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Research Article

Fractional Cauchy Problem with Caputo Nabla Derivative on Time Scales

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The definition of Caputo fractional derivative is given and some of its properties are discussed in detail. After then, the existence of the solution and the dependency of the solution upon the initial value for Cauchy type problem with fractional Caputo nabla derivative are studied. Also the explicit solutions to homogeneous equations and nonhomogeneous equations are derived by using Laplace transform method.

1. Introduction

Fractional differential equation theory has gained considerable popularity and importance due to their numerous applications in many fields of science and engineering including physics, population dynamics, chemical technology, biotechnology, aerodynamics, electrodynamics of complex medium, polymer rheology, control of dynamical systems, and so on (see, e.g., [1–4], and the references therein). On the other hand, in real applications, it is not always continuous case, but also discrete case. For example, in recent papers [5–8], in order to deeply understand the background of the discrete dynamics behaviors, some interesting results are obtained by applying the discrete fractional calculus to discrete chaos behaviors. In [9–12], the delta type discrete fractional calculus is studied. In [13, 14], the nabla type discrete fractional calculus is studied. In [15], the theory of fractional backward difference equations (i.e., the nabla type fractional difference equations) has been studied in detail. So how to unify continuous fractional calculus and discrete fractional calculus is a natural problem. In order to unify differential equations and difference equations, Hilger [16] proposed firstly the time scale and then some relevant basic theories are studied by some authors (see [17–22]). Recently, some authors studied fractional calculus on time scales (see [23–25]), where Williams [24] gives a definition of fractional integral and derivative on time scales to unify three cases of specific time scales, which improved the results in [23]. Bastos gives definition of fractional Δ-integral and Δ-derivative on time scales in [25]. The delta fractional calculus and Laplace transform on some specific discrete time scales are also discussed in [26–28]. In the light of the above work, we further studied the theory of fractional integral and derivative on general time scales in [29], where ∇-Laplace transform, fractional ∇-power function, ∇-Mittag-Leffler function, fractional ∇-integrals, and fractional ∇-differential on time scales are defined. Some of their properties are discussed in detail. After then, by using Laplace transform method, the existence of the solution and the dependency of the solution upon the initial value for Cauchy type problem with Riemann-Liouville fractional ∇-derivative are studied. Also the explicit solutions to homogeneous equations and nonhomogeneous equations are derived by using Laplace transform method. But there is a shortcoming for Riemann-Liouville fractional ∇-derivative. That is, Cauchy type problem with Riemann-Liouville fractional order derivative and the Laplace transform of Riemann-Liouville fractional order derivative require the initial conditions in terms of non-integer derivatives, which are very difficult to be interpreted from the physical point of view. Thus this paper's focus on defining nabla type Caputo fractional derivative on time scales proves some useful property about Caputo fractional derivative and then studies some Caputo fractional differential equations on time scales.

The structure of this paper is as follows. In Section 2, we give some preliminaries about time scales, generalized ∇-power function, and Riemann-Liouville ∇-integral and ∇-derivative. In Section 3, we present the definitions and the properties of the Caputo nabla derivative on time scales in detail. Then in Section 4, Cauchy type problem with Caputo fractional derivative is discussed. For the Caputo fractional differential initial value problem, we discuss the dependency of the solution upon the initial value. In Section 5, by applying the Laplace transform method, we study the fractional order linear differential equations with Caputo fractional derivative. We derive explicit solutions and fundamental system of solutions to homogeneous equations with constant coefficients and find particular solution and general solutions of the corresponding nonhomogeneous equations.

2. Preliminaries

First, we present some preliminaries about time scales in [17].

Definition 1 (see [17]). A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers.

Definition 2 (see [17]). For $t \in \mathbb{T}$ one defines the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \tag{1}$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}. \tag{2}$$

If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Finally, the graininess function $\nu : \mathbb{T} \to [0, \infty)$ is defined by

$$\nu(t) := t - \rho(t). \tag{3}$$

Definition 3 (see [17]). If $\mathbb{T}$ has a right-scattered minimum m, then one defines $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_k$. Then one defines $f^{\nabla}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|\left[f(\rho(t)) - f(s)\right] - f^{\nabla}(t)\left[\rho(t) - s\right]\right| \le \varepsilon \left|\rho(t) - s\right| \quad \forall s \in U. \tag{4}$$

We call $f^{\nabla}(t)$ the nabla derivative of f at t.

Definition 4 (see [17]). A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$.

Definition 5 (see [17, page 100]). The generalized nabla type polynomials are the functions $\widehat{h}_k : \mathbb{T}^2 := \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $k \in \mathbb{N}_0$, defined recursively as follows. The function $\widehat{h}_0$ is

$$\widehat{h}_0(t, s) = 1 \quad \forall s, t \in \mathbb{T}, \tag{5}$$

and given $\widehat{h}_k$ for $k \in \mathbb{N}_0$, the function $\widehat{h}_{k+1}$ is

$$\widehat{h}_{k+1}(t, s) = \int_s^t \widehat{h}_k(\tau, s)\, \nabla\tau \quad \forall s, t \in \mathbb{T}. \tag{6}$$

Definition 6 (see [18, page 38]). The generalized delta type polynomials are the functions $h_k : \mathbb{T}^2 := \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $k \in \mathbb{N}_0$, defined recursively as follows. The function h_0 is

$$h_0(t, s) = 1 \quad \forall s, t \in \mathbb{T}, \tag{7}$$

and given h_k for $k \in \mathbb{N}_0$, the function h_{k+1} is

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\, \Delta\tau \quad \forall s, t \in \mathbb{T}. \tag{8}$$

It is similar to the discussion in the reference [17, (page 103)] for $n \in \mathbb{N}_0$ and ld-continuous functions $p_i : \mathbb{T} \to \mathbb{R}$, $1 \le i \le n$, we consider the nth order linear dynamic equation

$$Ly = 0, \quad \text{where } Ly = y^{\nabla^n} + \sum_{i=1}^{n} p_i y^{\nabla^{n-i}}. \tag{9}$$

Definition 7 (see [17]). One defines the Cauchy function $y : \mathbb{T} \times \mathbb{T}_{k^n} \to \mathbb{R}$ for the linear dynamic equation (9) to be for each fixed $s \in \mathbb{T}_{k^n}$ the solution of the initial value problem

$$Ly = 0, \qquad y^{\nabla^i}(\rho(s), s) = 0, \quad 0 \le i \le n-2, \qquad y^{\nabla^{n-1}}(\rho(s), s) = 1. \tag{10}$$

Remark 8 (see [17]). Note that

$$y(t, s) := \widehat{h}_{n-1}(t, \rho(s)) \tag{11}$$

is the Cauchy function for y^{∇^n}.

Theorem 9 (see [17] (variation of constants)). *Let $\alpha \in \mathbb{T}_{k^n}$ and $t \in \mathbb{T}$. If $f \in C_{ld}$, then the solution of the initial value problem*

$$Ly = f(t), \qquad y^{\nabla^i}(\alpha) = 0, \quad 0 \le i \le n-1 \tag{12}$$

is given by

$$y(t) = \int_{\alpha}^{t} y(t, \tau) f(\tau)\, \nabla\tau, \tag{13}$$

where $y(t, \tau)$ is the Cauchy function for (9).

Theorem 10 (see [17] (Taylor's Formula)). *Let $n \in \mathbb{N}$. Suppose the function f is such that $f^{\nabla^{n+1}}$ is ld-continuous on $\mathbb{T}_{k^{n+1}}$. Let $\alpha \in \mathbb{T}_{k^n}$, $t \in \mathbb{T}$. Then one has*

$$f(t) = \sum_{k=0}^{n} \widehat{h}_k(t, \alpha) f^{\nabla^k}(\alpha) + \int_{\alpha}^{t} \widehat{h}_n(t, \rho(\tau)) f^{\nabla^{n+1}}(\tau)\, \nabla\tau. \tag{14}$$

Definition 11 (see [24]). A subset $I \subset \mathbb{T}$ is called a time scale interval, if it is of the form $I = A \cap \mathbb{T}$ for some real interval $A \subset \mathbb{R}$. For a time scale interval I, a function $f : I \to \mathbb{R}$ is said to be left-dense absolutely continuous if for all $\varepsilon > 0$ there exist $\delta > 0$ such that $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$ whenever a disjoint finite collection of subtime scale intervals $(a_k, b_k] \cap \mathbb{T} \subset I$ for $1 \leq k \leq n$ satisfies $\sum_{k=1}^{n} |b_k - a_k| < \delta$. One denotes $f \in AC_{\nabla}$. If $f^{\nabla^{m-1}} \in AC$, then one denotes $f \in AC_{\nabla}^{m}$.

Theorem 12 (see [4]). *Let X be a normed linear space, $\mathscr{C} \subset X$ a convex set, and U open in $\mathscr{C}$ with $\theta \in U$. Let $T : \overline{U} \to \mathscr{C}$ be a continuous and compact mapping. Then either*

(i) *the mapping T has a fixed point in $\overline{U}$, or*

(ii) *there exists $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda T u$.*

The following results can be found in our recent paper [29].

Lemma 13 (see [29]). *Let $E \subset \mathbb{T} - \{\max \mathbb{T}\}$ be a measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is integrable on E, then*

$$\int_E f^{\sigma}(s)\, \Delta s = \int_E f(s)\, \nabla s. \tag{15}$$

From now on, let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = \infty$ and fix $t_0 \in \mathbb{T}$.

Definition 14 (see [29]). Assume that $x : \mathbb{T} \to \mathbb{R}$ is regulated and $t_0 \in \mathbb{T}$. Then the Laplace transform of x is defined by

$$\mathscr{L}_{\nabla, t_0}\{x\}(z) = \int_{t_0}^{\infty} x(t)\, \hat{e}^{\rho}_{\ominus_\nu z}(t, t_0)\, \nabla t. \tag{16}$$

for $z \in \mathscr{D}\{x\}$, where $\mathscr{D}\{x\}$ consists of all complex numbers $z \in \mathscr{R}_\nu$ for which the improper integral exists.

Theorem 15 (see [29]). *Assume that $x : \mathbb{T} \to \mathbb{C}$ is such that x^{∇^k} is regulated. Then*

$$\mathscr{L}_{\nabla, t_0}\left\{x^{\nabla^k}\right\}(z) = z^k \mathscr{L}_{\nabla, t_0}\{x\}(z) - \sum_{i=0}^{k-1} z^{k-i-1} x^{\nabla^i}(t_0) \tag{17}$$

for those regressive $z \in \mathbb{C}$ satisfying $\lim_{t \to \infty}\{x^{\nabla^i}(t)\hat{e}_{\ominus_\nu z}(t, t_0)\} = 0$, $i = 0, 1, \ldots, k-1$.

Definition 16 (see [29]). One defines fractional generalized ∇-power function on time scales

$$\hat{h}_\alpha(t, t_0) = \mathscr{L}^{-1}_{\nabla, t_0}\left\{\frac{1}{z^{\alpha+1}}\right\}(t) \quad (\alpha > -1) \tag{18}$$

to those regressive $z \in \mathbb{C} \backslash \{0\}$, $t \geq t_0$; and for $t < t_0$, $\hat{h}_\alpha(t, t_0) = 0$.

Here we introduce generalized ∇-derivative on time scales:

$$\int f^{\nabla} g \nabla t = -\int f^{\rho} g^{\nabla} \nabla t. \tag{19}$$

Since $\hat{h}_\alpha(t, t_0)$ $(\alpha > -1)$ is integral, we can consider it as a generalized function, and thus we can define $\hat{h}_\alpha(t, t_0) = D_{\nabla}\hat{h}_{\alpha+1}(t, t_0)$ for $-2 < \alpha \leq -1$, where D_∇ here means a generalized derivative. In the same way, we can define $\hat{h}_\alpha(t, t_0)$ for $\alpha \leq -1$.

For $\alpha > 0$, we have

$$\hat{h}_\alpha(t_0, t_0) = 0. \tag{20}$$

Definition 17 (see [29]). For a given $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, the solution of the shifting problem

$$\begin{gathered} u^{\nabla_t}(t, \rho(s)) = -u^{\nabla_s}(t, s), \quad t, s \in \mathbb{T},\ t \geq s \geq t_0, \\ u(t, t_0) = f(t), \quad t \in \mathbb{T},\ t \geq t_0, \end{gathered} \tag{21}$$

is denoted by $\tilde{f}$ and is called the shift of f.

Definition 18 (see [29]). For given functions $f, g : \mathbb{T} \to \mathbb{R}$, their convolution $f * g$ is defined by

$$(f * g)(t) = \int_{t_0}^{t} \tilde{f}(t, \rho(\tau))\, g(\tau)\, \nabla \tau, \quad t \in \mathbb{T}, \tag{22}$$

where $\tilde{f}$ is the shift of f, which is introduced in Definition 17.

Definition 19 (see [29]). Fractional generalized ∇-power function $\hat{h}_\alpha(t, s)$ on time scales is defined as the shift of $\hat{h}_\alpha(t, t_0)$; that is,

$$\hat{h}_\alpha(t, s) = \widetilde{\hat{h}_\alpha(\cdot, t_0)}(t, s) \quad (t \geq s \geq t_0). \tag{23}$$

In this paper, we always denote $\Omega := [t_0, t_1]_{\mathbb{T}}$ a finite interval on a time scale $\mathbb{T}$ $(\sup \mathbb{T} = \infty)$.

