathbb{S})$.

Next, we show that $\tilde{x} \in (\mathbb{A}+\mathbb{B})^{-1}0$. Let $v \in \mathbb{B}u$. Note that $x_n = J_\tau^{\mathbb{B}}((1-t_n)\mathbb{S}x_n - \tau\mathbb{A}\mathbb{S}x_n)$ for all n. Then, we have

$$\left(1-t_n\right)\mathbb{S}x_n - \tau\mathbb{A}\mathbb{S}x_n \in (I + \tau\mathbb{B})\,x_n. \tag{36}$$

So,

$$\frac{1-t_n}{\tau}\mathbb{S}x_n - \mathbb{A}\mathbb{S}x_n - \frac{x_n}{\tau} \in \mathbb{B}x_n. \tag{37}$$

Since $\mathbb{B}$ is monotone, we have, for $(u, v) \in \mathbb{B}$,

$$\begin{aligned}
&\left\langle \frac{t_n \gamma f(x_n)}{\tau} + \frac{1-t_n}{\tau}\mathbb{S}x_n - \mathbb{A}\mathbb{S}x_n - \frac{x_n}{\tau} - v, x_n - u \right\rangle \geq 0 \\
&\Longrightarrow \langle (1-t_n)\mathbb{S}x_n - \tau\mathbb{A}\mathbb{S}x_n - x_n - \tau v, x_n - u\rangle \geq 0 \\
&\Longrightarrow \langle \mathbb{A}\mathbb{S}x_n + v, x_n - u\rangle \\
&\qquad \leq \frac{1}{\tau}\langle \mathbb{S}x_n - x_n, x_n - u\rangle - \frac{t_n}{\tau}\langle \mathbb{S}x_n, x_n - u\rangle \\
&\Longrightarrow \langle \mathbb{A}\mathbb{S}\tilde{x} + v, x_n - u\rangle \\
&\qquad \leq \frac{1}{\tau}\langle \mathbb{S}x_n - x_n, x_n - u\rangle - \frac{t_n}{\tau}\langle \mathbb{S}x_n, x_n - u\rangle \\
&\qquad\quad + \langle \mathbb{A}\mathbb{S}\tilde{x} - \mathbb{A}\mathbb{S}x_n, x_n - u\rangle \\
&\Longrightarrow \langle \mathbb{A}\mathbb{S}\tilde{x} + v, x_n - u\rangle \\
&\qquad \leq \frac{1}{\tau}\|\mathbb{S}x_n - x_n\|\|x_n - u\| + \frac{t_n}{\tau}\|\mathbb{S}x_n\|\|x_n - u\| \\
&\qquad\quad + \|\mathbb{A}\mathbb{S}\tilde{x} - \mathbb{A}\mathbb{S}x_n\|\|x_n - u\|.
\end{aligned} \tag{38}$$

It follows that

$$\begin{aligned}
\langle \mathbb{A}\mathbb{S}\tilde{x} + v, \tilde{x} - u\rangle \leq{} & \frac{1}{\tau}\left\|\mathbb{S}x_{n_j} - x_{n_j}\right\|\left\|x_{n_j} - u\right\| \\
&+ \frac{t_{n_j}}{\tau}\left\|\mathbb{S}x_{n_j}\right\|\left\|x_{n_j} - u\right\| \\
&+ \left\|\mathbb{A}\mathbb{S}\tilde{x} - \mathbb{A}\mathbb{S}x_{n_j}\right\|\left\|x_{n_j} - u\right\| \\
&+ \left\langle \mathbb{A}\mathbb{S}\tilde{x} + v, \tilde{x} - x_{n_j}\right\rangle.
\end{aligned} \tag{39}$$

Since

$$\left\langle x_{n_j} - \tilde{x}, \mathbb{A}\mathbb{S}x_{n_j} - \mathbb{A}\mathbb{S}\tilde{x}\right\rangle \geq \varrho\left\|\mathbb{A}\mathbb{S}x_{n_j} - \mathbb{A}\mathbb{S}\tilde{x}\right\|^2, \tag{40}$$

$\mathbb{A}\mathbb{S}x_{n_j} \to \mathbb{A}\mathbb{S}z$, and $x_{n_j} \rightharpoonup \tilde{x}$, we have $\mathbb{A}\mathbb{S}x_{n_j} \to \mathbb{A}\mathbb{S}\tilde{x}$. We also observe that $t_n \to 0$ and $\|\mathbb{S}x_n - x_n\| \to 0$. Then, from (39), we derive

$$\langle \mathbb{A}\mathbb{S}\tilde{x} + v, \tilde{x} - u\rangle \leq 0. \tag{41}$$

