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Research Article

On q-Bleimann, Butzer, and Hahn-Type Operators

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Sequence of q-Bleimann, Butzer, and Hahn operators which is based on a continuously differentiable function τ on $\mathbb{R}^+$, with $\tau(0) = 0$, $\inf \tau'(x) \geq 1$, has been considered. Uniform approximation by such a sequence has been studied and degree of approximation by the operators has been obtained. Moreover, shape preserving properties of the sequence of operators have been investigated.

1. Introduction

In 1987, Lupaş [1] introduced the first q-analogue of Bernstein operators and investigated its approximation and shape preserving properties. Another q-generalization of the classical Bernstein polynomials is due to Phillips [2]. After that many generalizations of positive linear operators based on q-integers were introduced and studied by several authors. Some are in [3–13].

Bleimann et al. [14] proposed a sequence of positive linear operators L_n defined by

$$L_n(f;x) = \frac{1}{(1+x)^n}\sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k}x^k, \quad x \geq 0,\ n \in \mathbb{N} \tag{1}$$

for $f \in C[0,\infty)$, where $C[0,\infty)$ denote the space of all continuous and real valued functions defined on $[0,\infty)$. Also the authors proved that $L_n(f;x) \to f(x)$ as $n \to \infty$ pointwise on $[0,\infty)$ for any $f \in C_B[0,\infty)$, where $C_B[0,\infty)$ denote the space of all bounded functions from $C[0,\infty)$. It is well known that

$$\|f\|_{C_B} = \sup_{x\geq 0}|f(x)| \tag{2}$$

defines a norm on $C_B[0,\infty)$. Moreover, they showed that the convergence is uniform on each compact subset of $[0,\infty)$. In [15], using test functions $(t/(1+t))^\nu$ for $\nu = 0,1,2$, Gadjiev and Çakar stated a Korovkin-type theorem for the uniform approximation of functions belonging to some suitable function spaces by some linear positive operators. As an application of this result, they proved uniform approximation by Bleimann, Butzer, and Hahn operators. Further results concerning such a sequence of operators and its generalizations may be found in [16–19].

Now, we recall some notations from q-analysis [20, 21].

The q-integer $[n]$ and the q-factorial $[n]!$ are defined by

$$[n] := [n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1, \end{cases} \quad \text{for } n \in \mathbb{N} \tag{3}$$

$[0] = 0$, and

$$[n]! := \begin{cases} [1]_q[2]_q \cdots [n]_q, & n = 1,2,\ldots \\ 1, & n = 0, \end{cases} \quad \text{for } n \in \mathbb{N},\ [0]! = 1, \tag{4}$$

respectively, where $q > 0$. For integers $n \geq r \geq 0$ the q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q!\,[n-r]_q!}. \tag{5}$$

Moreover, Euler identity is given by

$$\prod_{s=0}^{n-1}(1+q^s x) = \sum_{k=0}^{n} q^{k(k-1)/2}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k. \tag{6}$$

Aral and Doğru [22] constructed the q-Bleimann, Butzer, and Hahn operators as

$$L_{n,q}(f;x) = \frac{1}{\ell_n(x)} \sum_{k=0}^{n} f\left(\frac{[k]}{[n-k+1]q^k}\right) q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \tag{7}$$

where

$$\ell_n(x) = \prod_{s=0}^{n-1} (1 + q^s x) \tag{8}$$

and f is defined on the semiaxis $[0,\infty)$. The authors studied Korovkin-type approximation properties by using the test functions $(t/(1+t))^{\nu}$ for $\nu = 0,1,2$. Moreover, they obtained rate of convergence of the operators and proved that rate of the q-Bleimann, Butzer, and Hahn operators is better than the classical one. A generalization of the q-Bleimann, Butzer, and Hahn operators was introduced by Agratini and Nowak in [23]. In this paper, the authors gave representation of the operators in terms of q-differences and investigated some approximation properties.

A Voronovskaja-type result and monotonicity properties of these operators are investigated in [24].

In [25], the authors introduced a new generalization of Bernstein polynomials denoted by B_n^{τ} and defined as

$$\begin{aligned} B_n^{\tau}(f;x) &:= B_n\left(f \circ \tau^{-1}; \tau(x)\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} \tau^k(x)(1-\tau(x))^{n-k} \left(f \circ \tau^{-1}\right)\left(\frac{k}{n}\right), \end{aligned} \tag{9}$$

where B_n is the nth Bernstein polynomial, $f \in C[0,1]$, $x \in [0,1]$, and τ is a function that is continuously differentiable of infinite order on $[0,1]$ such that $\tau(0) = 0$, $\tau(1) = 1$, and $\tau'(x) > 0$ for $x \in [0,1]$. Also, the authors studied some shape preserving and convergence properties concerning the generalized Bernstein operators $B_n^{\tau}(f;x)$.