Definition 20 (see [29]). Let $t, t_0 \in \Omega$. The Riemann-Liouville fractional ∇-integral $I^{\alpha}_{\nabla, t_0} f$ of order $\alpha > 0$ is defined by

$$\begin{aligned} I^{\alpha}_{\nabla, t_0} f(t) &:= \hat{h}_{\alpha-1}(t, t_0) * f(t) \\ &= \int_{t_0}^{t} \widetilde{\hat{h}_{\alpha-1}(\cdot, t_0)}(t, \rho(\tau))\, f(\tau)\, \nabla \tau \\ &= \int_{t_0}^{t} \hat{h}_{\alpha-1}(t, \rho(\tau))\, f(\tau)\, \nabla \tau \quad (t > t_0). \end{aligned} \tag{24}$$

Definition 21 (see [29]). Let $t, t_0 \in \Omega$. The Riemann-Liouville fractional ∇-derivative $D^{\alpha}_{\nabla, t_0} f$ of order $\alpha \geq 0$ is defined by

$$D^{\alpha}_{\nabla, t_0} f(t) = D^{m}_{\nabla} I^{m-\alpha}_{\nabla, t_0} f(t) \quad (m = [\alpha] + 1;\ t > t_0). \tag{25}$$

Throughout this paper, we denote $f^{\nabla^n} = D^n_\nabla f = D^n_{\nabla, t_0} f$, $n \in \mathbb{N}$.

Property 1 (see [29]). Let $\alpha \geq 0$, $m = [\alpha] + 1$, $\beta > 0$, $t, t_0 \in \Omega_{k^m}$. Then

$$\begin{aligned} &(1)\ I^{\alpha}_{\nabla, t_0} \hat{h}_{\beta-1}(t, t_0) = \hat{h}_{\alpha+\beta-1}(t, t_0), \quad (\alpha > 0); \\ &(2)\ D^{\alpha}_{\nabla, t_0} \hat{h}_{\beta-1}(t, t_0) = \hat{h}_{\beta-\alpha-1}(t, t_0), \quad (\alpha \geq 0). \end{aligned} \tag{26}$$

Property 2 (see [29]). If $\alpha > 0$ and $\beta > 0$, then the equation

$$I^{\alpha}_{\nabla,t_0} I^{\beta}_{\nabla,t_0} f(t) = I^{\alpha+\beta}_{\nabla,t_0} f(t) \tag{27}$$

is satisfied at almost every point $t \in \Omega$ for $f(t) \in L_{\nabla,p}(\Omega)$ $(1 \leq p \leq \infty)$.

Property 3 (see [29]). If $\alpha > 0$ and $f(t) \in L_{\nabla,p}(\Omega)$ $(1 \leq p \leq \infty)$, then the following equality

$$D^{\alpha}_{\nabla,t_0} I^{\alpha}_{\nabla,t_0} f(t) = f(t) \tag{28}$$

holds almost everywhere on Ω.

Property 4 (see [29]). If $\alpha > \beta > 0$, then, for $f(t) \in L_{\nabla,p}(\Omega)$ $(1 \leq p \leq \infty)$, the relation

$$D^{\beta}_{\nabla,t_0} I^{\alpha}_{\nabla,t_0} f(t) = I^{\alpha-\beta}_{\nabla,t_0} f(t) \tag{29}$$

holds almost everywhere on Ω. In particular, when $\beta = k \in \mathbb{N}$ and $\alpha > k$, then

$$D^{k}_{\nabla,t_0} I^{\alpha}_{\nabla,t_0} f(t) = I^{\alpha-k}_{\nabla,t_0} f(t). \tag{30}$$

Property 5 (see [29]). Let $\alpha > 0$, $m = [\alpha] + 1$ and let $f_{m-\alpha}(t) = I^{m-\alpha}_{\nabla,t_0} f(t)$.

(1) If $1 \leq p \leq \infty$ and $f(t) \in I^{\alpha}_{\nabla,t_0}(L_{\nabla,p})$, then

$$I^{\alpha}_{\nabla,t_0} D^{\alpha}_{\nabla,t_0} f(t) = f(t). \tag{31}$$

(2) If $f(t) \in L_{\nabla,1}(\Omega)$ and $f_{m-\alpha}(t) \in AC^{m}_{\nabla}(\Omega)$, then the equality

$$I^{\alpha}_{\nabla,t_0} D^{\alpha}_{\nabla,t_0} f(t) = f(t) - \sum_{k=1}^{m} \widehat{h}_{\alpha-k}(t,t_0) D^{\alpha-k}_{\nabla,t_0} f(t_0) \tag{32}$$

holds almost everywhere on Ω, where $D^{\alpha-m}_{\nabla,t_0} y(t_0) = \lim_{t\to t_0^+} I^{m-\alpha}_{\nabla,t_0} y(t)$.

Lemma 22 (see [29]). *Let $\alpha > 0$, $m-1 < \alpha \leq m$ $(m \in \mathbb{N})$ and $f : \Omega \to \mathbb{R}$. For $t_0, t \in \Omega_{k^m}$ with $t_0 < t$. Then one has the following.*

(1) *If $f \in L_{\nabla,p}(\Omega)$, then*

$$\mathscr{L}_{\nabla,t_0}\left\{I^{\alpha}_{\nabla,t_0} f(t)\right\}(z) = \frac{1}{z^{\alpha}} \mathscr{L}_{\nabla,t_0}\{f(t)\}(z). \tag{33}$$

(2) *If $f \in AC^{m}_{\nabla}(\Omega)$, then*

$$\begin{aligned}&\mathscr{L}_{\nabla,t_0}\left\{D^{\alpha}_{\nabla,t_0} f(t)\right\}(z)\\ &\quad = z^{\alpha}\mathscr{L}_{\nabla,t_0}\{f(t)\}(z) - \sum_{j=1}^{m} z^{j-1} D^{\alpha-j}_{\nabla,t_0} f(t_0),\end{aligned} \tag{34}$$

for those regressive $z \in \mathbb{C}$ satisfying $\lim_{t\to\infty}\{D^{j}_{\nabla} I^{m-\alpha}_{\nabla,t_0} f(t) \widehat{e}_{\ominus_\nu z}(t,t_0)\} = 0$, $j = 0, 1, \ldots, m-1$.

Definition 23 (see [29]). ∇-Mittag-Leffler function is defined as

$$_{\nabla}F_{\alpha,\beta}(\lambda; t, t_0) = \sum_{j=0}^{\infty} \lambda^{j} \widehat{h}_{\alpha j+\beta-1}(t,t_0) \tag{35}$$

provided the right-hand series is convergent, where $\alpha, \beta > 0$, $\lambda \in \mathbb{R}$.

Theorem 24 (see [29]). *The Laplace transform of ∇-Mittag-Leffler function is*

$$\mathscr{L}_{\nabla,t_0}\left\{{}_{\nabla}F_{\alpha,\beta}(\lambda; t, t_0)\right\}(z) = \frac{z^{\alpha-\beta}}{z^{\alpha}-\lambda} \left(|\lambda| < |z|^{\alpha}\right). \tag{36}$$

By differentiating k times with respect to λ on both sides of the formula in the theorem above, we get the following result:

$$\mathscr{L}_{\nabla,t_0}\left\{\frac{\partial^k}{\partial\lambda^k}\, {}_{\nabla}F_{\alpha,\beta}(\lambda; t, t_0)\right\}(z) = \frac{k! z^{\alpha-\beta}}{(z^{\alpha}-\lambda)^{k+1}}. \tag{37}$$

3. Definition and Properties of Caputo Fractional Derivative on Time Scales

Definition 25. Let $t, t_0 \in \Omega$. The Caputo fractional derivative of order $\alpha \geq 0$ is defined via Riemann-Liouville fractional derivative by

$$\begin{aligned}{}^{C}D^{\alpha}_{\nabla,t_0} f(t) &:= D^{\alpha}_{\nabla,t_0}\left[f(t) - \sum_{k=0}^{m-1} \widehat{h}_k(t,t_0) f^{\nabla^k}(t_0)\right]\\ &\qquad (t > t_0),\end{aligned} \tag{38}$$

where

$$m = [\alpha] + 1 \quad \text{for } \alpha \notin \mathbb{N};\ m = \alpha \text{ for } \alpha \in \mathbb{N}. \tag{39}$$

In particular, when $0 < \alpha < 1$, the relation (38) takes the following forms:

$${}^{C}D^{\alpha}_{\nabla,t_0} f(t) = D^{\alpha}_{\nabla,t_0}\left[f(t) - f(t_0)\right]. \tag{40}$$

If $\alpha \notin \mathbb{N}$, then the Caputo fractional derivative coincides with the Riemann-Liouville fractional derivative in the following case:

$${}^{C}D^{\alpha}_{\nabla,t_0} f(t) = D^{\alpha}_{\nabla,t_0} f(t), \tag{41}$$

if $f^{\nabla^k}(t_0) = 0$ $(k = 0, 1, \ldots, m-1, m = [\alpha] + 1)$.

In particular, when $0 < \alpha < 1$, we have

$${}^{C}D^{\alpha}_{\nabla,t_0} f(t) = D^{\alpha}_{\nabla,t_0} f(t), \quad \text{when } f(t_0) = 0. \tag{42}$$

If $\alpha = m \in \mathbb{N}$ and the usual nabla derivative $f^{\nabla^m}(t)$ of order m exists, then ${}^{C}D^{m}_{\nabla,t_0} f(t)$ coincides with $f^{\nabla^m}(t)$:

$${}^{C}D^{m}_{\nabla,t_0} f(t) = f^{\nabla^m}(t) \quad (m \in \mathbb{N}). \tag{43}$$

The Caputo fractional derivative ${}^{C}D^{\alpha}_{\nabla,t_0}f(t)$ is defined for functions $f(t)$ for which the Riemann-Liouville fractional derivative of the right-hand sides of (38) exists. In particular, they are defined for $f(t)$ belonging to the space $AC^{m}_{\nabla}(\Omega)$ of absolutely continuous functions defined in Definition 11. Thus the following statement holds.

Property 6. Let $\alpha \geq 0$ and let m be given by (39). If $f(t) \in AC^{m}_{\nabla}(\Omega)$, then the Caputo fractional derivative ${}^{C}D^{\alpha}_{\nabla,t_0}f(t)$ exists almost everywhere on Ω_{k^m}.

(a) If $\alpha \notin \mathbb{N}$, ${}^{C}D^{\alpha}_{\nabla,t_0}f(t)$ is represented by

$$ {}^{C}D^{\alpha}_{\nabla,t_0}f(t) = \widehat{h}_{m-\alpha-1}(t,t_0) * f^{\nabla^m}(t) =: I^{m-\alpha}_{\nabla,t_0}D^{m}_{\nabla}f(t), \tag{44}$$

where $m = [\alpha]+1$. Thus when $\alpha \notin \mathbb{N}$, ${}^{C}D^{\alpha}_{\nabla,t_0}f(t_0) = 0$, where the notation ${}^{C}D^{\alpha}_{\nabla,t_0}f(t_0)$ denote the limit of ${}^{C}D^{\alpha}_{\nabla,t_0}f(t)$ as $t \to t_0^{+}$.

In particular, when $0<\alpha<1$ and $f(t) \in AC_{\nabla}(\Omega)$,

$$ {}^{C}D^{\alpha}_{\nabla,t_0}f(t) = \widehat{h}_{-\alpha}(t,t_0) * f^{\nabla}(t) =: I^{1-\alpha}_{\nabla,t_0}f^{\nabla}(t). \tag{45}$$

(b) If $\alpha = m \in \mathbb{N}$, then ${}^{C}D^{\alpha}_{\nabla,t_0}f(t)$ is represented by (43). In particular,

$$ {}^{C}D^{0}_{\nabla,t_0}f(t) = f(t). \tag{46}$$

Proof. (a) By Taylor's formula on time scales

$$\begin{aligned} f(t) &= \sum_{k=0}^{m-1}\widehat{h}_k(t,t_0) f^{\nabla^k}(t_0) + \int_{t_0}^{t}\widehat{h}_{m-1}(t,\rho(\tau)) f^{\nabla^m}(\tau)\nabla\tau \\ &= \sum_{k=0}^{m-1}\widehat{h}_k(t,t_0) f^{\nabla^k}(t_0) + I^{m}_{\nabla,t_0}f^{\nabla^m}(t) \end{aligned} \tag{47}$$

and using (29), we have

$$\begin{aligned} {}^{C}D^{\alpha}_{\nabla,t_0}f(t) &= D^{\alpha}_{\nabla,t_0}\left[f(t) - \sum_{k=0}^{m-1}\widehat{h}_k(t,t_0) f^{\nabla^k}(t_0)\right] \\ &= D^{\alpha}_{\nabla,t_0}I^{m}_{\nabla,t_0}f^{\nabla^m}(t) \\ &= I^{m-\alpha}_{\nabla,t_0}f^{\nabla^m}(t). \end{aligned} \tag{48}$$

(b) If $\alpha = m \in \mathbb{N}$, then (38) takes the form

$$ {}^{C}D^{m}_{\nabla,t_0}f(t) = D^{m}_{\nabla,t_0}\left[f(t) - \sum_{k=0}^{m-1}\widehat{h}_k(t,t_0) f^{\nabla^k}(t_0)\right], \tag{49}$$

and, from Taylor's formula and (28), we derive ${}^{C}D^{m}_{\nabla,t_0}f(t) = f^{\nabla^m}(t)$. □

Property 7. Let $\alpha > 0$ and let m be given by (39), $\beta > 0$, $t \in \Omega_{k^m}$. Then

$$ {}^{C}D^{\alpha}_{\nabla,t_0}\widehat{h}_{\beta-1}(t,t_0) = \widehat{h}_{\beta-\alpha-1}(t,t_0) \quad (\beta > m), \tag{50}$$

$$ {}^{C}D^{\alpha}_{\nabla,t_0}\widehat{h}_{k}(t,t_0) = 0 \quad (k = 0,1,\ldots,m-1). \tag{51}$$