That is, $\langle -\mathbb{A}\tilde{x} - v, \tilde{x} - u\rangle \geq 0$. Since $\mathbb{B}$ is maximal monotone, we have $-\mathbb{A}\tilde{x} \in \mathbb{B}\tilde{x}$. This shows that $0 \in (\mathbb{A} + \mathbb{B})\tilde{x}$. Hence, we have $\tilde{x} \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A} + \mathbb{B})^{-1}0$. Therefore, we can substitute $\tilde{x}$ for z in (34) to get

$$\|x_n - \tilde{x}\|^2 \leq \langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\rangle + (t_n + \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\|)M. \tag{42}$$

Consequently, the weak convergence of $\{x_n\}$ to $\tilde{x}$ actually implies that $x_n \to \tilde{x}$. This has proved the relative norm-compactness of the net $\{x_t\}$ as $t \to 0+$.

From (34), we get

$$\|\tilde{x} - z\|^2 \leq \langle -z, \tilde{x} - z\rangle, \quad \forall z \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A} + \mathbb{B})^{-1}0. \tag{43}$$

That is,

$$\langle \tilde{x}, \tilde{x} - z\rangle \leq 0, \quad \forall z \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A} + \mathbb{B})^{-1}0. \tag{44}$$

It follows that

$$\|\tilde{x}\| \leq \|z\|, \quad \forall z \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A} + \mathbb{B})^{-1}0. \tag{45}$$

It is obvious that $\tilde{x} = \mathrm{proj}_{\mathrm{Fix}(\mathbb{S})\cap(\mathbb{A}+\mathbb{B})^{-1}0}(0)$ by (44). This denotes that the entire net $\{x_t\}$ converges to $\tilde{x}$. This completes the proof. □

Next, we present another algorithm.

Algorithm 9. For given $x_0 \in \mathscr{C}$, define a sequence $\{x_n\} \subset \mathscr{C}$ iteratively by

$$x_{n+1} = \varsigma_n x_n + (1 - \varsigma_n) J^{\mathbb{B}}_{\tau_n}((1 - \varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n), \quad \forall n \geq 0, \tag{46}$$

where $\{\tau_n\} \subset (0, 2\varrho)$, $\{\varrho_n\} \subset (0, 1)$, and $\{\varsigma_n\} \subset (0, 1)$.

Theorem 10. *Suppose that* $\mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A} + \mathbb{B})^{-1}0 \neq \emptyset$. *Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\varrho_n = 0$ *and* $\sum_n \varrho_n = \infty$*;*

(ii) $0 < \underline{\lim}_{n\to\infty}\varsigma_n \leq \overline{\lim}_{n\to\infty}\varsigma_n < 1$*;*

(iii) $a(1 - \varrho_n) \leq \tau_n \leq b(1 - \varrho_n)$*, where* $[a, b] \subset (0, 2\varrho)$ *and* $\lim_{n\to\infty}(\tau_{n+1} - \tau_n) = 0$.

Then $\{x_n\}$ *generated by* (46) *converges strongly to a point* $\tilde{x} = \mathrm{proj}_{\mathrm{Fix}(\mathbb{S})\cap(\mathbb{A}+\mathbb{B})^{-1}(0)}(0)$ *which is the minimum norm element in* $\mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A} + \mathbb{B})^{-1}(0)$.

Proof. Let $z \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A} + \mathbb{B})^{-1}(0)$. We have $z = J^{\mathbb{B}}_{\tau_n}(z - \tau_n\mathbb{A}z) = J^{\mathbb{B}}_{\tau_n}(\varrho_n z + (1 - \varrho_n)(z - \tau_n\mathbb{A}z/(1 - \varrho_n)))$ for all $n \geq 0$. Since $J^{\mathbb{B}}_{\tau_n}$, $\mathbb{S}$, and $I - \tau_n\mathbb{A}/(1 - \varrho_n)$ are nonexpansive, we have

$$\begin{aligned}
&\left\|J^{\mathbb{B}}_{\tau_n}((1 - \varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n) - z\right\| \\
&\quad = \left\|J^{\mathbb{B}}_{\tau_n}\left((1 - \varrho_n)\left(\mathbb{S}x_n - \frac{\tau_n}{1 - \varrho_n}\mathbb{A}\mathbb{S}x_n\right)\right)\right. \\
&\qquad \left. - J^{\mathbb{B}}_{\tau_n}\left(\varrho_n z + (1 - \varrho_n)\left(z - \frac{\tau_n}{1 - \varrho_n}\mathbb{A}z\right)\right)\right\| \\
&\quad \leq \left\|\left((1 - \varrho_n)\left(\mathbb{S}x_n - \frac{\tau_n}{1 - \varrho_n}\mathbb{A}\mathbb{S}x_n\right)\right)\right. \\
&\qquad \left. - \left(\varrho_n z + (1 - \varrho_n)\left(z - \frac{\tau_n}{1 - \varrho_n}\mathbb{A}z\right)\right)\right\| \\
&\quad = \left\|(1 - \varrho_n)\left(\mathbb{S}x_n - \frac{\tau_n}{1 - \varrho_n}\mathbb{A}\mathbb{S}x_n\right.\right. \\
&\qquad \left.\left. - \left(z - \frac{\tau_n}{1 - \varrho_n}\mathbb{A}z\right)\right) + \varrho_n(-z)\right\| \\
&\quad \leq (1 - \varrho_n)\|x_n - z\| + \varrho_n\|z\|.
\end{aligned} \tag{47}$$