In [26], Aral et al. constructed sequences of Szasz-Mirakyan operators which are based on a function ρ. They studied weighted approximation properties and Voronovskaja-type results for these operators. They also showed that the sequence of the generalized Szász-Mirakyan operators is monotonically nonincreasing under the ρ-convexity of the original function. A similar generalization for Bleimann, Butzer, and Hahn operators is studied by Söylemez [27]. Also the class H_{ω}^{τ} was defined, a Korovkin-type theorem was given for the functions in this class, and uniform convergence of the generalized Bleimann, Butzer, and Hahn operators was obtained [27]. Moreover, the monotonicity properties of the operators were investigated.

Now we recall the definition of H_{ω}^{τ} that is a subspace of $C_B[0,\infty)$ [27].

Let ω be a general modulus of continuity, satisfying the following properties:

(a) ω is continuous, nonnegative, and increasing function on $[0,\infty)$,

(b) $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$,

(c) $\lim_{\delta \to 0} \omega(\delta) = 0$.

The space of all real valued functions f defined on $[0,\infty)$ satisfying

$$\left| f(x) - f(y) \right| \leq \omega \left| \frac{\tau(x)}{1+\tau(x)} - \frac{\tau(y)}{1+\tau(y)} \right| \tag{10}$$

for all $x, y \in [0,\infty)$ is denoted by H_{ω}^{τ}.

It is clear from condition (b) that we have

$$\omega(n\delta) \leq n\omega(\delta), \quad n \in \mathbb{N} \tag{11}$$

and one can get from the condition (a) that for any $\lambda > 0$

$$\omega(\lambda\delta) \leq \omega((1 + [|\lambda|])\lambda) \leq 1 + \lambda\omega(\delta), \tag{12}$$

where $[|\lambda|]$ denotes the greatest integer that is not greater than λ.

Now we define a new generalization of q-Bleimann, Butzer, and Hahn operators for $f \in C[0,\infty)$ by

$$\begin{aligned} L_{n,q}^{\tau}(f;x) &= \frac{1}{\ell_n^{\tau}(x)} \sum_{k=0}^{n} \left(f \circ \tau^{-1}\right)\left(\frac{[k]}{[n-k+1]q^k}\right) \\ &\quad \cdot q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \tau(x)^k, \end{aligned} \tag{13}$$

where

$$\ell_n^{\tau}(x) = \prod_{s=0}^{n-1} \left(1 + q^s \tau(x)\right), \tag{14}$$

and τ is a continuously differentiable function on $[0,\infty)$ such that

$$\begin{aligned} \tau(0) &= 0, \\ \inf_{x \in [0,\infty)} \tau'(x) &\geq 1. \end{aligned} \tag{15}$$

An example of such a function τ is given in [26]. Note that, in the setting of the operators (13), we have

$$L_{n,q}^{\tau} f := L_{n,q}\left(f \circ \tau^{-1}\right) \circ \tau, \tag{16}$$

where the operators $L_{n,q}$ are defined by (7). If $\tau = e_1$, then $L_{n,q}^{\tau} = L_{n,q}$. Obviously, we have

$$\begin{aligned} L_{n,q}^{\tau}(1;x) &= 1, \\ L_{n,q}^{\tau}\left(\frac{\tau}{1+\tau};x\right) &= \frac{[n]}{[n+1]} \frac{\tau(x)}{1+\tau(x)}, \\ L_{n,q}^{\tau}\left(\left(\frac{\tau}{1+\tau}\right)^2;x\right) &= \frac{[n][n-1]}{[n+1]^2} q^2 \frac{\tau^2(x)}{(1+\tau(x))(1+q\tau(x))} \\ &\quad + \frac{[n]}{[n+1]^2} \frac{\tau(x)}{1+\tau(x)}. \end{aligned} \tag{17}$$

In this study, we consider a generalization of q-Bleimann, Butzer, and Hahn operators in the sense of [26], we investigate uniform convergence of $\{L_{n,q}^{\tau}(f;x)\}_{n\in\mathbb{N}}$ to $f(x)$ on $[0,\infty)$ for $f \in H_{\omega}^{\tau}$, and we obtain the degree of approximation. Moreover, we study shape preserving properties under τ-convexity of the function. Our results show that the new operators are sensitive to the rate of convergence to f, depending on the selection of τ. For the particular case $\tau(x) = x$, the previous results for q-Bleimann Butzer and Hahn operators are obtained.