In particular,

$$ {}^{C}D^{\alpha}_{\nabla,t_0}1 = 0. \tag{52}$$

Proof. From Property 6 and (26), it is obtained that for $\alpha \notin \mathbb{N}$,

$$\begin{aligned} {}^{C}D^{\alpha}_{\nabla,t_0}\widehat{h}_{\beta-1}(t,t_0) &= I^{m-\alpha}_{\nabla,t_0}D^{m}_{\nabla}\widehat{h}_{\beta-1}(t,t_0) \\ &= I^{m-\alpha}_{\nabla,t_0}\widehat{h}_{\beta-m-1}(t,t_0) = \widehat{h}_{\beta-\alpha-1}(t,t_0), \\ {}^{C}D^{\alpha}_{\nabla,t_0}\widehat{h}_{k}(t,t_0) &= I^{m-\alpha}_{\nabla,t_0}D^{m}_{\nabla}\widehat{h}_{k}(t,t_0) = I^{m-\alpha}_{\nabla,t_0}0 = 0 \\ &\qquad (k = 0,1,\ldots,m-1,\ t > t_0), \end{aligned} \tag{53}$$

while for $\alpha = m \in \mathbb{N}$,

$$\begin{aligned} {}^{C}D^{m}_{\nabla,t_0}\widehat{h}_{\beta-1}(t,t_0) &= D^{m}_{\nabla}\widehat{h}_{\beta-1}(t,t_0) = \widehat{h}_{\beta-m-1}(t,t_0), \\ {}^{C}D^{m}_{\nabla,t_0}\widehat{h}_{k}(t,t_0) &= D^{m}_{\nabla}\widehat{h}_{k}(t,t_0) = 0 \\ &\qquad (k = 0,1,\ldots,m-1,\ t > t_0). \end{aligned} \tag{54}$$

□

Property 8. Let $\alpha > 0$ and let $f(t) \in L_{\nabla,\infty}(\Omega)$ or $f(t) \in AC_{\nabla}(\Omega)$. Then

$$ {}^{C}D^{\alpha}_{\nabla,t_0}I^{\alpha}_{\nabla,t_0}f(t) = f(t). \tag{55}$$

Proof. Let $f(t) \in L_{\nabla,\infty}(\Omega)(f(t) \in AC_{\nabla}(\Omega))$, and let $\alpha > 0$ and $k = 0,1,\ldots,m-1$. Since $f(t) \in L_{\nabla,\infty}(\Omega)(f(t) \in AC_{\nabla}(\Omega))$, then for a.e. (for any) $t \in \Omega_{k^m}$, we get

$$\begin{gathered} I^{\alpha-k}_{\nabla,t_0}f(t) = \int_{t_0}^{t}\widehat{h}_{\alpha-k-1}(t,\rho(\tau)) f(\tau)\nabla\tau \leq K^2\widehat{h}_1(t,t_0) \\ \left(\text{where } \max_{t,\tau\in\Omega}\left\{\|f\|_{L_{\nabla,\infty}}, \|f\|_{AC_{\nabla}}, \left|\widehat{h}_{\alpha-k-1}(t,\rho(\tau))\right|\right\} \leq K\right) \end{gathered} \tag{56}$$

for any $k = 0,1,\ldots,m-1 = [\alpha]$, and hence

$$ D^{k}_{\nabla}I^{\alpha}_{\nabla,t_0}f(t_0) = I^{\alpha-k}_{\nabla,t_0}f(t_0) = 0 \quad (k = 0,1,\ldots,m-1). \tag{57}$$

Thus using (41) for $\alpha \notin \mathbb{N}$ with $f(t)$ replaced by $I^{\alpha}_{\nabla,t_0}f(t)$ and (28), we derive

$$ {}^{C}D^{\alpha}_{\nabla,t_0}I^{\alpha}_{\nabla,t_0}f(t) = f(t). \tag{58}$$

For $\alpha = m \in \mathbb{N}$,

$$ {}^{C}D^{\alpha}_{\nabla,t_0}I^{\alpha}_{\nabla,t_0}f(t) = D^{m}_{\nabla,t_0}I^{m}_{\nabla,t_0}f(t) = f(t). \tag{59}$$

□

Property 9. Let $\alpha > 0$ and let m be given by (39). If $f(t) \in AC_{\nabla}^{m}(\Omega)$, then

$$I_{\nabla,t_0}^{\alpha}\,{}^{C}D_{\nabla,t_0}^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} \widehat{h}_k(t,t_0)\, D_{\nabla,t_0}^{k} f(t_0). \quad (60)$$

In particular, if $0 < \alpha \le 1$ and $f(t) \in AC_{\nabla}(\Omega)$, then

$$I_{\nabla,t_0}^{\alpha}\,{}^{C}D_{\nabla,t_0}^{\alpha} f(t) = f(t) - f(t_0). \quad (61)$$

Proof. Let $\alpha \notin \mathbb{N}$. If $f(t) \in AC_{\nabla}^{m}(\Omega)$, then using Property 6, (27) and (32), we have

$$\begin{aligned} I_{\nabla,t_0}^{\alpha}\,{}^{C}D_{\nabla,t_0}^{\alpha} f(t) &= I_{\nabla,t_0}^{\alpha} I_{\nabla,t_0}^{m-\alpha} D_{\nabla}^{m} f(t) = I_{\nabla,t_0}^{m} D_{\nabla}^{m} f(t) \\ &= f(t) - \sum_{k=0}^{m-1} \widehat{h}_k(t,t_0)\, D_{\nabla,t_0}^{k} f(t_0). \end{aligned} \quad (62)$$

For $\alpha = m \in \mathbb{N}$, the result is obvious from Property 6 and (32). □

Property 10. Assume that $f(t) \in AC_{\nabla}^{m}(\Omega)$ and $m-1 < \beta < \alpha < m$. Then, for all $k \in \{1,\dots,m-1\}$,

$${}^{C}D_{\nabla,t_0}^{\alpha-m+k} D_{\nabla,t_0}^{m-k} f(t) = {}^{C}D_{\nabla,t_0}^{\alpha} f(t), \quad (63)$$

$${}^{C}D_{\nabla,t_0}^{\alpha-\beta}\,{}^{C}D_{\nabla,t_0}^{\beta} f(t) = {}^{C}D_{\nabla,t_0}^{\alpha} f(t) \quad (64)$$

for all $t \in \Omega_{k^m}$.

Proof. For each $k \in \{1,\dots,m-1\}$, by Property 6,

$$\begin{aligned} {}^{C}D_{\nabla,t_0}^{\alpha} f(t) &= I_{\nabla,t_0}^{m-\alpha} D_{\nabla}^{m} f(t) \\ &= \int_{t_0}^{t} \widehat{h}_{m-\alpha-1}(t,\rho(s))\, D_{\nabla}^{m} f(s)\,\nabla s \\ &= \int_{t_0}^{t} \widehat{h}_{k-(\alpha-(m-k))-1}(t,\rho(s))\, D_{\nabla}^{k} D_{\nabla}^{m-k} f(s)\,\nabla s. \end{aligned} \quad (65)$$

Noting that $\alpha-(m-k) \in (k-1,k)$ and according to Property 6, we have

$$\begin{aligned} &\int_{t_0}^{t} \widehat{h}_{k-(\alpha-(m-k))-1}(t,\rho(s))\, D_{\nabla}^{k} D_{\nabla}^{m-k} f(s)\,\nabla s \\ &\quad = {}^{C}D_{\nabla,t_0}^{\alpha-m+k} D_{\nabla,t_0}^{m-k} f(t). \end{aligned} \quad (66)$$

Thus (63) holds.

Now, for all $\alpha_0, \beta_0 \in (0,1)$ with $\alpha_0+\beta_0 < 1$, we have

$${}^{C}D_{\nabla,t_0}^{\alpha_0}\,{}^{C}D_{\nabla,t_0}^{\beta_0} f(t) = {}^{C}D_{\nabla,t_0}^{\alpha_0+\beta_0} f(t) = {}^{C}D_{\nabla,t_0}^{\beta_0}\,{}^{C}D_{\nabla,t_0}^{\alpha_0} f(t). \quad (67)$$

In fact, from Property 6, we can get ${}^{C}D_{\nabla,t_0}^{\beta_0} f(t_0) = 0$. Since $\alpha_0, \beta_0 \in (0,1)$, then by (41), (29), and Property 6

$$\begin{aligned} {}^{C}D_{\nabla,t_0}^{\alpha_0}\,{}^{C}D_{\nabla,t_0}^{\beta_0} f(t) &= D_{\nabla,t_0}^{\alpha_0}\,{}^{C}D_{\nabla,t_0}^{\beta_0} f(t) = D_{\nabla,t_0}^{\alpha_0} I_{\nabla,t_0}^{1-\beta_0} f^{\nabla}(t) \\ &= I_{\nabla,t_0}^{1-\beta_0-\alpha_0} f^{\nabla}(t) = {}^{C}D_{\nabla,t_0}^{\alpha_0+\beta_0} f(t). \end{aligned} \quad (68)$$

Similarly, we have ${}^{C}D_{\nabla,t_0}^{\beta_0}\,{}^{C}D_{\nabla,t_0}^{\alpha_0} f(t) = {}^{C}D_{\nabla,t_0}^{\alpha_0+\beta_0} f(t)$. Thus (67) holds. Then, by using (63) and (67), we have that

$$\begin{aligned} {}^{C}D_{\nabla,t_0}^{\alpha} f(t) &= {}^{C}D_{\nabla,t_0}^{\alpha-m+1} D_{\nabla}^{m-1} f(t) \\ &= {}^{C}D_{\nabla,t_0}^{(\alpha-\beta)+(\beta-m+1)} D_{\nabla}^{m-1} f(t) \\ &= {}^{C}D_{\nabla,t_0}^{\alpha-\beta}\,{}^{C}D_{\nabla,t_0}^{\beta-m+1} D_{\nabla}^{m-1} f(t) \\ &= {}^{C}D_{\nabla,t_0}^{\alpha-\beta}\,{}^{C}D_{\nabla,t_0}^{\beta} f(t). \end{aligned} \quad (69)$$

That is, (64) holds. The results follow. □

The next assertion yields the Laplace transform of the Caputo fractional nabla derivative.

Property 11. Let $\alpha > 0$, $m-1 < \alpha \le m$ $(m \in \mathbb{N})$ be such that $f(t) \in AC_{\nabla}^{m}(\Omega)$. Then

$$\begin{aligned} &\mathscr{L}_{\nabla,t_0}\left\{{}^{C}D_{\nabla,t_0}^{\alpha} f(t)\right\}(z) \\ &\quad = z^{\alpha} \mathscr{L}_{\nabla,t_0}\{f(t)\}(z) - \sum_{k=0}^{m-1} z^{\alpha-k-1} f^{\nabla^k}(t_0) \end{aligned} \quad (70)$$

for those regressive $z \in \mathbb{C}$ satisfying $\lim_{t\to\infty}\{f^{\nabla k}(t)\widehat{e}_{\ominus_{\nu} z}(t,t_0)\} = 0$ $(k = 0,1,\dots,m-1)$.

In particular, if $0 < \alpha \le 1$, then

$$\mathscr{L}_{\nabla,t_0}\left\{{}^{C}D_{\nabla,t_0}^{\alpha} f(t)\right\}(z) = z^{\alpha} \mathscr{L}_{\nabla,t_0}\{f(t)\}(z) - z^{\alpha-1} f(t_0) \quad (71)$$

for those regressive $z \in \mathbb{C}$ satisfying $\lim_{t\to\infty}\{f(t)\widehat{e}_{\ominus_{\nu} z}(t,t_0)\} = 0$.

Proof. By Property 6, (33), and (17), for $\alpha \notin \mathbb{N}$, we have

$$\begin{aligned} &\mathscr{L}_{\nabla,t_0}\left\{{}^{C}D_{\nabla,t_0}^{\alpha} f(t)\right\}(z) \\ &\quad = \mathscr{L}_{\nabla,t_0}\left\{I_{\nabla,t_0}^{m-\alpha} D_{\nabla}^{m} f(t)\right\}(z) \\ &\quad = \frac{1}{z^{m-\alpha}} \mathscr{L}_{\nabla,t_0}\left\{D_{\nabla}^{m} f(t)\right\}(z) \\ &\quad = \frac{1}{z^{m-\alpha}}\left[z^{m} \mathscr{L}_{\nabla,t_0}\{f(t)\}(z) - \sum_{k=0}^{m-1} z^{m-k-1} f^{\nabla^k}(t_0)\right] \\ &\quad = z^{\alpha} \mathscr{L}_{\nabla,t_0}\{f(t)\}(z) - \sum_{k=0}^{m-1} z^{\alpha-k-1} f^{\nabla^k}(t_0), \end{aligned} \quad (72)$$

and for $\alpha = m \in \mathbb{N}$, we have

$$\begin{aligned} \mathscr{L}_{\nabla,t_0}\left\{{}^{C}D_{\nabla,t_0}^{m} f(t)\right\}(z) &= \mathscr{L}_{\nabla,t_0}\left\{f^{\nabla^m}(t)\right\}(z) \\ &= z^{m} \mathscr{L}_{\nabla,t_0}\{f(t)\}(z) \\ &\quad - \sum_{k=0}^{m-1} z^{m-k-1} f^{\nabla^k}(t_0). \end{aligned} \quad (73)$$

The result follows. □

Remark 26. (1) For Riemann-Liouville fractional derivative,

$$D^{\alpha}_{\nabla,t_0} 1 = \widehat{h}_{-\alpha}(t,t_0) \quad (0<\alpha<1), \tag{74}$$

while for the Caputo fractional derivative,

$${}^{C}D^{\alpha}_{\nabla,t_0} 1 = 0, \tag{75}$$

which shows that the Caputo fractional derivative is more near to the usual sense derivative than Riemann-Liouville fractional derivative.