Thus,

$$\begin{aligned}\|x_{n+1}-z\| &\le \varsigma_n \|x_n - z\| + (1-\varsigma_n)(1-\varrho_n)\|x_n - z\| \\ &\quad + (1-\varsigma_n)\varrho_n\|z\| \\ &= [1-\varrho_n(1-\varsigma_n)]\|x_n - z\| + (1-\varsigma_n)\varrho_n\|z\|.\end{aligned} \tag{48}$$

By induction, we have

$$\|x_{n+1}-z\| \le \max\{\|x_0 - z\|, \|z\|\}. \tag{49}$$

Therefore, $\{x_n\}$ is bounded.

From (4) and (47), we derive

$$\begin{aligned}&\left\|(1-\varrho_n)\left(\left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n\right) - \left(z - \frac{\tau_n}{1-\varrho_n}\mathbb{A}z\right)\right)\right. \\ &\qquad \left. + \varrho_n(-z)\right\|^2 \\ &\le (1-\varrho_n)\left\|\left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n\right) - \left(z - \frac{\tau_n}{1-\varrho_n}\mathbb{A}z\right)\right\|^2 \\ &\qquad + \varrho_n\|z\|^2 \\ &= (1-\varrho_n)\left\|(\mathbb{S}x_n - z) - \frac{\tau_n}{1-\varrho_n}(\mathbb{A}\mathbb{S}x_n - Az)\right\|^2 + \varrho_n\|z\|^2 \\ &= (1-\varrho_n)\left(\|\mathbb{S}x_n - z\|^2 - \frac{2\tau_n}{1-\varrho_n}\langle \mathbb{A}\mathbb{S}x_n - \mathbb{A}z, \mathbb{S}x_n - z\rangle\right. \\ &\qquad \left. + \frac{\tau_n^2}{(1-\varrho_n)^2}\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2\right) + \varrho_n\|z\|^2 \\ &\le (1-\varrho_n)\left(\|x_n - z\|^2 - \frac{2\varrho\tau_n}{1-\varrho_n}\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2\right. \\ &\qquad \left. + \frac{\tau_n^2}{(1-\varrho_n)^2}\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2\right) + \varrho_n\|z\|^2 \\ &= (1-\varrho_n)\left(\|x_n - z\|^2 + \frac{\tau_n}{(1-\varrho_n)^2}(\tau_n - 2(1-\varrho_n)\varrho)\right. \\ &\qquad \left. \times \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2\right) + \varrho_n\|z\|^2.\end{aligned} \tag{50}$$

Set $u_n = (1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n$ for all $n \ge 0$. Since $\tau_n - 2(1-\varrho_n)\varrho \le 0$ for all $n \ge 0$, we obtain

$$\begin{aligned}&\left\|J_{\tau_n}^{\mathbb{B}}u_n - z\right\|^2 \\ &\quad \le (1-\varrho_n)\left(\|x_n - z\|^2 + \frac{\tau_n}{(1-\varrho_n)^2}(\tau_n - 2(1-\varrho_n)\varrho)\right. \\ &\qquad \left. \times \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2\right) + \varrho_n\|z\|^2.\end{aligned} \tag{51}$$

From (46), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \left\|\varsigma_n(x_n - z) + (1-\varsigma_n)\left(J_{\tau_n}^{\mathbb{B}}u_n - z\right)\right\|^2 \\ &\le \varsigma_n\|x_n - z\|^2 + (1-\varsigma_n)\left\|J_{\tau_n}^{\mathbb{B}}u_n - z\right\|^2.\end{aligned} \tag{52}$$