In order to ensure that the convergence properties holds, the author will assume $q = q_n$ is a sequence such that $q_n \to 1$ as $n \to \infty$ for $0 < q_n < 1$, as in [22].

Definition 1. Let $x_0, x_1, \ldots, x_n$ be distinct points in the domain of f. Denote

$$f[x_0, x_1, \ldots, x_n] = \sum_{r=0}^{n} \frac{f(x_r)}{\prod_{j\neq r}^{n}(x_r - x_j)}, \tag{18}$$

where r remains fixed and j takes all values from 0 to n, excluding r.

Definition 2. A continuous, real valued function f is said to be convex in $D \subseteq [0,\infty)$, if

$$f\left(\sum_{i=1}^{m}\alpha_i x_i\right) \leq \sum_{i=1}^{m}\alpha_i f(x_i) \tag{19}$$

for every $x_1, x_2, \ldots, x_m \in D$ and for every nonnegative number of $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 1$.

In [25] Cárdenas-Morales et al. introduced the following definition of τ-convexity of a continuous function.

Definition 3. A continuous, real valued function f is said to be τ-convex in D, if $f \circ \tau^{-1}$ is convex in the sense of Definition 2.

2. Approximation Properties

In this section we deal with the promised approximation properties of the sequence of q-Bleimann, Butzer, and Hahn operators. In [27], the following Korovkin-type theorem was given.

Theorem 4. *Let $\{T_n^{\tau}(f)\}_{n\in\mathbb{N}}$ be a sequence of linear positive operators from H_{ω}^{τ} to $C_B[0,\infty)$. If*

$$\lim_{n\to\infty}\left\| T_n^{\tau}\left(\left(\frac{\tau(t)}{1+\tau(t)}\right)^{\nu}; x\right) - \left(\frac{\tau(x)}{1+\tau(x)}\right)^{\nu}\right\|_{C_B} = 0 \tag{20}$$

is satisfied for $\nu = 0, 1, 2$, then for $f \in H_{\omega}^{\tau}$ one has

$$\lim_{n\to\infty}\left\|T_n^{\tau} f - f\right\|_{C_B} = 0. \tag{21}$$

Now we are ready to give the following theorem.

Theorem 5. *Suppose that $q = q_n$, $0 < q_n < 1$, and let $q_n \to 1$ as $n \to \infty$. If $L_{n,q}^{\tau}$ is the operator defined by (13), then for any $f \in H_{\omega}^{\tau}$ one has*

$$\lim_{n\to\infty}\left\|L_{n,q}^{\tau} f - f\right\|_{C_B} = 0. \tag{22}$$

Proof. According to Theorem 4 we will show that (20) holds for $L_{n,q}^{\tau}$. Obviously, from (17) we easily obtain that

$$\begin{aligned}
&L_{n,q}^{\tau}(1;x) = 1.\\
&\left\|L_{n,q}^{\tau}\left(\frac{\tau(t)}{1+\tau(t)}; x\right) - \frac{\tau(x)}{1+\tau(x)}\right\|_{C_B} = \left|\frac{[n]}{[n+1]} - 1\right|\\
&\quad \leq \left|\frac{1}{q_n} - \frac{1}{q_n[n+1]} - 1\right|,\\
&\left\|L_{n,q}^{\tau}\left(\left(\frac{\tau(t)}{1+\tau(t)}\right)^2; x\right) - \left(\frac{\tau(x)}{1+\tau(x)}\right)^2\right\|_{C_B}\\
&\quad = \sup_{x\geq 0}\left(\frac{\tau(x)}{1+\tau(x)}\right)^2\\
&\quad \cdot \left(\frac{[n][n-1]}{[n+1]^2}q_n^2\frac{(1+\tau(x))}{(1+q_n\tau(x))} - 1\right) + \frac{[n]}{[n+1]^2}\\
&\quad \cdot \frac{\tau(x)}{1+\tau(x)} \leq \frac{1}{q_n^2}\left(1 - q_n^2 - \frac{2}{[n+1]} + \frac{1}{[n+1]^2}\right).
\end{aligned} \tag{23}$$

Therefore, the conditions (20) are satisfied. By Theorem 4, the proof is completed. □

Theorem 6. *Let $q = q_n$, $0 < q_n < 1$, and let $q_n \to 1$ as $n \to \infty$. Then one has*

$$\left|L_{n,q}^{\tau}(f;x) - f(x)\right| \leq 2\omega\left(\sqrt{\mu_n^{\tau}(x)}\right), \tag{24}$$

for all $f \in H_{\omega}^{\tau}$. Here,

$$\begin{aligned}
\mu_n^{\tau}(x) &= \left(\frac{\tau(x)}{1+\tau(x)}\right)^2\\
&\quad \cdot \left(1 - 2\frac{[n]}{[n+1]} + \frac{[n][n-1]}{[n+1]^2}q_n^2\frac{(1+\tau(x))}{(1+q_n\tau(x))}\right)\\
&\quad + \frac{[n]}{[n+1]^2}\frac{\tau(x)}{1+\tau(x)}.
\end{aligned} \tag{25}$$