(2) Comparing (34) and (70), we know that the Laplace transform of the Caputo fractional derivative involves only initial value with integer order derivative, such as $f^{\nabla^k}(t_0), k = 0,1,\ldots,m-1$, while the Laplace transform of the Riemann-Liouville fractional derivative is related to initial value with fractional order derivative which is difficult to understand the physics background, such as $D^{\alpha-k}_{\nabla,t_0} f(t_0), k = 1,\ldots,m$. Thus, the Caputo fractional derivative is used more widely in realistic applications.

4. The Cauchy Problem with Caputo Fractional Derivative

4.1. Existence and Uniqueness of the Solution to the Cauchy Type Problem. In this section we consider the nonlinear differential equation of order $\alpha > 0$:

$${}^{C}D^{\alpha}_{\nabla,t_0} y(t) = f(t, y(t)) \tag{76}$$

involving the Caputo fractional derivative ${}^{C}D^{\alpha}_{\nabla,t_0} y(t)$, defined in (38), with the initial conditions

$$D^{k}_{\nabla} y(t_0) = b_k, \quad b_k \in \mathbb{R} \ \ (k = 0,1,\ldots,m-1;\ m = -[-\alpha]). \tag{77}$$

We give the conditions for a unique solution $y(t)$ to this problem in the space $AC^{m}_{\nabla}(\Omega)$. Our investigations are based on reducing the problem (76)-(77) to the integral equation

$$y(t) = \sum_{j=0}^{m-1} \widehat{h}_j(t,t_0) b_j + \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t,\rho(\tau)) f(\tau, y(\tau)) \nabla\tau. \tag{78}$$

First we establish an equivalence between the problem (76)-(77) and the integral equation (78).

Theorem 27. *Let $\alpha > 0$ and let m be given by* (39)*. Let G be an open set in $\mathbb{R}$ and let $f : \Omega \times G \to \mathbb{R}$ be a function such that, for any $y \in G, f(t,y) \in AC_{\nabla}(\Omega)$. If $y(t) \in AC^{m}_{\nabla}(\Omega)$, then $y(t)$ satisfies the relation* (76)-(77) *if and only if $y(t)$ satisfies the Volterra integral equation* (78).

Proof. First we prove the necessity. Let $y(t)$ be the solution to the Cauchy problem (76)-(77). Applying the operator I^{α}_{∇,t_0} to (76) and taking into account

$$I^{\alpha}_{\nabla,t_0}\, {}^{C}D^{\alpha}_{\nabla,t_0} y(t) = y(t) - \sum_{j=0}^{m-1} \widehat{h}_j(t,t_0) D^{j}_{\nabla,t_0} y(t_0) \tag{79}$$

and (77), we arrive at the integral equation (78) since $y(t) \in AC^{m}_{\nabla}(\Omega)$.

Inversely, if $y(t)$ satisfies (78), for $f(t,y) \in AC_{\nabla}(\Omega)$, applying the operator ${}^{C}D^{\alpha}_{\nabla,t_0}$ to both sides of (78) and taking into account (51) and (55), we have

$$\begin{aligned} {}^{C}D^{\alpha}_{\nabla,t_0} y(t) &= \sum_{j=0}^{m-1} {}^{C}D^{\alpha}_{\nabla,t_0} \widehat{h}_j(t,t_0) b_j + {}^{C}D^{\alpha}_{\nabla,t_0} I^{\alpha}_{\nabla,t_0} f(t, y(t)) \\ &= f(t, y(t)). \end{aligned} \tag{80}$$

In addition, by term-by-term differentiation of (78) and using (51), we have

$$\begin{aligned} D^{k}_{\nabla} y(t) &= \sum_{j=0}^{m-1} D^{k}_{\nabla} \widehat{h}_j(t,t_0) b_j + D^{k}_{\nabla} I^{\alpha}_{\nabla,t_0} f(t, y(t)) \\ &= \sum_{j=0}^{m-1} D^{k}_{\nabla} \widehat{h}_j(t,t_0) b_j \\ &\quad + \int_{t_0}^{t} \widehat{h}_{\alpha-k-1}(t,\rho(\tau)) f(\tau, y(\tau)) \nabla\tau \\ &= \sum_{j=0}^{k-1} D^{k}_{\nabla} \widehat{h}_j(t,t_0) b_j + \sum_{j=k}^{m-1} D^{k}_{\nabla} \widehat{h}_j(t,t_0) b_j \\ &\quad + \int_{t_0}^{t} \widehat{h}_{\alpha-k-1}(t,\rho(\tau)) f(\tau, y(\tau)) \nabla\tau \\ &= \sum_{j=k}^{m-1} \widehat{h}_{j-k}(t,t_0) b_j \\ &\quad + \int_{t_0}^{t} \widehat{h}_{\alpha-k-1}(t,\rho(\tau)) f(\tau, y(\tau)) \nabla\tau \end{aligned} \tag{81}$$

for $k = 0,1,\ldots,m-1$. Thus we obtain relations in (77) by letting $t = t_0$ in (81). □

In the following, we bring into Lipschitzian-type condition:

$$\left| f(t, y_1(t)) - f(t, y_2(t)) \right| \le A \left| y_1(t) - y_2(t) \right|, \tag{82}$$

where $A > 0$ does not depend on $t \in \Omega$. We will derive a unique solution to the Cauchy problem (76)-(77).

Theorem 28. *Let $\alpha > 0$ and let m be given by* (39)*. Let G be an open set in $\mathbb{R}$ and $f : \Omega \times G \to \mathbb{R}$ a function such that, for any $y \in G, f(t,y) \in AC_{\nabla}(\Omega), y(t) \in AC^{m}_{\nabla}(\Omega)$. Let $f(t,y)$ satisfies the Lipschitzian condition* (82)*, and $\max_{y\in G, t,s\in\Omega}\{|f(t,y)|, |\widehat{h}_{\alpha-1}(t,s)|\} \le M$. Then there exists a unique solution $y(t)$ to initial value problem* (76)-(77).

Proof. Since the Cauchy type problem (76)-(77) and the nonlinear Volterra integral equation (78) are equivalent, we only need to prove there exists a unique solution to (78).

We define function sequences:

$$y_l(t) = y_0(t) + \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t,\rho(\tau)) f(\tau, y_{l-1}(\tau)) \nabla\tau \quad (l = 1, 2, \ldots), \tag{83}$$

where

$$y_0(t) = \sum_{j=0}^{m-1} \widehat{h}_j(t,t_0) b_j. \tag{84}$$

To simplify our proof, without loss of generality, we assume that G is large enough such that $y_l(t) \in G, \forall t \in \Omega, \forall l$.

We obtain by inductive method that

$$\left|y_l(t) - y_{l-1}(t)\right| \le A^{l-1} M^{l+1} \widehat{h}_l(t,t_0). \tag{85}$$

In fact, for $l = 1$, since $\max_{y\in G, t,s\in\Omega}\{|f(t,y)|, |\widehat{h}_{\alpha-1}(t,s)|\} \le M$, we have

$$\begin{aligned}\left|y_1(t) - y_0(t)\right| &\le \int_{t_0}^{t} \left|\widehat{h}_{\alpha-1}(t,\rho(\tau))\right| \left|f(\tau, y_0(\tau))\right| \nabla\tau \\ &\le M^2 \int_{t_0}^{t} \nabla\tau = M^2 \widehat{h}_1(t,t_0).\end{aligned} \tag{86}$$

If

$$\left|y_{l-1}(t) - y_{l-2}(t)\right| \le A^{l-2} M^{l} \widehat{h}_{l-1}(t,t_0), \tag{87}$$

then

$$\begin{aligned}&\left|y_l(t) - y_{l-1}(t)\right| \\ &\quad\le \int_{t_0}^{t} \left|\widehat{h}_{\alpha-1}(t,\rho(\tau))\right| \left|f(\tau, y_{l-1}(\tau)) - f(\tau, y_{l-2}(\tau))\right| \nabla\tau \\ &\quad\le AM \int_{t_0}^{t} \left|y_{l-1}(\tau) - y_{l-2}(\tau)\right| \nabla\tau \\ &\quad\le AM \int_{t_0}^{t} A^{l-2} M^{l} \widehat{h}_{l-1}(\tau,t_0) \nabla\tau \\ &\quad= A^{l-1} M^{l+1} \widehat{h}_l(t,t_0).\end{aligned} \tag{88}$$

According to

$$\begin{aligned}\sum_{l=1}^{\infty} \left|y_l(t) - y_{l-1}(t)\right| &\le \sum_{l=1}^{\infty} A^{l-1} M^{l+1} \widehat{h}_l(t,t_0) \\ &\le \frac{M}{A} \sum_{l=1}^{\infty} (AM)^l h_l(\sigma(t), t_0) \\ &\le \frac{M}{A} \sum_{l=1}^{\infty} (AM)^l \frac{(\sigma(t) - t_0)^l}{l!}\end{aligned} \tag{89}$$

and by Weierstrass discriminance, we obtain $y_l(t)$ convergent uniformly and the limit is the solution. Thus we prove the existence of solution.

Next we will show the uniqueness. Assume $z(t)$ is another solution to (78); that is,

$$z(t) = y_0(t) + \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t,\rho(\tau)) f(\tau, z(\tau)) \nabla\tau. \tag{90}$$

Since

$$\max_{y\in G, t,s\in\Omega} \left\{\left|f(t,y)\right|, \left|\widehat{h}_{\alpha-1}(t,s)\right|\right\} \le M, \tag{91}$$

we have

$$\begin{aligned}\left|y_0(t) - z(t)\right| &\le \int_{t_0}^{t} \left|\widehat{h}_{\alpha-1}(t,\rho(\tau))\right| \left|f(\tau, z(\tau))\right| \nabla\tau \\ &\le M^2 \int_{t_0}^{t} \nabla\tau = M^2 \widehat{h}_1(t,t_0).\end{aligned} \tag{92}$$

If

$$\left|y_{l-1}(t) - z(t)\right| \le A^{l-1} M^{l+1} \widehat{h}_l(t,t_0), \tag{93}$$

then

$$\begin{aligned}&\left|y_l(t) - z(t)\right| \\ &\quad\le \int_{t_0}^{t} \left|\widehat{h}_{\alpha-1}(t,\rho(\tau))\right| \left|f(\tau, y_{l-1}(\tau)) - f(\tau, z(\tau))\right| \nabla\tau \\ &\quad\le AM \int_{t_0}^{t} \left|y_{l-1}(\tau) - z(\tau)\right| \nabla\tau \\ &\quad\le AM \int_{t_0}^{t} A^{l-1} M^{l+1} \widehat{h}_l(\tau,t_0) \nabla\tau \\ &\quad\le A^{l} M^{l+2} \widehat{h}_{l+1}(t,t_0).\end{aligned} \tag{94}$$

By mathematical induction, we have

$$\left|y_l(t) - z(t)\right| \le A^{l} M^{l+2} \widehat{h}_{l+1}(t,t_0). \tag{95}$$

and then we get that

$$\begin{aligned}\sum_{l=0}^{\infty} \left|y_l(t) - z(t)\right| &\le \sum_{l=0}^{\infty} A^{l} M^{l+2} \widehat{h}_{l+1}(t,t_0) \\ &\le \frac{M}{A} \sum_{l=0}^{\infty} (AM)^{l+1} h_{l+1}(\sigma(t), t_0) \\ &\le \frac{M}{A} \sum_{l=0}^{\infty} (AM)^{l+1} \frac{(\sigma(t) - t_0)^{l+1}}{(l+1)!}.\end{aligned} \tag{96}$$

Thus, $\lim_{l\to\infty}|y_l(t) - z(t)| = 0$, and then we have $z(t) = y(t)$ owing to the uniqueness of the limit. The result follows. □

In the following, we consider generalized Cauchy type problems:

$$\begin{gathered}{}^{C}D^{\alpha}_{\nabla,t_0} y(t) = f\left(t, y(t), {}^{C}D^{\alpha_1}_{\nabla,t_0} y(t), \ldots, {}^{C}D^{\alpha_l}_{\nabla,t_0} y(t)\right) \\ \left(0 = \alpha_0 \le \alpha_1 \le \cdots \le \alpha_l \le \alpha\right), \\ D^{k}_{\nabla,t_0} y(t_0) = b_k \quad (k = 1, \ldots, m,\ \alpha = -[-\alpha]).\end{gathered} \tag{97}$$

Theorem 29. *Let $\alpha > 0, G$ be an open set and let $f : \Omega \times G \to \mathbb{R}$ be a function such that, for any $(y, y_1, \ldots, y_l) \in G$, $f(t, y, y_1, \ldots, y_l) \in AC_\nabla(\Omega)$. If $y(t) \in AC_\nabla^m(\Omega)$, then $y(t)$ satisfies* (97) *if and only if $y(t)$ satisfies the integral equation*

$$y(t) = \sum_{j=0}^{m-1} \widehat{h}_j(t, t_0) b_j + \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t, \rho(\tau)) \times f\left(\tau, y(\tau), {}^{C}D_{\nabla,t_0}^{\alpha_1} y(\tau), \ldots, {}^{C}D_{\nabla,t_0}^{\alpha_l} y(\tau)\right) \nabla\tau. \tag{98}$$

Suppose that f satisfies generalized Lipschitzian condition:

$$\left|f(t, y_0, y_1, \ldots, y_l) - f(t, z_0, z_1, \ldots, z_l)\right| \le A\left[\sum_{j=0}^{l} \left|y_j - z_j\right|\right] \quad (A > 0). \tag{99}$$

According to the theorem above and the proof of Theorem 28, we have the following theorem.