Set $y_n = J_{\tau_n}^{\mathbb{B}}((1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n)$ for all $n \ge 0$. Then $x_{n+1} = \varsigma_n x_n + (1-\varsigma_n)y_n$ for all $n \ge 0$. Next, we estimate $\|x_{n+1} - x_n\|$. In fact, we have

$$\begin{aligned}\|y_{n+1}-y_n\| &= \left\|J_{\tau_{n+1}}^{\mathbb{B}}u_{n+1} - J_{\tau_n}^{\mathbb{B}}u_n\right\| \\ &\le \left\|J_{\tau_{n+1}}^{\mathbb{B}}u_{n+1} - J_{\tau_{n+1}}^{\mathbb{B}}u_n\right\| + \left\|J_{\tau_{n+1}}^{\mathbb{B}}u_n - J_{\tau_n}^{\mathbb{B}}u_n\right\| \\ &\le \|((1-\varrho_{n+1})\mathbb{S}x_{n+1} - \tau_{n+1}\mathbb{A}\mathbb{S}x_{n+1}) \\ &\qquad - ((1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n)\| \\ &\qquad + \left\|J_{\tau_{n+1}}^{\mathbb{B}}u_n - J_{\tau_n}^{\mathbb{B}}u_n\right\| \\ &= \|(I-\tau_{n+1}\mathbb{A})\mathbb{S}x_{n+1} - (I-\tau_{n+1}\mathbb{A})\mathbb{S}x_n \\ &\qquad + (\tau_n - \tau_{n+1})\mathbb{A}\mathbb{S}x_n + \varrho_n\mathbb{S}x_n - \varrho_{n+1}\mathbb{S}x_{n+1}\| \\ &\qquad + \left\|J_{\tau_{n+1}}^{\mathbb{B}}u_n - J_{\tau_n}^{\mathbb{B}}u_n\right\| \\ &\le \|(I-\tau_{n+1}\mathbb{A})\mathbb{S}x_{n+1} - (I-\tau_{n+1}\mathbb{A})\mathbb{S}x_n\| \\ &\qquad + |\tau_{n+1} - \tau_n|\,\|\mathbb{A}\mathbb{S}x_n\| + \varrho_n\|\mathbb{S}x_n\| \\ &\qquad + \varrho_{n+1}\|\mathbb{S}x_{n+1}\| + \left\|J_{\tau_{n+1}}^{\mathbb{B}}u_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|.\end{aligned} \tag{53}$$

Since $I - \tau_{n+1}\mathbb{A}$ is nonexpansive for $\tau_{n+1} \in (0, 2\varrho)$, we have

$$\begin{aligned}&\|(I-\tau_{n+1}\mathbb{A})\mathbb{S}x_{n+1} - (I-\tau_{n+1}\mathbb{A})\mathbb{S}x_n\| \\ &\quad \le \|\mathbb{S}x_{n+1} - \mathbb{S}x_n\| \le \|x_{n+1} - x_n\|.\end{aligned} \tag{54}$$

From (13), we have

$$J_{\tau_{n+1}}^{\mathbb{B}}u_n = J_{\tau_n}^{\mathbb{B}}\left(\frac{\tau_n}{\tau_{n+1}}u_n + \left(1 - \frac{\tau_n}{\tau_{n+1}}\right)J_{\tau_{n+1}}^{\mathbb{B}}u_n\right). \tag{55}$$

It follows that

$$\begin{aligned}&\left\|J_{\tau_{n+1}}^{\mathbb{B}}u_n - J_{\tau_n}^{\mathbb{B}}u_n\right\| \\ &\quad = \left\|J_{\tau_n}^{\mathbb{B}}\left(\frac{\tau_n}{\tau_{n+1}}u_n + \left(1 - \frac{\tau_n}{\tau_{n+1}}\right)J_{\tau_{n+1}}^{\mathbb{B}}u_n\right) - J_{\tau_n}^{\mathbb{B}}u_n\right\| \\ &\quad \le \left\|\left(\frac{\tau_n}{\tau_{n+1}}u_n + \left(1 - \frac{\tau_n}{\tau_{n+1}}\right)J_{\tau_{n+1}}^{\mathbb{B}}u_n\right) - u_n\right\| \\ &\quad \le \frac{|\tau_{n+1} - \tau_n|}{\tau_{n+1}}\left\|u_n - J_{\tau_{n+1}}^{\mathbb{B}}u_n\right\|.\end{aligned} \tag{56}$$

So,

$$\begin{aligned}\|y_{n+1}-y_n\| \le{}& \|x_{n+1}-x_n\| + |\tau_{n+1}-\tau_n| \, \|\mathbb{A}\mathbb{S}x_n\| \\ &+ \varrho_n \|\mathbb{S}x_n\| \\ &+ \varrho_{n+1}\|\mathbb{S}x_{n+1}\| + \frac{|\tau_{n+1}-\tau_n|}{\tau_{n+1}} \left\|u_n - J^{\mathbb{B}}_{\tau_{n+1}}u_n\right\|.\end{aligned} \tag{57}$$

Then,

$$\begin{aligned}&\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\| \\ &\quad\le |\tau_{n+1}-\tau_n| \, \|\mathbb{A}\mathbb{S}x_n\| + \varrho_n \|\mathbb{S}x_n\| \\ &\qquad+ \varrho_{n+1}\|\mathbb{S}x_{n+1}\| + \frac{|\tau_{n+1}-\tau_n|}{\tau_{n+1}} \left\|u_n - J^{\mathbb{B}}_{\tau_{n+1}}u_n\right\|.\end{aligned} \tag{58}$$