Proof. Firstly, from (12) and (10) we can write

$$\begin{aligned}
|f(t) - f(x)| &\leq \omega\left(\left|\frac{\tau(t)}{1+\tau(t)} - \frac{\tau(x)}{1+\tau(x)}\right|\right)\\
&\leq \left(1 + \frac{|\tau(t)/(1+\tau(t)) - \tau(x)/(1+\tau(x))|}{\delta}\right)\omega(\delta)
\end{aligned} \tag{26}$$

and considering (17), we get

$$\left|L_{n,q}^{\tau}(f;x) - f(x)\right| \leq L_{n,q}^{\tau}\left(|f(t) - f(x)|; x\right). \tag{27}$$

Equations (27) and (26) together imply that

$$\left|L_{n,q}^{\tau}(f;x)-f(x)\right| \leq \omega(\delta)\left(1+\frac{1}{\delta}L_{n,q}^{\tau}\left|\frac{\tau(t)}{1+\tau(t)}-\frac{\tau(x)}{1+\tau(x)}\right|;x\right). \tag{28}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\left|L_{n,q}^{\tau}(f;x)-f(x)\right| \leq \omega(\delta)\left(1+\frac{1}{\delta}\left[L_{n,q}^{\tau}\left(\left(\frac{\tau(t)}{1+\tau(t)}-\frac{\tau(x)}{1+\tau(x)}\right)^{2};x\right)\right]^{1/2}\right). \tag{29}$$

By choosing $\delta=\sqrt{\mu_n^{\tau}(x)}=[L_{n,q}^{\tau}((\tau(t)/(1+\tau(t))-\tau(x)/(1+\tau(x)))^{2};x)]^{1/2}$, we get

$$\left|L_{n,q}^{\tau}(f;x)-f(x)\right| \leq 2\omega\left(\sqrt{\mu_n^{\tau}(x)}\right), \tag{30}$$

where

$$\sup_{x\geq 0}\mu_n^{\tau}(x)\leq \frac{1}{[n+1]^{2}} \tag{31}$$

which concludes the proof. □

3. Shape Preserving Properties

Theorem 7. *Let f be a τ-convex function that is nonincreasing on $[0,\infty)$; then one has*

$$L_{n,q}^{\tau}(f;x)\geq L_{n+1,q}^{\tau}(f;x) \tag{32}$$

for $n\in\mathbb{N}$.