Theorem 30. *Let the condition of Theorem 29 be valid. If f satisfies Lipschitzian condition* (99) *and* $\max_{y\in G, t,s\in\Omega}\{|f(t, y, y_1, \ldots, y_l)|, |\widehat{h}_{\alpha-1}(t, s)|\} \le M$ *holds, then there exists a unique solution $y(t)$ to initial value problem* (97).

4.2. The Dependency of the Solution upon the Initial Value. We consider Caputo fractional differential initial value problem again:

$$\begin{aligned} {}^{C}D_{\nabla,t_0}^{\alpha} y(t) &= f(t, y(t)), \\ D_\nabla^k y(t_0) &= b_k \quad (k = 0, 1, \ldots, m-1;\ m = -[-\alpha]), \end{aligned} \tag{100}$$

where $\alpha > 0$.

Using Theorem 27, we have

$$y(t) = y_0(t) + \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau, y(\tau)) \nabla\tau, \tag{101}$$

where

$$y_0(t) = \sum_{j=0}^{m-1} \widehat{h}_j(t, t_0) b_j. \tag{102}$$

Suppose $z(t)$ is the solution to the initial value problem:

$$\begin{aligned} {}^{C}D_{\nabla,t_0}^{\alpha} y(t) &= f(t, y(t)), \\ D_\nabla^k y(t_0) &= c_k \quad (k = 0, 1, \ldots, m-1;\ m = -[-\alpha]). \end{aligned} \tag{103}$$

We denote $\|y\| := \sup_{t\in\Omega} y(t)$. We can derive the dependency of the solution upon the initial value.

Theorem 31. *Let $y(t), z(t)$ be the solutions to* (100) *and* (103), *respectively, and let $t_0, t, s \in \Omega$, $|\widehat{h}_{\alpha-1}(t, s)| \le M$. Suppose f satisfies the Lipschitz condition; that is,*

$$\left|f(t, z) - f(t, y)\right| \le A\left|z - y\right| \quad (A > 0). \tag{104}$$

Then we have

$$\left|z(t) - y(t)\right| \le \left\|z_0 - y_0\right\| \sum_{j=0}^{\infty} (AM)^j \frac{(\sigma(t) - t_0)^j}{j!}, \quad \forall t \in \Omega. \tag{105}$$

Proof. By the proof of Theorem 28, we know that $y(t) = \lim_{m\to\infty} y_m(t), z(t) = \lim_{m\to\infty} z_m(t)$, where

$$\begin{aligned} y_0(t) &= \sum_{j=0}^{m-1} \widehat{h}_j(t, t_0) b_j, \\ y_m(t) &= y_0(t) + \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau, y_{m-1}(\tau)) \nabla\tau, \\ z_0(t) &= \sum_{j=0}^{m-1} \widehat{h}_j(t, t_0) c_j, \\ z_m(t) &= z_0(t) + \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau, z_{m-1}(\tau)) \nabla\tau. \end{aligned} \tag{106}$$

Using the Lipschitz condition, we have

$$\begin{aligned} &\left|z_1(t) - y_1(t)\right| \\ &\quad \le \left\|z_0 - y_0\right\| + \int_{t_0}^{t} \left|\widehat{h}_{\alpha-1}(t, \rho(\tau))\right| \left|f(\tau, z_0(\tau)) - f(\tau, y_0(\tau))\right| \nabla\tau \\ &\quad \le \left\|z_0 - y_0\right\| + M \int_{t_0}^{t} A\left|z_0(\tau) - y_0(\tau)\right| \nabla\tau \\ &\quad \le \left\|z_0 - y_0\right\| + \left\|z_0 - y_0\right\| AM \int_{t_0}^{t} \nabla\tau \\ &\quad = \left\|z_0 - y_0\right\| + \left\|z_0 - y_0\right\| AM \widehat{h}_1(t, t_0) \\ &\quad = \left\|z_0 - y_0\right\| \left[1 + AM \widehat{h}_1(t, t_0)\right]. \end{aligned} \tag{107}$$

Suppose

$$\left|z_{m-1}(t) - y_{m-1}(t)\right| \le \left\|z_0 - y_0\right\| \sum_{j=0}^{m-1} (AM)^j \widehat{h}_j(t, t_0), \tag{108}$$

then

$$\begin{aligned}
&\left|z_m(t) - y_m(t)\right| \\
&\quad \le \|z_0 - y_0\| \\
&\qquad + \int_{t_0}^{t} \left|\widehat{h}_{\alpha-1}(t,\rho(\tau))\right| \\
&\qquad\qquad \times \left|f(\tau, z_{m-1}(\tau)) - f(\tau, y_{m-1}(\tau))\right| \nabla\tau \\
&\quad \le \|z_0 - y_0\| \\
&\qquad + M \int_{t_0}^{t} A \left|z_{m-1}(\tau) - y_{m-1}(\tau)\right| \nabla\tau \\
&\quad \le \|z_0 - y_0\| \\
&\qquad + M \int_{t_0}^{t} A \|z_0 - y_0\| \sum_{j=0}^{m-1} (AM)^j \widehat{h}_j(\tau, t_0) \nabla\tau \\
&\quad = \|z_0 - y_0\| + \|z_0 - y_0\| \sum_{j=0}^{m-1} (AM)^{j+1} \int_{t_0}^{t} \widehat{h}_j(\tau, t_0) \nabla\tau \\
&\quad = \|z_0 - y_0\| + \|z_0 - y_0\| \sum_{j=0}^{m-1} (AM)^{j+1} \widehat{h}_{j+1}(t, t_0) \\
&\quad = \|z_0 - y_0\| \sum_{j=0}^{m} (AM)^j \widehat{h}_j(t, t_0).
\end{aligned} \tag{109}$$

According to mathematical induction, we have

$$\begin{aligned}
\left|z_m(t) - y_m(t)\right| &\le \|z_0 - y_0\| \sum_{j=0}^{m} (AM)^j \widehat{h}_j(t, t_0) \\
&\le \|z_0 - y_0\| \sum_{j=0}^{m} (AM)^j h_j(\sigma(t), t_0) \\
&\le \|z_0 - y_0\| \sum_{j=0}^{m} (AM)^j \frac{(\sigma(t) - t_0)^j}{j!}.
\end{aligned} \tag{110}$$

Taking the limit $m \to \infty$, we obtain that

$$\left|z(t) - y(t)\right| \le \|z_0 - y_0\| \sum_{j=0}^{\infty} (AM)^j \frac{(\sigma(t) - t_0)^j}{j!}, \tag{111}$$

and the proof is completed. □

4.3. Initial Value Problems for Nonlinear Term Containing Fractional Derivative. In this section, we are interested in the nonlinear differential equation

$${}^{C}D^{\alpha}_{\nabla,t_0} u(t) = f\left(t, {}^{C}D^{\beta}_{\nabla,t_0} u(t)\right) \quad (t \in \Omega, t > t_0), \tag{112}$$

of fractional order $\alpha \in (m-1, m)$, where $\beta \in (n-1, n)$, $m, n \in \mathbb{N}$, and $\alpha > \beta$, with the initial conditions

$$D^{k}_{\nabla,t_0} u(t_0) = \eta_k, \quad k = 0, \ldots, m-1. \tag{113}$$

We obtain the existence of at least one solution for integral equations using the Leray-Schauder Nonlinear Alternative for several types of initial value problems and establish sufficient conditions for unique solutions using the Banach contraction principle.

Our objective is to find solutions to the initial value problem (112) and (113) in the space $AC^{m}_{\nabla}(\Omega)$. There are two cases to investigate: $n-1 < \beta < n \le m-1 < \alpha < m$ and $n-1 < \beta < \alpha < n$.

Throughout this section, we suppose that the following are satisfied:

(H_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a ld-continuously and nabla differentiable function;

(H_2) there exist nonnegative functions $a_1, a_2 \in AC_{\nabla}(\Omega)$ such that $|f(t,z)| \le a_1(t) + a_2(t)|z|$;

(H_3) $f(t_0, 0) = 0$ and $f(t, 0) \ne 0$ on a compact subinterval of $\Omega \setminus \{t_0\}$.

The following shows that the solvability of the initial value problem (112) and (113) is equivalent to that of the Volterra-type integral equation (115) in the space $AC_{\nabla}(\Omega)$.

Lemma 32. *Let $n-1 < \beta < n \le m-1 < \alpha < m$ and assume that (H_1) and (H_3) hold. A function $u(t) \in AC^{m}_{\nabla}(\Omega)$ is a solution of the initial value problem* (112) *and* (113) *if and only if*

$$u(t) = \sum_{k=0}^{n-1} \widehat{h}_k(t, t_0) \eta_k + \int_{t_0}^{t} \widehat{h}_{n-1}(t, \rho(s)) v(s) \nabla s, \quad t \in \Omega, \tag{114}$$

where $v \in AC_{\nabla}(\Omega)$ is a solution of the integral equation

$$\begin{aligned}
v(t) = &\sum_{i=0}^{m-n-1} \widehat{h}_i(t, t_0) \eta_{n+i} \\
&+ \int_{t_0}^{t} \widehat{h}_{\alpha-n-1}(t, \rho(s)) \\
&\quad \times f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s, \rho(\tau)) v(\tau) \nabla\tau\right) \nabla s.
\end{aligned} \tag{115}$$

Proof. By (63), we have

$${}^{C}D^{\alpha-n}_{\nabla,t_0} D^{n}_{\nabla} u(t) = {}^{C}D^{\alpha}_{\nabla,t_0} u(t) = f\left(t, {}^{C}D^{\beta}_{\nabla,t_0} u(t)\right). \tag{116}$$

By using Property 6, ${}^{C}D^{\beta}_{\nabla,t_0}u(t) = I^{n-\beta}_{\nabla,t_0}D^{n}_{\nabla}u(t)$, thus we have

$$ {}^{C}D^{\alpha-n}_{\nabla,t_0}D^{n}_{\nabla}u(t) = f\left(t, \int_{t_0}^{t} \widehat{h}_{n-\beta-1}(t,\rho(\tau)) D^{n}_{\nabla}u(\tau)\nabla\tau\right). \tag{117}$$

Let $v(t) = D^{n}_{\nabla}u(t)$, by using Theorem 27, we obtain

$$\begin{aligned} v(t) = &\sum_{i=0}^{m-n-1} \widehat{h}_i(t,t_0) D^{i}_{\nabla}v(t_0) \\ &+ \int_{t_0}^{t} \widehat{h}_{\alpha-n-1}(t,\rho(s)) \\ &\quad\times f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau)) v(\tau)\nabla\tau\right)\nabla s. \end{aligned} \tag{118}$$

As $D^{i}_{\nabla}v(t) = D^{n+i}_{\nabla}u(t)$ and by (113), the above equation transforms into (115). An application of Definition 7 and Theorem 9 yields (114) in view of $v(t) = D^{n}_{\nabla}u(t)$ and (113).

To prove the converse, let $v \in AC_{\nabla}(\Omega)$ be a solution of (115). Since $v \in AC_{\nabla}(\Omega)$, the function

$$ s \longrightarrow \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau)) v(\tau)\nabla\tau \tag{119}$$

is ld-continuous on $\Omega \setminus \{t_0\}$ and so is

$$ s \longrightarrow f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau)) v(\tau)\nabla\tau\right). \tag{120}$$

We have

$$\begin{aligned} D^{n}_{\nabla}u(t) &= v(t) \\ &= \sum_{i=0}^{m-n-1} \widehat{h}_i(t,t_0)\eta_{n+i} \\ &\quad+ \int_{t_0}^{t} \widehat{h}_{\alpha-n-1}(t,\rho(s)) \\ &\qquad\times f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau)) D^{n}_{\nabla}u(\tau)\nabla\tau\right)\nabla s \\ &= \sum_{i=0}^{m-n-1} \widehat{h}_i(t,t_0)\eta_{n+i} + I^{\alpha-n}_{\nabla,t_0} f\left(t, {}^{C}D^{\beta}_{\nabla,t_0}u(t)\right). \end{aligned} \tag{121}$$

Since $\alpha - n \in (m-n-1, m-n)$, by

$$\begin{aligned} {}^{C}D^{\alpha}_{\nabla,t_0}u(t) &= {}^{C}D^{\alpha-n}_{\nabla,t_0}D^{n}_{\nabla}u(t) \\ &= {}^{C}D^{\alpha-n}_{\nabla,t_0}\left(\sum_{i=0}^{m-n-1} \widehat{h}_i(t,t_0)\eta_{n+i}\right) \\ &\quad+ {}^{C}D^{\alpha-n}_{\nabla,t_0} I^{\alpha-n}_{\nabla,t_0} f\left(t, {}^{C}D^{\beta}_{\nabla,t_0}u(t)\right) \\ &= f\left(t, {}^{C}D^{\beta}_{\nabla,t_0}u(t)\right), \end{aligned} \tag{122}$$

so u is a solution of (112) in view of (H_1). By absolute continuity of the integral, differentiating (115), we obtain

$$\begin{aligned} D^{k}_{\nabla}v(t) = &\sum_{i=0}^{m-n-1} D^{k}_{\nabla}\widehat{h}_i(t,t_0)\eta_{n+i} \\ &+ D^{k}_{\nabla} I^{\alpha-n}_{\nabla,t_0} f\left(t, {}^{C}D^{\beta}_{\nabla,t_0}u(t)\right) \\ = &\sum_{i=0}^{k-1} D^{k}_{\nabla}\widehat{h}_i(t,t_0)\eta_{n+i} + \sum_{i=k}^{m-n-1} D^{k}_{\nabla}\widehat{h}_i(t,t_0)\eta_{n+i} \\ &+ D^{k}_{\nabla} I^{\alpha-n}_{\nabla,t_0} f\left(t, {}^{C}D^{\beta}_{\nabla,t_0}u(t)\right) \\ = &\, 0 + \sum_{i=k}^{m-n-1} \widehat{h}_{i-k}(t,t_0)\eta_{n+i} + I^{\alpha-n-k}_{\nabla,t_0} f\left(t, {}^{C}D^{\beta}_{\nabla,t_0}u(t)\right) \end{aligned} \tag{123}$$

for each $k = 0,\ldots,m-n-1$. Thus, $D^{n+k}_{\nabla}u(t_0) = D^{k}_{\nabla}v(t_0) = \eta_{n+k}, k = 0,\ldots,m-n-1$; that is, $D^{i}_{\nabla}u(t_0) = \eta_i, i = n,\ldots,m-1$. On the other hand, from (114),

$$\begin{aligned} D^{i}_{\nabla}u(t) &= \sum_{k=0}^{n-1} D^{i}_{\nabla}\widehat{h}_k(t,t_0)\eta_k + D^{i}_{\nabla} I^{n}_{\nabla,t_0} v(t) \\ &= \sum_{k=0}^{i-1} D^{i}_{\nabla}\widehat{h}_k(t,t_0)\eta_k + \sum_{k=i}^{n-1} D^{i}_{\nabla}\widehat{h}_k(t,t_0)\eta_k + I^{n-i}_{\nabla,t_0} v(t) \\ &= \sum_{k=i}^{n-1} D^{i}_{\nabla}\widehat{h}_k(t,t_0)\eta_k + I^{n-i}_{\nabla,t_0} v(t) \end{aligned} \tag{124}$$

and thus $D^{i}_{\nabla}u(t_0) = \eta_i$ $(i = 0,\ldots,n-1)$. Also it is easy to see that $D^{m-n}_{\nabla}v(t) = D^{m}_{\nabla}u(t) \in AC_{\nabla}(\Omega)$. □

For the sake of brevity, by ϕ, we denote the first term in the right-hand side of (115).