Since $\varrho_n \to 0$, $\tau_{n+1}-\tau_n \to 0$ and $\underline{\lim}_{n\to\infty}\tau_n > 0$, we obtain

$$\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|\right) \le 0. \tag{59}$$

By Lemma 6, we get

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{60}$$

Consequently, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\varsigma_n)\|y_n - x_n\| = 0. \tag{61}$$

From (51) and (52), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 \le{}& \varsigma_n\|x_n-z\|^2 + (1-\varsigma_n)\left\|J^{\mathbb{B}}_{\tau_n}u_n - z\right\|^2 \\ \le{}& (1-\varsigma_n)(1-\varrho_n) \\ &\times\left(\|x_n-z\|^2 + \frac{\tau_n}{(1-\varrho_n)^2}(\tau_n - 2(1-\varrho_n)\varrho) \right.\\ &\left.\times\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2\right) \\ &+ (1-\varsigma_n)\varrho_n\|z\|^2 + \varsigma_n\|x_n-z\|^2 \\ ={}& [1-(1-\varsigma_n)\varrho_n]\|x_n-z\|^2 \\ &+ \frac{(1-\varsigma_n)\tau_n}{1-\varrho_n}(\tau_n - 2(1-\varrho_n)\varrho)\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2 \\ &+ (1-\varsigma_n)\varrho_n\|z\|^2 \\ \le{}& \|x_n-z\|^2 + \frac{(1-\varsigma_n)\tau_n}{1-\varrho_n}(\tau_n - 2(1-\varrho_n)\varrho) \\ &\times\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2 + (1-\varsigma_n)\varrho_n\|z\|^2.\end{aligned} \tag{62}$$

Then, we obtain

$$\begin{aligned}&\frac{(1-\varsigma_n)\tau_n}{(1-\varrho_n)}(2(1-\varrho_n)\varrho - \tau_n)\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2 \\ &\quad\le \|x_n-z\|^2 - \|x_{n+1}-z\|^2 + (1-\varsigma_n)\varrho_n\|z\|^2 \\ &\quad\le (\|x_n-z\| - \|x_{n+1}-z\|)\|x_{n+1}-x_n\| \\ &\qquad+ (1-\varsigma_n)\varrho_n\|z\|^2.\end{aligned} \tag{63}$$

Since $\lim_{n\to\infty}\varrho_n = 0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, and $\underline{\lim}_{n\to\infty}((1-\varsigma_n)\tau_n/(1-\varrho_n))(2(1-\varrho_n)\varrho-\tau_n) > 0$, we have

$$\lim_{n\to\infty}\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\| = 0. \tag{64}$$

Next, we show $\|x_n - \mathbb{S}x_n\| \to 0$. By using the firm nonexpansivity of $J^{\mathbb{B}}_{\tau_n}$, we have

$$\begin{aligned}&\left\|J^{\mathbb{B}}_{\tau_n}u_n - z\right\|^2 \\ &\quad= \left\|J^{\mathbb{B}}_{\tau_n}((1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n) - J^{\mathbb{B}}_{\tau_n}(z-\tau_n\mathbb{A}z)\right\|^2 \\ &\quad\le \left\langle (1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n - (z-\tau_n\mathbb{A}z), J^{\mathbb{B}}_{\tau_n}u_n - z\right\rangle \\ &\quad= \frac{1}{2}\left(\|(1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n - (z-\tau_n\mathbb{A}z)\|^2\right. \\ &\qquad+ \left\|J^{\mathbb{B}}_{\tau_n}u_n - z\right\|^2 \\ &\qquad\left.- \left\|(1-\varrho_n)\mathbb{S}x_n - \tau_n(\mathbb{A}\mathbb{S}x_n - \mathbb{A}z) - J^{\mathbb{B}}_{\tau_n}u_n\right\|^2\right).\end{aligned} \tag{65}$$

Observe that

$$\begin{aligned}&\|(1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n - (z-\tau_n\mathbb{A}z)\|^2 \\ &\quad= \left\|(1-\varrho_n)\left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n\right.\right. \\ &\qquad\left.\left.-\left(z - \frac{\tau_n}{1-\varrho_n}\mathbb{A}z\right)\right) + \varrho_n(-z)\right\|^2 \\ &\quad\le (1-\varrho_n)\left\|\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n\right. \\ &\qquad\left.-\left(z - \frac{\tau_n}{1-\varrho_n}\mathbb{A}z\right)\right\|^2 + \varrho_n\|z\|^2 \\ &\quad\le (1-\varrho_n)\|x_n-z\|^2 + \varrho_n\|z\|^2.\end{aligned} \tag{66}$$