Proof. From (13), one can write

$$\begin{aligned}
L_{n,q}^{\tau}(f;x)-L_{n+1,q}^{\tau}(f;x) &= \frac{1}{\prod_{s=0}^{n}(1+q^{s}\tau(x))} \\
&\cdot\sum_{k=0}^{n}\left(f\circ\tau^{-1}\right)\left(\frac{[k]}{[n-k+1]q^{k}}\right)q^{k(k-1)/2}\begin{bmatrix}n\\k\end{bmatrix}\tau(x)^{k} \\
&\cdot(1+q^{n}\tau(x))-\frac{1}{\prod_{s=0}^{n}(1+q^{s}\tau(x))}\sum_{k=0}^{n+1}\left(f\circ\tau^{-1}\right) \\
&\cdot\left(\frac{[k]}{[n-k+2]q^{k}}\right)q^{k(k-1)/2}\begin{bmatrix}n+1\\k\end{bmatrix}\tau(x)^{k} \\
&=\frac{(\tau(x))^{n+1}}{\ell_{n+1}^{\tau}(x)}q^{n(n+1)/2}\left[\left(f\circ\tau^{-1}\right)\left(\frac{[n]}{q^{n}}\right)\right. \\
&\left.-\left(f\circ\tau^{-1}\right)\left(\frac{[n+1]}{q^{n+1}}\right)\right]+\frac{1}{\ell_{n+1}^{\tau}(x)}\sum_{k=1}^{n}\left(f\circ\tau^{-1}\right) \\
&\cdot\left(\frac{[k]}{[n-k+1]q^{k}}\right)q^{k(k-1)/2}\begin{bmatrix}n\\k\end{bmatrix}\tau(x)^{k}+\frac{1}{\ell_{n+1}^{\tau}(x)} \\
&\cdot\sum_{k=0}^{n-1}\left(f\circ\tau^{-1}\right)\left(\frac{[k]}{[n-k+1]q^{k}}\right) \\
&\cdot q^{k(k-1)/2}q^{n}\begin{bmatrix}n\\k\end{bmatrix}\tau(x)^{k+1}-\frac{1}{\ell_{n+1}^{\tau}(x)}\sum_{k=1}^{n}\left(f\circ\tau^{-1}\right) \\
&\cdot\left(\frac{[k]}{[n-k+2]q^{k}}\right)q^{k(k-1)/2}\begin{bmatrix}n+1\\k\end{bmatrix}\tau(x)^{k} \\
&=\frac{(\tau(x))^{n+1}}{\ell_{n+1}^{\tau}(x)}q^{n(n+1)/2}\left[\left(f\circ\tau^{-1}\right)\left(\frac{[n]}{q^{n}}\right)\right. \\
&\left.-\left(f\circ\tau^{-1}\right)\left(\frac{[n+1]}{q^{n+1}}\right)\right]+\frac{1}{\ell_{n+1}^{\tau}(x)}\sum_{k=0}^{n-1}\left(f\circ\tau^{-1}\right) \\
&\cdot\left(\frac{[k+1]}{[n-k]q^{k+1}}\right)q^{k(k-1)/2}q^{k}\begin{bmatrix}n\\k+1\end{bmatrix}\tau(x)^{k+1} \\
&+\frac{1}{\ell_{n+1}^{\tau}(x)}\sum_{k=0}^{n-1}\left(f\circ\tau^{-1}\right)\left(\frac{[k]}{[n-k+1]q^{k}}\right) \\
&\cdot q^{k(k-1)/2}q^{n}\begin{bmatrix}n\\k\end{bmatrix}\tau(x)^{k+1}-\frac{1}{\ell_{n+1}^{\tau}(x)}\sum_{k=0}^{n-1}\left(f\circ\tau^{-1}\right) \\
&\cdot\left(\frac{[k+1]}{[n-k+1]q^{k+1}}\right)q^{k(k-1)/2}q^{k}\begin{bmatrix}n+1\\k+1\end{bmatrix}\tau(x)^{k+1}.
\end{aligned} \tag{33}$$

Moreover, we have the following equalities that are proved in Lemma 3.1 of [24]:

$$\begin{aligned}
\begin{bmatrix}n+1\\k+1\end{bmatrix}&=\frac{[n][n+1]}{[n-k][k+1]}\begin{bmatrix}n-1\\k\end{bmatrix}, \\
\begin{bmatrix}n\\k\end{bmatrix}&=\frac{[n]}{[n-k]}\begin{bmatrix}n-1\\k\end{bmatrix}, \\
\begin{bmatrix}n\\k+1\end{bmatrix}&=\frac{[n]}{[k+1]}\begin{bmatrix}n-1\\k\end{bmatrix},
\end{aligned} \tag{34}$$

which imply

$$\begin{aligned}
L_{n,q}^{\tau}(f;x)-L_{n+1,q}^{\tau}(f;x)&=\frac{(\tau(x))^{n+1}}{\ell_{n+1}^{\tau}(x)} \\
&\cdot q^{n(n+1)/2}\left[\left(f\circ\tau^{-1}\right)\left(\frac{[n]}{q^{n}}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&-\left(f\circ\tau^{-1}\right)\left(\frac{[n+1]}{q^{n+1}}\right)\Bigg]+\frac{1}{\ell_{n+1}^{\tau}(x)}\\
&\cdot\sum_{k=0}^{n-1}q^{k}q^{k(k-1)/2}\begin{bmatrix}n-1\\k\end{bmatrix}\tau(x)^{k+1}\\
&\cdot\Bigg\{\left(f\circ\tau^{-1}\right)\left(\frac{[k+1]}{[n-k]\,q^{k+1}}\right)\frac{[n]}{[k+1]}\\
&+\left(f\circ\tau^{-1}\right)\left(\frac{[k]}{[n-k+1]\,q^{k}}\right)q^{n-k}\frac{[n]}{[n-k]}\\
&-\left(f\circ\tau^{-1}\right)\left(\frac{[k+1]}{[n-k+1]\,q^{k+1}}\right)\frac{[n]\,[n+1]}{[n-k]\,[k+1]}\Bigg\}\\
&=\frac{1}{\ell_{n+1}^{\tau}(x)}\\
&\cdot\sum_{k=0}^{n-1}q^{k}q^{k(k-1)/2}\frac{[n]\,[n+1]}{[n-k]\,[k+1]}\begin{bmatrix}n-1\\k\end{bmatrix}\tau(x)^{k+1}\\
&\cdot\Bigg\{\left(f\circ\tau^{-1}\right)\left(\frac{[k+1]}{[n-k]\,q^{k+1}}\right)\frac{[n-k]}{[n+1]}\\
&+\left(f\circ\tau^{-1}\right)\left(\frac{[k]}{[n-k+1]\,q^{k}}\right)q^{n-k}\frac{[k+1]}{[n+1]}\\
&-\left(f\circ\tau^{-1}\right)\left(\frac{[k+1]}{[n-k+1]\,q^{k+1}}\right)\Bigg\}.
\end{aligned}
\tag{35}
$$