Theorem 33. *Suppose that (H_1)–(H_3) hold. Then the integral equation (115) has a solution in $AC_{\nabla}(\Omega)$ provided*

$$\begin{aligned} A = \sup_{t\in\Omega} \int_{t_0}^{t} &\left(\left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right| \right. \\ &\left. \times \int_{t_0}^{s} \left|\widehat{h}_{n-\beta-1}(s,\rho(\tau))\right| \nabla\tau \right) a_2(s)\nabla s < 1, \\ 0 < B = \sup_{t\in\Omega} &\left(|\phi(t)| + \int_{t_0}^{t} \left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right| a_1(s)\nabla s \right) < \infty. \end{aligned} \tag{125}$$

Proof. In the normed space $(AC_{\nabla}(\Omega), \|\cdot\|_0)$ with the sup-norm $\|\cdot\|_0$, we define the mapping T by

$$(Tv)(t) = \phi(t)$$

$$+\int_{t_0}^{t}\widehat{h}_{\alpha-n-1}(t,\rho(s))\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\nabla\tau\right)\nabla s \tag{126}$$

for all $t\in\Omega$. Indeed, one can easily verify that the mapping T is well defined and $T: AC_{\nabla}(\Omega)\to AC_{\nabla}(\Omega)$.

Let

$$U=\{v\in AC_{\nabla}(\Omega):\|v\|_0<R\} \tag{127}$$

with

$$R=\frac{B}{1-A}>0. \tag{128}$$

Let $\mathscr{C}\subset AC_{\nabla}(\Omega)$ be defined by $\mathscr{C}=\overline{U}$.

Let $v\in\overline{U}$; that is, $\|v\|_0\le R$. Then

$$\begin{aligned}
&\|Tv\|_0\\
&=\sup_{t\in\Omega}\left|\phi(t)+\int_{t_0}^{t}\widehat{h}_{\alpha-n-1}(t,\rho(s))\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\nabla\tau\right)\nabla s\right|\\
&\le\sup_{t\in\Omega}\left(|\phi(t)|+\int_{t_0}^{t}\left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right|\times\left|f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\nabla\tau\right)\right|\nabla s\right)\\
&\le\sup_{t\in\Omega}\left(|\phi(t)|+\int_{t_0}^{t}\left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right|\times\left(a_1(s)+a_2(s)\times\int_{t_0}^{s}\left|\widehat{h}_{n-\beta-1}(s,\rho(\tau))\right||v(\tau)|\nabla\tau\right)\nabla s\right)\\
&\le\sup_{t\in\Omega}\left(|\phi(t)|+\int_{t_0}^{t}\left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right|a_1(s)\nabla s\right)\\
&\quad+\sup_{t\in\Omega}\int_{t_0}^{t}\left(\left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right|\times\int_{t_0}^{s}\left|\widehat{h}_{n-\beta-1}(s,\rho(\tau))\right|\nabla\tau\right)a_2(s)\nabla s\|v\|_0\\
&=B+A\|v\|_0\\
&\le B+AR\\
&=R,
\end{aligned} \tag{129}$$

which shows that $Tv\in\mathscr{C}$.

In addition,

$$\begin{aligned}
&|(Tv)(t_1)-(Tv)(t_2)|\\
&\le|\phi(t_1)-\phi(t_2)|\\
&\quad+\left|\int_{t_0}^{t_1}\widehat{h}_{\alpha-n-1}(t_1,\rho(s))\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\nabla\tau\right)\nabla s\right.\\
&\qquad\left.-\int_{t_0}^{t_2}\widehat{h}_{\alpha-n-1}(t_2,\rho(s))\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\nabla\tau\right)\nabla s\right|\\
&=\left|\sum_{i=0}^{m-n-1}\widehat{h}_i(t_1,t_0)\,\eta_{n+i}-\sum_{i=0}^{m-n-1}\widehat{h}_i(t_2,t_0)\,\eta_{n+i}\right|\\
&\quad+\left|\int_{t_0}^{t_1}\int_{\rho(s)}^{t_1}\widehat{h}_{\alpha-n-2}(\theta,\rho(s))\nabla\theta\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\nabla\tau\right)\nabla s\right.\\
&\qquad\left.-\int_{t_0}^{t_2}\int_{\rho(s)}^{t_2}\widehat{h}_{\alpha-n-2}(\theta,\rho(s))\nabla\theta\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\nabla\tau\right)\nabla s\right|\\
&\le\left|\sum_{i=0}^{m-n-1}\int_{t_0}^{t_1}\widehat{h}_{i-1}(\tau,t_0)\nabla\tau\eta_{n+i}-\sum_{i=0}^{m-n-1}\int_{t_0}^{t_2}\widehat{h}_{i-1}(\tau,t_0)\nabla\tau\eta_{n+i}\right|\\
&\quad+\left|\int_{t_0}^{t_1}\int_{\rho(s)}^{t_1}\widehat{h}_{\alpha-n-2}(\theta,\rho(s))\nabla\theta\right.
\end{aligned}$$

$$\begin{aligned}
&\times f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau))\, v(\tau)\,\nabla\tau\right)\nabla s \\
&- \int_{t_0}^{t_1}\int_{\rho(s)}^{t_2} \widehat{h}_{\alpha-n-2}(\theta,\rho(s))\,\nabla\theta \\
&\times f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau))\, v(\tau)\,\nabla\tau\right)\nabla s\Bigg| \\
&+ \Bigg|\int_{t_0}^{t_1}\int_{\rho(s)}^{t_2} \widehat{h}_{\alpha-n-2}(\theta,\rho(s))\,\nabla\theta \\
&\times f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau))\, v(\tau)\,\nabla\tau\right)\nabla s \\
&- \int_{t_0}^{t_2}\int_{\rho(s)}^{t_2} \widehat{h}_{\alpha-n-2}(\theta,\rho(s))\,\nabla\theta \\
&\times f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau))\, v(\tau)\,\nabla\tau\right)\nabla s\Bigg| \\
&\le \sum_{i=0}^{m-n-1}\left|\int_{t_2}^{t_1} \widehat{h}_{i-1}(\tau,t_0)\,\nabla\tau\,\eta_{n+i}\right| \\
&+ \int_{t_0}^{t_1}\left|\int_{t_2}^{t_1} \widehat{h}_{\alpha-n-2}(\theta,\rho(s))\,\nabla\theta\right| \\
&\times \left|f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau))\, v(\tau)\,\nabla\tau\right)\right|\nabla s \\
&+ \int_{t_2}^{t_1}\left|\int_{\rho(s)}^{t_2} \widehat{h}_{\alpha-n-2}(\theta,\rho(s))\,\nabla\theta\right| \\
&\times \left|f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau))\, v(\tau)\,\nabla\tau\right)\right|\nabla s \\
&\le M\sum_{i=0}^{m-n-1}\left|\eta_{n+i}\right|\left|t_1-t_2\right| + M^2\left|t_1-t_2\right| + M\left|t_1-t_2\right| \\
&= \left(M\sum_{i=0}^{m-n-1}\left|\eta_{n+i}\right| + M^2 + M\right)\left|t_1-t_2\right|,
\end{aligned} \tag{130}$$

where $\max_{\tau,\theta,s,t_1,t_2\in\Omega}\{\widehat{h}_{i-1}(\tau,t_0)\ (i = 0,\dots,m-n-1)$, $\widehat{h}_{\alpha-n-2}(\theta,\rho(s))$, $\int_{t_0}^{t_1}|f(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))v(\tau)\nabla\tau)|\nabla s$, $\int_{t_2}^{t_1}|f(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))v(\tau)\nabla\tau)|\nabla s\} \le M$.

Thus, Tv is equicontinuous on Ω. This shows that T is a compact mapping.

Consider the eigenvalue problem

$$v = \lambda T v, \quad \lambda \in (0,1). \tag{131}$$

Under the assumption that v is a solution of (131) for a $\lambda \in (0,1)$, one obtains

$$\begin{aligned}
&\|v\|_0 \\
&= \sup_{t\in\Omega}\Bigg|\lambda\phi(t) \\
&\quad + \lambda\int_{t_0}^{t}\widehat{h}_{\alpha-n-1}(t,\rho(s)) \\
&\quad \times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\,\nabla\tau\right)\nabla s\Bigg| \\
&< \sup_{t\in\Omega}\Bigg(\left|\phi(t)\right| \\
&\quad + \int_{t_0}^{t}\left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right| \\
&\quad \times \left|f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\,\nabla\tau\right)\right|\nabla s\Bigg) \\
&\le B + A\|v\|_0 \le R,
\end{aligned} \tag{132}$$

which shows that $v \notin \partial U$. By Theorem 12, T has a fixed point in $\overline{U}$, which we denote by v_0, such that $\|v_0\|_0 \le R$. □

It follows from Lemma 32 that

$$u_0(t) = \sum_{k=0}^{n-1}\widehat{h}_k(t,t_0)\,\eta_k + \int_{t_0}^{t}\widehat{h}_{n-1}(t,\rho(s))\,v_0(s)\,\nabla s, \quad t\in\Omega, \tag{133}$$

is a solution of (112) and (113).

In the following, we will discuss another case: $n-1 < \beta < \alpha < n$.

Lemma 34. *Let $n-1 < \beta < \alpha < n$ and suppose that (H_1) and (H_3) hold. A function $u \in AC_{\nabla}^{n}(\Omega)$ is a solution of the initial value problem* (112) *and* (113) *if and only if*

$$u(t) = \sum_{k=0}^{n-1}\widehat{h}_k(t,t_0)\,\eta_k + \int_{t_0}^{t}\widehat{h}_{\beta-1}(t,\rho(s))\,v(s)\,\nabla s, \quad t\in\Omega, \tag{134}$$

where $v \in AC_{\nabla}(\Omega)$ is a solution of

$$v(t) = \int_{t_0}^{t}\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\,f(s,v(s))\,\nabla s. \tag{135}$$

Proof. Let $u \in AC_{\nabla}^{n}(\Omega)$ be a solution of the

$${}^{C}D_{\nabla,t_0}^{\alpha-\beta}\,{}^{C}D_{\nabla,t_0}^{\beta}u(t) = {}^{C}D_{\nabla,t_0}^{\alpha}u(t) = f\left(t,{}^{C}D_{\nabla,t_0}^{\beta}u(t)\right), \tag{136}$$

which, after the substitution $v(t) = {}^{C}D_{\nabla,t_0}^{\beta}u(t)$, becomes

$${}^{C}D_{\nabla,t_0}^{\alpha-\beta}v(t) = f(t,v(t)). \tag{137}$$

Next, by Property 9 and (113)

$$u(t)=\sum_{k=0}^{n-1}\widehat{h}_k(t,t_0)\eta_k+\int_{t_0}^{t}\widehat{h}_{\beta-1}(t,\rho(s))v(s)\nabla s,$$
$$v(t)=v(t_0)+\int_{t_0}^{t}\widehat{h}_{\alpha-\beta-1}(t,\rho(s))f(s,v(s))\nabla s. \tag{138}$$

By Property 6, we have that $v(t_0)={}^{C}D^{\beta}_{\nabla,t_0}u(t_0)=0$, and thus the above equation becomes (135).