Hence,

$$\begin{aligned}\left\|J^{\mathbb{B}}_{\tau_n}u_n - z\right\|^2 \le{}& \frac{1}{2}\left((1-\varrho_n)\|x_n-z\|^2 + \varrho_n\|z\|^2 + \left\|J^{\mathbb{B}}_{\tau_n}u_n - z\right\|^2\right. \\ &\left.- \left\|(1-\varrho_n)\mathbb{S}x_n - J^{\mathbb{B}}_{\tau_n}u_n - \tau_n(\mathbb{A}\mathbb{S}x_n - \mathbb{A}z)\right\|^2\right).\end{aligned} \tag{67}$$

It follows that

$$\begin{aligned}\left\|J_{\tau_n}^{\mathbb{B}}u_n - z\right\|^2 &\leq (1-\varrho_n)\left\|x_n - z\right\|^2 + \varrho_n\|z\|^2 \\ &\quad - \left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n - \tau_n(\mathbb{A}\mathbb{S}x_n - \mathbb{A}z)\right\|^2 \\ &= (1-\varrho_n)\left\|x_n - z\right\|^2 + \varrho_n\|z\|^2 \\ &\quad - \left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|^2 \\ &\quad + 2\tau_n\left\langle(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n, \mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\rangle \\ &\quad - \tau_n^2\left\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\|^2 \\ &\leq (1-\varrho_n)\left\|x_n - z\right\|^2 + \varrho_n\|z\|^2 \\ &\quad - \left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|^2 \\ &\quad + 2\tau_n\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|\left\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\|.\end{aligned} \tag{68}$$

This together with (52) implies that

$$\begin{aligned}\left\|x_{n+1} - z\right\|^2 &\leq \varsigma_n\left\|x_n - z\right\|^2 + (1-\varsigma_n)(1-\varrho_n)\left\|x_n - z\right\|^2 \\ &\quad + (1-\varsigma_n)\varrho_n\|z\|^2 \\ &\quad - (1-\varsigma_n)\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|^2 \\ &\quad + 2\tau_n(1-\varsigma_n)\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\| \\ &\quad \times\left\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\| \\ &= \left[1-(1-\varsigma_n)\varrho_n\right]\left\|x_n - z\right\|^2 + (1-\varsigma_n)\varrho_n\|z\|^2 \\ &\quad - (1-\varsigma_n)\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|^2 \\ &\quad + 2\tau_n(1-\varsigma_n)\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\| \\ &\quad \times\left\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\|.\end{aligned} \tag{69}$$

Hence,

$$\begin{aligned}&(1-\varsigma_n)\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|^2 \\ &\quad\leq \left\|x_n - z\right\|^2 - \left\|x_{n+1} - z\right\|^2 - (1-\varsigma_n)\varrho_n\left\|x_n - z\right\|^2 \\ &\qquad + (1-\varsigma_n)\varrho_n\|z\|^2 + 2\tau_n(1-\varsigma_n) \\ &\qquad \times\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|\left\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\| \\ &\quad\leq \left(\left\|x_n - z\right\| + \left\|x_{n+1} - z\right\|\right)\left\|x_{n+1} - x_n\right\| \\ &\qquad + (1-\varsigma_n)\varrho_n\|z\|^2 + 2\tau_n(1-\varsigma_n) \\ &\qquad \times\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|\left\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\|.\end{aligned} \tag{70}$$

Since $\overline{\lim}_{n\to\infty}\varsigma_n < 1$, $\|x_{n+1}-x_n\| \to 0$, $\varrho_n \to 0$, and $\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\| \to 0$ (by (60)), we deduce

$$\lim_{n\to\infty}\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\| = 0. \tag{71}$$

This indicates that

$$\lim_{n\to\infty}\left\|\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\| = 0. \tag{72}$$

Combining (60) and (72), we get

$$\lim_{n\to\infty}\left\|x_n - \mathbb{S}x_n\right\| = 0. \tag{73}$$

Put $\tilde{x} = \lim_{t\to 0+}x_t = \mathrm{proj}_{\mathrm{Fix}(\mathbb{S})\cap(\mathbb{A}+\mathbb{B})^{-1}(0)}(0)$, where x_t is the net defined by (17). We will finally show that $x_n \to \tilde{x}$.