From $[n+1]/q^{n+1}-[n]/q^{n}=1/q^{n+1}>0$ and by hypothesis, we have

$$
\left(f\circ\tau^{-1}\right)\left(\frac{[n]}{q^{n}}\right)-\left(f\circ\tau^{-1}\right)\left(\frac{[n+1]}{q^{n+1}}\right)>0. \tag{36}
$$

Since f is τ-convex, by using Lemma 3.2 in [24] we obtain

$$
L_{n,q}^{\tau}(f;x)-L_{n+1,q}^{\tau}(f;x)\geq 0 \tag{37}
$$

for $x\in[0,\infty)$ and $n\in\mathbb{N}$. This proves the theorem. □

Theorem 8. *Suppose that $x\in[0,\infty)\setminus\{[k]/[n-k+1]q^{k}:k=0,1,\ldots,n\}$, and τ is linear. Then one has*

$$
\begin{aligned}
L_{n,q}^{\tau}(f;x)-f\left(\frac{x}{q}\right)&=-\frac{\tau(x)^{n+1}}{\ell_{n}^{\tau}(x)}\left[\frac{\tau(x)}{q},\frac{[n]}{q^{n}};\left(f\circ\tau^{-1}\right)\right]q^{n(n-1)/2-1}+\frac{\tau(x)}{\ell_{n}^{\tau}(x)}\\
&\cdot\sum_{k=0}^{n-1}\left[\frac{\tau(x)}{q},\frac{[k]}{[n-k+1]\,q^{k}},\frac{[k+1]}{[n-k]\,q^{k+1}};\left(f\circ\tau^{-1}\right)\right]\\
&\cdot\frac{q^{k(k-1)/2-k-2}}{[n-k]}\begin{bmatrix}n+1\\k\end{bmatrix}\tau(x)^{k}.
\end{aligned}
\tag{38}
$$

Proof. From (17), we have

$$
\begin{aligned}
L_{n,q}^{\tau}(f;x)-f\left(\frac{x}{q}\right)&=\frac{1}{\prod_{s=0}^{n-1}\left(1+q^{s}\tau(x)\right)}\\
&\cdot\sum_{k=0}^{n}\left[\left(f\circ\tau^{-1}\right)\left(\frac{[k]}{[n-k+1]\,q^{k}}\right)-f\left(\frac{x}{q}\right)\right]\\
&\cdot q^{k(k-1)/2}\begin{bmatrix}n\\k\end{bmatrix}\tau(x)^{k}=-\frac{1}{\ell_{n}^{\tau}(x)}\\
&\cdot\sum_{k=0}^{n}\left(\frac{\tau(x)}{q}-\frac{[k]}{[n-k+1]\,q^{k}}\right)\\
&\cdot\left[\frac{\tau(x)}{q},\frac{[k]}{[n-k+1]\,q^{k}};\left(f\circ\tau^{-1}\right)\right]\\
&\cdot q^{k(k-1)/2}\begin{bmatrix}n\\k\end{bmatrix}\tau(x)^{k}.
\end{aligned}
\tag{39}
$$

Using the equality $([k]/[n-k+1])\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]=\left[\begin{smallmatrix}n\\k-1\end{smallmatrix}\right]$, we now get