Conversely, let $v\in AC_{\nabla}(\Omega)$ be a solution of the integral equation (135); that is,

$$v(t)=\int_{t_0}^{t}\widehat{h}_{\alpha-\beta-1}(t,\rho(s))f(s,v(s))\nabla s, \tag{139}$$

$$u(t)=\sum_{k=0}^{n-1}\widehat{h}_k(t,t_0)\eta_k+\int_{t_0}^{t}\widehat{h}_{\beta-1}(t,\rho(s))v(s)\nabla s. \tag{140}$$

Then, by (H_1),

$${}^{C}D^{\alpha-\beta}_{\nabla,t_0}v(t)=f(t,v(t)),\quad t\in\Omega,\ t>t_0, \tag{141}$$

and ${}^{C}D^{\beta}_{\nabla,t_0}u(t)=v(t)$. Hence

$${}^{C}D^{\alpha-\beta C}_{\nabla,t_0}D^{\beta}_{\nabla,t_0}u(t)=f\left(t,{}^{C}D^{\beta}_{\nabla,t_0}u(t)\right)\quad(t\in\Omega,t>t_0), \tag{142}$$

and we obtain (112). Also, it follows from (140) that $u\in AC^{n}_{\nabla}(\Omega)$ and (113) are satisfied since, for $i=0,\dots,n-1$,

$$\begin{aligned}D^{i}_{\nabla}u(t)&=\sum_{k=0}^{n-1}D^{i}_{\nabla}\widehat{h}_k(t,t_0)\eta_k+D^{i}_{\nabla}I^{\beta}_{\nabla,t_0}v(t)\\&=\sum_{k=0}^{i-1}D^{i}_{\nabla}\widehat{h}_k(t,t_0)\eta_k+\sum_{k=i}^{n-1}D^{i}_{\nabla}\widehat{h}_k(t,t_0)\eta_k+D^{i}_{\nabla}I^{\beta}_{\nabla,t_0}v(t)\\&=\sum_{k=i}^{n-1}D^{i}_{\nabla}\widehat{h}_k(t,t_0)\eta_k+I^{\beta-i}_{\nabla,t_0}v(t).\end{aligned} \tag{143}$$

□

Our next existence result corresponds to the case $n-1<\beta<\alpha<n$.

Theorem 35. *Suppose that (H_1)–(H_3) are satisfied. Then the integral equation (135) has a solution in $AC_{\nabla}(\Omega)$ provided*

$$A=\sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|a_2(s)\nabla s<1,$$
$$0<B=\sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|a_1(s)\nabla s<\infty. \tag{144}$$

Proof. We endow $AC_{\nabla}(\Omega)$ with the sup-norm and define, for $v\in AC_{\nabla}(\Omega)$, the mapping T by

$$Tv(t)=\int_{t_0}^{t}\widehat{h}_{\alpha-\beta-1}(t,\rho(s))f(s,v(s))\nabla s. \tag{145}$$

The mapping T is well defined and $T:AC_{\nabla}(\Omega)\to AC_{\nabla}(\Omega)$. Let

$$U=\{v\in AC_{\nabla}(\Omega):\|v\|_0<R\} \tag{146}$$

with

$$R=\frac{B}{1-A}>0. \tag{147}$$

Let $\mathscr{C}\subset AC_{\nabla}(\Omega)$ be defined by $\mathscr{C}=\overline{U}$.

If $v\in\overline{U}$, then

$$\begin{aligned}\|Tv\|_0&=\sup_{t\in\Omega}\left|\int_{t_0}^{t}\widehat{h}_{\alpha-\beta-1}(t,\rho(s))f(s,v(s))\nabla s\right|\\&\le\sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|\left|f(s,v(s))\right|\nabla s\\&\le\sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|(a_1(s)+a_2(s)|v(s)|)\nabla s\\&\le\sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|a_1(s)\nabla s\\&\quad+\sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|a_2(s)\nabla s\|v\|_0\\&=B+A\|v\|_0\\&\le R;\end{aligned} \tag{148}$$

that is, $T:\overline{U}\to\mathscr{C}$. Certainly, $T:\overline{U}\to\mathscr{C}$ is continuous and compact. Consider

$$v=\lambda Tv,\quad \lambda\in(0,1). \tag{149}$$

The rest of the proof is the same as the corresponding part of the proof of Theorem 30. □

The uniqueness results are based on applications of the Banach contraction principle.

The main assumption in the existence theorems below is that

(H_4) for each $R>0$, there exists a nonnegative function γ such that $|f(t,z_1(t))-f(t,z_2(t))|\le\gamma(t)|z_1-z_2|$, $t\in\Omega$, $z_1,z_2\in\mathbb{R}$.

The first uniqueness result is for the case $n-1<\beta<\alpha<n$.

Theorem 36. *Suppose that (H_1), (H_3), and (H_4) hold. Assume that*

$$\zeta=\sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|\gamma(s)\nabla s<1,$$
$$0<\sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|\left|f(s,0)\right|\nabla s<\infty. \tag{150}$$

Then the integral equation (135) has a unique solution.

Proof. In the Banach space $\mathscr{B} = (AC_{\nabla}(\Omega), \|\cdot\|_0)$ we define $\mathscr{C}$ by

$$\mathscr{C} = \{v \in \mathscr{B} : \|v\|_0 \le R\}, \tag{151}$$

where

$$R = \frac{1}{1-\zeta} \sup_{t\in\Omega} \int_{t_0}^{t} \left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right| |f(s,0)| \nabla s. \tag{152}$$

We define the mapping $T : AC_{\nabla}(\Omega) \to AC_{\nabla}(\Omega)$ as in the proof of Theorem 31.

If $v \in \mathscr{C}$, then

$$\begin{aligned}\|Tv\|_0 &\le \|Tv - T\theta\|_0 + \|T\theta\|_0 \\ &\le \zeta\|v\|_0 + \sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right||f(s,0)|\nabla s \\ &= \zeta\|v\|_0 + (1-\zeta)R \\ &\le R;\end{aligned} \tag{153}$$

that is, $T : \mathscr{C} \to \mathscr{C}$.

Let $v_1, v_2 \in \mathscr{C}$. Then

$$\begin{aligned}&\|Tv_1 - Tv_2\|_0 \\ &= \sup_{t\in\Omega}|Tv_1 - Tv_2| \\ &\le \sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right||f(s,v_1(s)) - f(s,v_2(s))|\nabla s \\ &\le \sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|\gamma(s)|v_1(s) - v_2(s)|\nabla s \\ &\le \sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|\gamma(s)\nabla s\|v_1 - v_2\|_0 \\ &\le \zeta\|v_1 - v_2\|_0;\end{aligned} \tag{154}$$

that is, T is a contraction since $\zeta < 1$.

By the Banach contraction principle, T has a unique fixed point, which is a solution of the integral equation (135). □

For the case $n-1 < \beta < n \le m-1 < \alpha < m$, the uniqueness result is given without proof.

Theorem 37. *Suppose that (H_1), (H_3), and (H_4) hold and assume that*

$$\zeta = \sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|\gamma(s) \times\left(\int_{t_0}^{s}\left|\widehat{h}_{n-\beta-1}(s,\rho(\tau))\right|\nabla\tau\right)\nabla s < 1. \tag{155}$$

Assume further that

$$0 < \sup_{t\in\Omega}\left(|\phi(t)| + \int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right||f(s,0)|\nabla s\right) < \infty. \tag{156}$$

Then the integral equation (115) *has a unique solution.*

5. Laplace Transform Method for Solving Ordinary Differential Equations with Caputo Fractional Derivatives

5.1. Homogeneous Equations with Constant Coefficients. In this section we apply the Laplace transform method to derive explicit solutions to homogeneous equations of the form

$$\begin{gathered}\sum_{k=1}^{m} A_k\left[{}^{C}D_{\nabla,t_0}^{\alpha_k}y(t)\right] + A_0 y(t) = 0 \\ (m\in\mathbb{N};\ 0<\alpha_1<\cdots<\alpha_m; \\ l-1<\alpha_m<l,\ l\in\mathbb{N},\ t_0,t\in\Omega_{k^l},\ t>t_0)\end{gathered} \tag{157}$$

involving the Caputo fractional derivatives ${}^{C}D_{\nabla,t_0}^{\alpha_k}y$ $(k = 1,\dots,m)$, with real constants $A_k \in \mathbb{R}$ $(k = 0,\dots,m-1)$ and $A_m = 1$.

The Laplace transform method is based on the formula:

$$\begin{aligned}&\mathscr{L}_{\nabla,t_0}\left\{{}^{C}D_{\nabla,t_0}^{\alpha}y(t)\right\}(z) \\ &\quad= z^{\alpha}\mathscr{L}_{\nabla,t_0}\{y(t)\}(z) \\ &\quad\quad-\sum_{j=0}^{l-1}d_j z^{\alpha-j-1} \quad (l-1<\alpha\le l\in\mathbb{N}),\end{aligned} \tag{158}$$

$$d_j = D_{\nabla}^{j}y(t_0) \quad (j = 0,\dots,l-1). \tag{159}$$

First, we derive explicit solutions to (157) with $m = 1$:

$$\begin{gathered}{}^{C}D_{\nabla,t_0}^{\alpha}y(t) - \lambda y(t) = 0 \\ (t>t_0;\ l-1<\alpha\le l;\ l\in\mathbb{N};\ \lambda\in\mathbb{R}).\end{gathered} \tag{160}$$

In order to prove our result, we also need the following definition and lemma.

Definition 38. The function $W(t)$ is defined by

$$W(t) = \det\left(\left(D_{\nabla}^{k}y_j\right)(t)\right)_{k,j=1}^{n} \quad (t\in\Omega_{k^n}). \tag{161}$$

Lemma 39. *The solutions $y_1(t), y_2(t), \dots, y_n(t)$ are linearly independent if and only if $W(t^*) \ne 0$ at some point $t^* \in \Omega$.*

Proof. We first prove sufficiency. If, to the contrary, $y_j(t)$ $(j = 1,2,\dots,n)$ are linearly dependent in Ω, then there exist n constants $\{c_j\}_{j=1}^{n}$, not all zero, such that

$$\left(\left(D_{\nabla,t_0}^{k}y_j\right)(t)\right)_{k,j=1}^{n}\begin{pmatrix}c_1\\ c_2\\ \vdots\\ c_n\end{pmatrix} \equiv 0 \tag{162}$$

holds, and thus, $W(t) \equiv 0$ which leads to a contradiction. Therefore, if $W(t^*) \ne 0$ at some point $t^* \in \Omega$, then $y_1(t), y_2(t), \dots, y_n(t)$ are linearly independent. Now we prove the necessity. Suppose, to the contrary, for any $t \in \Omega$, $W(t) = 0$. Consider the equations

$$\left(\left(D^k_{\nabla,t_0} y_j\right)(t^*)\right)^n_{k,j=1} C = 0, \tag{163}$$

where $t^* \in \Omega$, $C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$. As $W(t^*) = 0$, the equations have nontrivial solution c_j $(j = 1, 2, \ldots, n)$. Now we construct a function using these constants:

$$y(t) = \sum_{j=1}^{n} c_j y_j(t), \tag{164}$$

and we get $y(t)$ as a solution. From (163), we obtain that $y(t)$ satisfies initial value condition

$$D^k_{\nabla,t_0} y(t^*) = 0, \quad k = 1, \ldots, n. \tag{165}$$

However, $y(t) = 0$ is also a solution to equation satisfying the initial value condition. By the uniqueness of solution, we have

$$\sum_{j=1}^{n} c_j y_j(t) = 0, \tag{166}$$

and thus, $y_j(t)$ $(j = 1, 2, \ldots, n)$ are linearly dependant which leads to a contradiction. Thus, if the solutions $y_1(t), y_2(t), \ldots, y_n(t)$ are linearly independent, then $W(t^*) \neq 0$ at some point $t^* \in \Omega$. The result follows. □

The following statements hold.

Theorem 40. *Let $l - 1 < \alpha \le l$ $(l \in \mathbb{N})$ and $\lambda \in \mathbb{R}$. Then the functions*

$$y_j(t) = {}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0) \quad (j = 0, \ldots, l-1) \tag{167}$$

yield the fundamental system of solutions to (160).

Proof. Applying the Laplace transform to (160) and taking (158) into account, we have

$$\mathscr{L}_{\nabla,t_0}\{y(t)\}(z) = \sum_{j=0}^{l-1} d_j \frac{z^{\alpha-j-1}}{z^\alpha - \lambda}, \tag{168}$$

where d_j $(j = 0, \ldots, l-1)$ are given by (159).