Set $v_n = x_n - (\tau_n/(1-\varrho_n))(\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x})$ for all n. Take $z = \tilde{x}$ in (64) to get $\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\| \to 0$. First, we prove $\overline{\lim}_{n\to\infty}\langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\rangle \leq 0$. We take a subsequence $\{\mathbb{S}x_{n_i}\}$ of $\{\mathbb{S}x_n\}$ such that

$$\overline{\lim_{n\to\infty}}\left\langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\right\rangle = \lim_{i\to\infty}\left\langle -\tilde{x}, \mathbb{S}x_{n_i} - \tilde{x}\right\rangle. \tag{74}$$

It is clear that $\{\mathbb{S}x_{n_i}\}$ is bounded due to the boundedness of $\{\mathbb{S}x_n\}$ and $\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\| \to 0$. Then, there exists a subsequence $\{\mathbb{S}x_{n_{i_j}}\}$ of $\{\mathbb{S}x_{n_i}\}$ which converges weakly to some point $w \in \mathscr{C}$. Hence, $\{x_{n_{i_j}}\}$ and $\{y_{n_{i_j}}\}$ also converge weakly to w because of $\|\mathbb{S}x_{n_{i_j}} - x_{n_{i_j}}\| \to 0$ and $\|x_{n_{i_j}} - y_{n_{i_j}}\| \to 0$. By the demiclosedness principle of the nonexpansive mapping (see Lemma 5) and (73), we deduce $w \in \mathrm{Fix}(\mathbb{S})$. Furthermore, by similar argument as that of Theorem 8, we can show that w is also in $(\mathbb{A}+\mathbb{B})^{-1}(0)$. Hence, we have $w \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}(0)$. This implies that

$$\begin{aligned}\overline{\lim_{n\to\infty}}\left\langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\right\rangle &= \lim_{j\to\infty}\left\langle -\tilde{x}, \mathbb{S}x_{n_{i_j}} - \tilde{x}\right\rangle \\ &= \left\langle -\tilde{x}, w - \tilde{x}\right\rangle.\end{aligned} \tag{75}$$

Note that $\tilde{x} = \mathrm{proj}_{\mathrm{Fix}(\mathbb{S})\cap(\mathbb{A}+\mathbb{B})^{-1}(0)}(0)$. Then, $\langle -\tilde{x}, w - \tilde{x}\rangle \leq 0$, $w \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}(0)$. Therefore,

$$\overline{\lim_{n\to\infty}}\left\langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\right\rangle \leq 0. \tag{76}$$

From (46), we have

$$\begin{aligned}&\left\|x_{n+1} - \tilde{x}\right\|^2 \\ &\quad\leq \varsigma_n\left\|x_n - \tilde{x}\right\|^2 + (1-\varsigma_n)\left\|J_{\tau_n}^{\mathbb{B}}u_n - \tilde{x}\right\|^2 \\ &\quad= \varsigma_n\left\|x_n - \tilde{x}\right\|^2 + (1-\varsigma_n)\left\|J_{\tau_n}^{\mathbb{B}}u_n - J_{\tau_n}^{\mathbb{B}}(\tilde{x} - \tau_n A\tilde{x})\right\|^2 \\ &\quad\leq \varsigma_n\left\|x_n - \tilde{x}\right\|^2 + (1-\varsigma_n)\left\|u_n - (\tilde{x} - \tau_n\mathbb{A}\tilde{x})\right\|^2\end{aligned}$$