$$
\begin{aligned}
L_{n,q}^{\tau}(f;x)-f\left(\frac{x}{q}\right)&=-\frac{1}{\ell_{n}^{\tau}(x)}\\
&\cdot\sum_{k=0}^{n}\left[\frac{\tau(x)}{q},\frac{[k]}{[n-k+1]\,q^{k}};\left(f\circ\tau^{-1}\right)\right]\\
&\cdot q^{k(k-1)/2-1}\begin{bmatrix}n\\k\end{bmatrix}\tau(x)^{k+1}+\frac{1}{\ell_{n}^{\tau}(x)}\\
&\cdot\sum_{k=1}^{n}\left[\frac{\tau(x)}{q},\frac{[k]}{[n-k+1]\,q^{k}};\left(f\circ\tau^{-1}\right)\right]\\
&\cdot q^{k(k-1)/2-k}\begin{bmatrix}n\\k-1\end{bmatrix}\tau(x)^{k}=-\frac{\tau(x)^{n+1}}{\ell_{n}^{\tau}(x)}\Bigg[\frac{\tau(x)}{q},\\
&\frac{[n]}{q^{n}};\left(f\circ\tau^{-1}\right)\Bigg]q^{n(n-1)/2-1}+\frac{1}{\ell_{n}^{\tau}(x)}\\
&\cdot\sum_{k=0}^{n-1}\Bigg\{\left[\frac{\tau(x)}{q},\frac{[k+1]}{[n-k]\,q^{k+1}};\left(f\circ\tau^{-1}\right)\right]\\
&-\left[\frac{\tau(x)}{q},\frac{[k]}{[n-k+1]\,q^{k}};\left(f\circ\tau^{-1}\right)\right]\Bigg\}\\
&\cdot q^{k(k-1)/2-1}\begin{bmatrix}n\\k\end{bmatrix}\tau(x)^{k+1}.
\end{aligned}
\tag{40}
$$

On the other hand we can write

$$
\begin{aligned}
&\frac{[k+1]}{[n-k]\,q^{k+1}}-\frac{[k]}{[n-k+1]\,q^{k}}\\
&\quad=\frac{[n+1]}{[n-k]\,[n-k+1]\,q^{k+1}},
\end{aligned}
\tag{41}
$$

which implies

$$\begin{aligned}&\left[\frac{\tau(x)}{q},\frac{[k+1]}{[n-k]\,q^{k+1}};\left(f\circ\tau^{-1}\right)\right]\\&\quad-\left[\frac{\tau(x)}{q},\frac{[k]}{[n-k+1]\,q^{k}};\left(f\circ\tau^{-1}\right)\right]\\&\quad=\left[\frac{\tau(x)}{q},\frac{[k]}{[n-k+1]\,q^{k}},\frac{[k+1]}{[n-k]\,q^{k+1}};\left(f\circ\tau^{-1}\right)\right]\\&\quad\cdot\frac{[n+1]}{[n-k]\,[n-k+1]\,q^{k+1}}.\end{aligned}\tag{42}$$

Now taking (42) in (40), we have

$$\begin{aligned}&L^{\tau}_{n,q}(f;x)-f\left(\frac{x}{q}\right)=-\frac{\tau(x)^{n+1}}{\ell^{\tau}_{n}(x)}\left[\frac{\tau(x)}{q},\frac{[n]}{q^{n}};\left(f\right.\right.\\&\quad\left.\left.\circ\,\tau^{-1}\right)\right]q^{n(n-1)/2-1}+\frac{\tau(x)}{\ell^{\tau}_{n}(x)}\\&\quad\cdot\sum_{k=0}^{n-1}\left[\frac{\tau(x)}{q},\frac{[k]}{[n-k+1]\,q^{k}},\frac{[k+1]}{[n-k]\,q^{k+1}};\left(f\circ\tau^{-1}\right)\right]\\&\quad\cdot\frac{[n+1]}{[n-k]\,[n-k+1]\,q^{k+1}}q^{k(k-1)/2-1}\begin{bmatrix}n\\k\end{bmatrix}\tau(x)^{k+1}.\end{aligned}\tag{43}$$

Thus the proof is completed. □

Now, from Theorem 8, we have the following corollary immediately.

Corollary 9. *Let f be a τ-convex, nonincreasing function on $[0,\infty)$ and τ is linear. Then one has*

$$f\left(\frac{x}{q}\right)\le L^{\tau}_{n,q}(f;x)\tag{44}$$

for any $x\in[0,\infty)\setminus\{[k]/[n-k+1]q^{k}:k=0,1,\dots,n\}$, $n\in\mathbb{N}$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] A. Lupaş, "A q-analogue of the Bernstein operators," in *Seminar on Numerical and Statistical Calculus*, vol. 9, Technical University of Cluj-Napoca, Cluj-Napoca, Romania, 1987.

[2] G. M. Phillips, "Bernstein polynomials based on the q-integers," *Annals of Numerical Mathematics*, vol. 4, pp. 511–518, 1997.

[3] A. Il'inskii and S. Ostrovska, "Convergence of generalized Bernstein polynomials," *Journal of Approximation Theory*, vol. 116, no. 1, pp. 100–112, 2002.

[4] H. Oruç and N. Tuncer, "On the convergence and iterates of q-Bernstein polynomials," *Journal of Approximation Theory*, vol. 117, no. 2, pp. 301–313, 2002.