$$
\begin{aligned}
&= \varsigma_n \|x_n - \tilde{x}\|^2 + (1-\varsigma_n) \\
&\quad \times \left\| (1-\varrho_n) \mathbb{S}x_n - \tau_n \mathbb{A}\mathbb{S}x_n - (\tilde{x} - \tau_n \mathbb{A}\tilde{x}) \right\|^2 \\
&= (1-\varsigma_n) \left\| (1-\varrho_n) \left(\left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n} \mathbb{A}\mathbb{S}x_n \right) \right.\right. \\
&\qquad \left.\left. - \left(\tilde{x} - \frac{\tau_n}{1-\varrho_n} \mathbb{A}\tilde{x} \right) \right) \right. \\
&\qquad \left. + \varrho_n (-\tilde{x}) \right\|^2 + \varsigma_n \|x_n - \tilde{x}\|^2 \\
&= \varsigma_n \|x_n - \tilde{x}\|^2 + (1-\varsigma_n) \\
&\quad \times \left((1-\varrho_n)^2 \left\| \left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n} \mathbb{A}\mathbb{S}x_n \right) \right.\right. \\
&\qquad \left. - \left(\tilde{x} - \frac{\tau_n}{1-\varrho_n} \mathbb{A}\tilde{x} \right) \right\|^2 \\
&\qquad + 2\varrho_n (1-\varrho_n) \left\langle -\tilde{x}, \left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n} \mathbb{A}\mathbb{S}x_n \right) \right. \\
&\qquad \left.\left. - \left(\tilde{x} - \frac{\tau_n}{1-\varrho_n} \mathbb{A}\tilde{x} \right) \right\rangle + \varrho_n^2 \|\tilde{x}\|^2 \right) \\
&\le \varsigma_n \|x_n - \tilde{x}\|^2 + (1-\varsigma_n) \\
&\quad \times \left((1-\varrho_n)^2 \|x_n - \tilde{x}\|^2 + 2\varrho_n \tau_n \langle -\tilde{x}, \mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x} \rangle \right. \\
&\qquad \left. + 2\varrho_n (1-\varrho_n) \langle -\tilde{x}, \mathbb{S}x_n - \tilde{x} \rangle + \varrho_n^2 \|\tilde{x}\|^2 \right) \\
&\le \varsigma_n \|x_n - \tilde{x}\|^2 + (1-\varsigma_n) \\
&\quad \times \left((1-\varrho_n)^2 \|x_n - \tilde{x}\|^2 + 2\varrho_n \tau_n \|\tilde{x}\| \, \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\| \right. \\
&\qquad \left. + 2\varrho_n (1-\varrho_n) \langle -\tilde{x}, \mathbb{S}x_n - \tilde{x} \rangle + \varrho_n^2 \|\tilde{x}\|^2 \right) \\
&\le [1 - 2(1-\varsigma_n)\varrho_n] \|x_n - \tilde{x}\|^2 \\
&\quad + 2\varrho_n (1-\varsigma_n) \tau_n \|\tilde{x}\| \, \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\| \\
&\quad + 2\varrho_n (1-\varsigma_n)(1-\varrho_n) \langle -\tilde{x}, \mathbb{S}x_n - \tilde{x} \rangle \\
&\quad + (1-\varsigma_n) \varrho_n^2 \left(\|\tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 \right) \\
&= [1 - 2(1-\varsigma_n)\varrho_n] \|x_n - \tilde{x}\|^2 \\
&\quad + 2(1-\varsigma_n)\varrho_n \left\{ \tau_n \|\tilde{x}\| \, \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\| \right. \\
&\qquad + (1-\varrho_n) \langle -\tilde{x}, \mathbb{S}x_n - \tilde{x} \rangle \\
&\qquad \left. + \varrho_n \left(\|\tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 \right) \right\}.
\end{aligned} \tag{77}
$$

It is clear that $\sum_n 2(1-\varsigma_n)\varrho_n = \infty$ and

$$
\begin{aligned}
&\limsup_{n\to\infty} \left\{ \tau_n \|\tilde{x}\| \, \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\| + (1-\varrho_n) \right. \\
&\qquad \left. \times \langle -\tilde{x}, \mathbb{S}x_n - \tilde{x} \rangle + \varrho_n \left(\|\tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 \right) \right\} \le 0.
\end{aligned} \tag{78}
$$

By Lemma 7, we conclude that $x_n \to \tilde{x}$. This completes the proof. □

Corollary 11. *Suppose that $(\mathbb{A} + \mathbb{B})^{-1}(0) \neq \emptyset$. Let τ be a constant satisfying $a \le \tau \le b$, where $[a, b] \subset (0, 2\varrho)$. For $t \in (0, 1 - \tau/(2\varrho))$, let $\{x_t\} \subset \mathscr{C}$ be a net generated by*

$$
x_t = J_\tau^{\mathbb{B}} \left((1-t) x_t - \tau \mathbb{A} x_t \right). \tag{79}
$$

Then the net $\{x_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x} = \mathrm{proj}_{(\mathbb{A}+\mathbb{B})^{-1}(0)}(0)$ which is the minimum norm element in $(\mathbb{A} + \mathbb{B})^{-1}(0)$.

Corollary 12. *Suppose that $(\mathbb{A} + \mathbb{B})^{-1}(0) \neq \emptyset$. For given $x_0 \in \mathscr{C}$, let $\{x_n\} \subset \mathbb{C}$ be a sequence generated by*

$$
x_{n+1} = \varsigma_n x_n + (1-\varsigma_n) J_{\tau_n}^{\mathbb{B}} \left((1-\varrho_n) x_n - \tau_n \mathbb{A} x_n \right) \tag{80}
$$

for all $n \ge 0$, where $\{\tau_n\} \subset (0, 2\varrho)$, $\{\varrho_n\} \subset (0, 1)$, and $\{\varsigma_n\} \subset (0, 1)$ satisfy

(i) $\lim_{n\to\infty} \varrho_n = 0$ *and* $\sum_n \varrho_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \varsigma_n \le \limsup_{n\to\infty} \varsigma_n < 1$;

(iii) $a(1-\varrho_n) \le \tau_n \le b(1-\varrho_n)$, *where* $[a, b] \subset (0, 2\varrho)$ *and* $\lim_{n\to\infty}(\tau_{n+1} - \tau_n) = 0$.

Then $\{x_n\}$ converges strongly to a point $\tilde{x} = \mathrm{proj}_{(\mathbb{A}+\mathbb{B})^{-1}(0)}(0)$ which is the minimum norm element in $(\mathbb{A} + \mathbb{B})^{-1}(0)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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