[5] S. Ostrovska, "q-Bernstein polynomials and their iterates," *Journal of Approximation Theory*, vol. 123, no. 2, pp. 232–255, 2003.

[6] A. Aral and V. Gupta, "On q-Baskakov type operators," *Demonstratio Mathematica*, vol. 42, no. 1, 2009.

[7] A. Aral and V. Gupta, "q-derivative and applications to the q-Szász Mirakyan operators," *Calcolo*, vol. 43, pp. 151–170, 2006.

[8] M. Mursaleen and K. J. Ansari, "Approximation of q-Stancu-Beta operators which preserve x^2," *Bulletin of the Malaysian Mathematical Sciences Society*, 2015.

[9] M. Mursaleen, F. Khan, A. Khan, and A. Kiliçman, "Some approximation results for generalized Kantorovich-type operators," *Journal of Inequalities and Applications*, vol. 2013, article 585, 2013.

[10] M. Mursaleen, F. Khan, and A. Khan, "Approximation properties for King's type modified q-Bernstein-Kantorovich operators," *Mathematical Methods in the Applied Sciences*, 2015.

[11] M. Mursaleen, F. Khan, A. Khan, and A. Kiliçman, "Some approximation properties of Bivariate q-stancu-β operators," *Journal of Function Spaces*, vol. 2014, Article ID 270673, 7 pages, 2014.

[12] M. Mursaleen and A. Khan, "Generalized q-Bernstein-Schurer operators and some approximation theorems," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 719834, 7 pages, 2013.

[13] M. Mursaleen and A. Khan, "Statistical approximation properties of modified q-Stancu-β operators," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 36, no. 3, pp. 683–690, 2013.

[14] G. Bleimann, P. L. Butzer, and L. Hahn, "A Bernstein-type operator approximating continuous functions on the semi-axis," *Indagationes Mathematicae (Proceedings)*, vol. 83, no. 3, pp. 255–262, 1980.

[15] A. D. Gadjiev and Ö. Çakar, "On uniform approximation by Bleimann, Butzer and Hahn operators on all positive semi-axis," *Transactions of AS Azerbaijan. Series of Physical Technical and Mathematical Sciences*, vol. 19, no. 5, pp. 21–26, 1999.

[16] U. Abel and M. Ivan, "Some identities for the operator of Bleimann, Butzer and Hahn involving divided differences," *Calcolo*, vol. 36, no. 3, pp. 143–160, 1999.

[17] R. A. Khan, "A note on a Bernstein-type operator of Bleimann, Butzer, and Hahn," *Journal of Approximation Theory*, vol. 53, no. 3, pp. 295–303, 1988.

[18] J. A. Adell, J. De la Cal, and M. San Miguel, "Inverse beta and generalized Bleimann-Butzer-Hahn operators," *Journal of Approximation Theory*, vol. 76, no. 1, pp. 54–64, 1994.

[19] B. Della Vecciha, *Some Properties of a Rational Operator of Bernstein-Type*, Istituto per Applicazioni della Matematica-C.N.R. Via P. Castellino, Napoli, Italy, 1991.

[20] T. Ernst, "The history of q-calculus and a new method," U.U.U.D.M Report 16, Department of Mathematics, Upsala University, 2000.

[21] G. M. Phillips, *Interpolation and Approximation by Polynomials*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC 14, Springer, New York, NY, USA, 2003.

[22] A. Aral and O. Doğru, "Bleimann, butzer and hahn operators on the q-Integers," *Journal of Inequalities and Applications*, vol. 2007, Article ID 79410, 12 pages, 2007.

[23] O. Agratini and G. Nowak, "On the generalizations of Bleimann, Butzer and Hahn operators based on q-integers," *Mathematical and Computer Modelling*, vol. 53, pp. 699–706, 2011.

[24] O. Doğru and V. Gupta, "Monotonicity and the asymptotic estimate of Bleimann, Butzer and Hahn operators based on q-integers," *Georgian Mathematical Journal*, vol. 12, no. 3, pp. 415–422, 2005.

[25] D. Cárdenas-Morales, P. Garrancho, and I. Raşa, "Bernstein-type operators which preserve polynomials," *Computers & Mathematics with Applications*, vol. 62, no. 1, pp. 158–163, 2011.

[26] A. Aral, D. Inoan, and I. Raşa, "On the generalized Szász-Mirakyan operators," *Results in Mathematics*, vol. 65, no. 3-4, pp. 441–452, 2014.

[27] D. Söylemez, "Some properties of the generalized Bleimann, Butzer and Hahn operators," *Communications Faculty of Science University of Ankara Series A1*, vol. 64, no. 2, pp. 55–62, 2015.