Formula (36) with $\beta = j + 1$ yields

$$\mathscr{L}_{\nabla,t_0}\left\{{}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0)\right\}(z) = \frac{z^{\alpha-j-1}}{z^\alpha - \lambda} \quad \left(|\lambda| < |z|^\alpha\right). \tag{169}$$

Thus, from (168), we derive the following solution to (160):

$$y(t) = \sum_{j=0}^{l-1} d_j y_j(t), \qquad y_j(t) = {}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0). \tag{170}$$

It is easily verified that the functions $y_j(t)$ are solutions to (160):

$$\begin{gathered} {}^{C}D^{\alpha}_{\nabla,t_0}\left[{}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0)\right] = \lambda {}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0) \\ (j = 0, \ldots, l-1). \end{gathered} \tag{171}$$

In fact,

$$\begin{aligned} &{}^{C}D^{\alpha}_{\nabla,t_0}\left[{}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0)\right] \\ &\quad = {}^{C}D^{\alpha}_{\nabla,t_0}\left[\sum_{k=0}^{\infty} \lambda^k \widehat{h}_{k\alpha+j}(t, t_0)\right] \\ &\quad = {}^{C}D^{\alpha}_{\nabla,t_0} \lambda^0 \widehat{h}_j(t, t_0) + {}^{C}D^{\alpha}_{\nabla,t_0} \sum_{k=1}^{\infty} \lambda^k \widehat{h}_{k\alpha+j}(t, t_0) \\ &\quad = 0 + \sum_{k=1}^{\infty} \lambda^k \widehat{h}_{(k-1)\alpha+j}(t, t_0) \\ &\quad = \sum_{k=0}^{\infty} \lambda^{k+1} \widehat{h}_{k\alpha+j}(t, t_0) \\ &\quad = \lambda \sum_{k=0}^{\infty} \lambda^k \widehat{h}_{k\alpha+j}(t, t_0) \\ &\quad = \lambda {}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0). \end{aligned} \tag{172}$$

Moreover,

$$\begin{aligned} D^k_{\nabla} y_j(t) &= D^k_{\nabla} {}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0) \\ &= D^k_{\nabla}\left[\sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s\alpha+j}(t, t_0)\right] = \sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s\alpha+j-k}(t, t_0). \end{aligned} \tag{173}$$

It follows from (173) and (20) that

$$\begin{gathered} D^k_{\nabla} y_j(t_0) = 0 \quad (k, j = 0, \ldots, l-1;\ j > k), \\ D^k_{\nabla} y_k(t_0) = 1 \quad (k = 0, \ldots, l-1). \end{gathered} \tag{174}$$

If $j < k$, then

$$\begin{aligned} D^k_{\nabla} y_j(t_0) &= D^k_{\nabla} \widehat{h}_j(t, t_0) + D^k_{\nabla} \sum_{s=1}^{\infty} \lambda^s \widehat{h}_{s\alpha+j}(t, t_0) \\ &= 0 + \sum_{s=1}^{\infty} \lambda^s \widehat{h}_{s\alpha+j-k}(t, t_0) \\ &= \sum_{s=0}^{\infty} \lambda^{s+1} \widehat{h}_{s\alpha+\alpha+j-k}(t, t_0), \end{aligned} \tag{175}$$

and, since $\alpha + j - k \ge \alpha + 1 - l > 0$ for any $k, j = 0, \ldots, l-1$, the following relations hold:

$$D^k_{\nabla} y_j(t_0) = 0 \quad (k, j = 0, \ldots, l-1;\ j < k). \tag{176}$$

By (174) and (176), the Wronskian function

$$W(t) = \det\left(D^k_{\nabla} y_j(t)\right)^{l-1}_{k,j=0} \tag{177}$$

at t_0 is equal to 1: $W(t_0) = 1$. Then $y_j(t)$ $(j = 0, \ldots, l-1)$ yield the fundamental system of solutions to (160). □

Corollary 41. *The equation*

$$ {}^{C}D^{\alpha}_{\nabla,t_0}y(t) - \lambda y(t) = 0 \quad (t > t_0;\ 0 < \alpha \le 1;\ \lambda \in \mathbb{R}) \tag{178}$$

has its solution given by

$$ y(t) = {}_{\nabla}F_{\alpha,1}(\lambda; t, t_0), \tag{179}$$

while the equation

$$ {}^{C}D^{\alpha}_{\nabla,t_0}y(t) - \lambda y(t) = 0 \quad (t > t_0;\ 1 < \alpha \le 2;\ \lambda \in \mathbb{R}) \tag{180}$$

has the fundamental system of solutions given by

$$ y_0(t) = {}_{\nabla}F_{\alpha,1}(\lambda; t, t_0), \qquad y_1(t) = {}_{\nabla}F_{\alpha,2}(\lambda; t, t_0). \tag{181}$$

Next we derive the explicit solutions to (157) with $m = 2$:

$$ {}^{C}D^{\alpha}_{\nabla,t_0}y(t) - \lambda {}^{C}D^{\beta}_{\nabla,t_0}y(t) - \mu y(t) = 0 \\ (t > t_0;\ l-1 < \alpha \le l;\ l \in \mathbb{N};\ 0 < \beta < \alpha) \tag{182}$$

with $\lambda, \mu \in \mathbb{R}$.

Theorem 42. *Let $l-1 < \alpha \le l$ ($l \in \mathbb{N}$), $0 < \beta < \alpha$, and $\lambda, \mu \in \mathbb{R}$. Then the functions*

$$ \begin{aligned} y_j(t) &= \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta,\beta n+j+1}(\lambda; t, t_0) \\ &\quad - \lambda \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta,\beta n+j+1+\alpha-\beta}(\lambda; t, t_0), \\ &\qquad j = 0, \ldots, m-1; \end{aligned} \tag{183}$$

$$ y_j(t) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta,\beta n+j+1}(\lambda; t, t_0), \quad j = m, \ldots, l-1 \tag{184}$$

yield the fundamental system of solutions to (182)*, provided that the series in* (183) *and* (184) *are convergent.*

Proof. Let $m-1 < \beta \le m$ ($m \le l$; $m \in \mathbb{N}$). Applying the Laplace transform to (182) and using (158), we obtain

$$ \mathscr{L}_{\nabla,t_0}\{y(t)\}(z) = \sum_{j=0}^{l-1} d_j \frac{z^{\alpha-j-1}}{z^{\alpha} - \lambda z^{\beta} - \mu} - \lambda \sum_{j=0}^{m-1} d_j \frac{z^{\beta-j-1}}{z^{\alpha} - \lambda z^{\beta} - \mu}, \tag{185}$$

where d_j ($j = 0, \ldots, l-1$) are given by (159).

For $z \in \mathbb{C}$ and $|\mu z^{-\beta}/(z^{\alpha-\beta} - \lambda)| < 1$, we have

$$ \begin{aligned} \frac{1}{z^{\alpha} - \lambda z^{\beta} - \mu} &= \frac{z^{-\beta}}{z^{\alpha-\beta} - \lambda} \cdot \frac{1}{1 - \left(\mu z^{-\beta}/\left(z^{\alpha-\beta} - \lambda\right)\right)} \\ &= \sum_{n=0}^{\infty} \mu^n \frac{z^{-\beta-n\beta}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}}. \end{aligned} \tag{186}$$

In addition, for $z \in \mathbb{C}$ and $|\lambda z^{\beta-\alpha}| < 1$, we have

$$ \begin{aligned} &\frac{z^{\alpha-j-1-\beta-n\beta}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\ &\quad = \frac{z^{(\alpha-\beta)-(\beta n+j+1)}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\ &\quad = \frac{1}{n!} \mathscr{L}_{\nabla,t_0}\left\{ \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta,\beta n+j+1}(\lambda; t, t_0) \right\}(z), \\ &\frac{z^{\beta-j-1-\beta-n\beta}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\ &\quad = \frac{z^{(\alpha-\beta)-(\beta n+j+1+\alpha-\beta)}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\ &\quad = \frac{1}{n!} \mathscr{L}_{\nabla,t_0}\left\{ \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta,\beta n+j+1+\alpha-\beta}(\lambda; t, t_0) \right\}(z). \end{aligned} \tag{187}$$

From (185) and (187), we derive the solution to (182):

$$ y(t) = \sum_{j=0}^{l-1} d_j y_j(t), \tag{188}$$

where $y_j(t)$ ($j = 0, \ldots, l-1$) are given by (183) for $j = 0, \ldots, m-1$ and by (184) for $j = m, \ldots, l-1$. For $k = 0, \ldots, l-1$, the direct evaluation yields

$$ \begin{aligned} &D^k_{\nabla} y_j(t) \\ &\quad = D^k_{\nabla}\left[\sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta,\beta n+j+1}(\lambda; t, t_0) \right. \\ &\qquad \left. - \lambda \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta,\beta n+j+1+\alpha-\beta}(\lambda; t, t_0) \right] \\ &\quad = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} D^k_{\nabla}\left[\sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\beta n+j}(t, t_0) \right] \\ &\qquad - \lambda \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} D^k_{\nabla}\left[\sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\beta n+j+\alpha-\beta}(t, t_0) \right] \\ &\quad = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} \sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\beta n+j-k}(t, t_0) \\ &\qquad - \lambda \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} \sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\beta n+j+\alpha-\beta-k}(t, t_0) \\ &\qquad\qquad (j = 0, \ldots, m-1), \\ &D^k_{\nabla} y_j(t) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} \sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\beta n+j-k}(t, t_0) \\ &\qquad\qquad (j = m, \ldots, l-1). \end{aligned} \tag{189}$$

For $j > k$, $D^k_{\nabla} y_j(t_0) = 0$, and for $j = k$, $D^k_{\nabla} y_j(t_0) = 1$. Thus we have $W(t_0) = 1$. Thus the functions $y_j(t)$ ($j = 0, \ldots, l-1$)

in (183) and (184) are linearly independent solutions to (182). The result follows. □

Corollary 43. *The equation*

$$ {}^{C}D^{\alpha}_{\nabla,t_0}y(t) - \lambda {}^{C}D^{\beta}_{\nabla,t_0}y(t) = 0 \quad (t > t_0;\ l-1 < \alpha \le l;\ l \in \mathbb{N};\ 0 < \beta < \alpha) \tag{190}$$

has its fundamental system of solutions given by

$$\begin{aligned} y_j(t) &= {}_{\nabla}F_{\alpha-\beta,j+1}(\lambda;t,t_0) - \lambda {}_{\nabla}F_{\alpha-\beta,\alpha-\beta+j+1} \quad (j = 0,\dots,m-1), \\ y_j(t) &= {}_{\nabla}F_{\alpha-\beta,j+1}(\lambda;t,t_0) \quad (j = m,\dots,l-1). \end{aligned}\tag{191}$$

Corollary 44. *The equation*

$$ {}^{C}D^{\alpha}_{\nabla,t_0}y(t) - \lambda {}^{C}D^{\beta}_{\nabla,t_0}y(t) - \mu y(t) = 0 \quad (t > t_0;\ 0 < \beta < \alpha \le 1;\ \lambda,\mu \in \mathbb{R}) \tag{192}$$

has its solution by

$$\begin{aligned} y_0(t) = {} & \sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\beta n+1}(\lambda;t,t_0) \\ & - \lambda\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\beta n+1+\alpha-\beta}(\lambda;t,t_0). \end{aligned}\tag{193}$$

In particular,

$$ y_0(t) = {}_{\nabla}F_{\alpha-\beta,1}(\lambda;t,t_0) - \lambda {}_{\nabla}F_{\alpha-\beta,\alpha-\beta+1}(\lambda;t,t_0) \tag{194}$$

is a solution to the equation

$$ {}^{C}D^{\alpha}_{\nabla,t_0}y(t) - \lambda {}^{C}D^{\beta}_{\nabla,t_0}y(t) = 0 \quad (t > t_0;\ 0 < \beta < \alpha \le 1;\ \lambda \in \mathbb{R}). \tag{195}$$

Corollary 45. *The equation*

$$ {}^{C}D^{\alpha}_{\nabla,t_0}y(t) - \lambda {}^{C}D^{\beta}_{\nabla,t_0}y(t) - \mu y(t) = 0, \tag{196}$$

where $t > t_0$; $1 < \alpha \le 2$, $0 < \beta < \alpha$; $\lambda,\mu \in \mathbb{R}$, has one solution $y_0(t)$, given by (193), *and a second solution $y_1(t)$ given by*

$$\begin{aligned} y_1(t) = {} & \sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\beta n+2}(\lambda;t,t_0) \\ & - \lambda\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\beta n+\alpha-\beta+2}(\lambda;t,t_0) \end{aligned}\tag{197}$$

for $1 < \beta < \alpha$, while, for $0 < \beta \le 1$, by

$$ y_1(t) = \sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\beta n+2}(\lambda;t,t_0). \tag{198}$$

In particular, the equation

$$ {}^{C}D^{\alpha}_{\nabla,t_0}y(t) - \lambda {}^{C}D^{\beta}_{\nabla,t_0}y(t) = 0 \quad (t > t_0;\ 1 < \alpha \le 2,\ 0 < \beta < \alpha;\ \lambda \in \mathbb{R}) \tag{199}$$

has one solution $y_0(t)$ given by (194), *and a second $y_1(t)$ given by*

$$ y_1(t) = {}_{\nabla}F_{\alpha-\beta,2}(\lambda;t,t_0) - \lambda {}_{\nabla}F_{\alpha-\beta,\alpha-\beta+2}(\lambda;t,t_0) \tag{200}$$

for $1 < \beta < \alpha$, while for $0 < \beta \le 1$, by

$$ y_1(t) = {}_{\nabla}F_{\alpha-\beta,2}(\lambda;t,t_0). \tag{201}$$

Finally, we find explicit solutions to (157) with any $m \in \mathbb{N}\setminus\{1,2\}$. It is convenient to rewrite (157) in the form (202)

$$\begin{aligned} & {}^{C}D^{\alpha}_{\nabla,t_0}y(t) - \lambda {}^{C}D^{\beta}_{\nabla,t_0}y(t) - \sum_{k=0}^{m-2}A_k {}^{C}D^{\alpha_k}_{\nabla,t_0}y(t) = 0 \\ & (t > t_0;\ m \in \mathbb{N}\setminus\{1,2\};\ 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{m-2} < \beta < \alpha; \\ & \lambda, A_0,\dots,A_{m-2} \in \mathbb{R}). \end{aligned}\tag{202}$$

Theorem 46. *Let $\alpha,\beta,\alpha_{m-2},\dots,\alpha_0$ and $l,l_{m-1},\dots,l_0 \in \mathbb{N}_0$ ($m \in \mathbb{N}\setminus\{1,2\}$) be such that*

$$\begin{gathered} 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{m-2} < \beta < \alpha, \\ 0 = l_0 \le l_1 \le \cdots \le l_{m-1} \le l, \\ l-1 < \alpha \le l, \\ l_{m-1}-1 < \beta \le l_{m-1}, \\ l_k - 1 < \alpha_k \le l_k \\ (k = 0,\dots,m-2), \end{gathered}\tag{203}$$

and let $\lambda, A_0,\dots,A_{m-2} \in \mathbb{R}$. Then the fundamental system of solutions to (202) *is given by the formulas*

$$\begin{aligned} y_j(t) = {} & \sum_{n=0}^{\infty}\left(\sum_{k_0+\cdots+k_{m-2}=n}\right)\frac{1}{k_0!\cdots k_{m-2}!}\left[\prod_{\nu=0}^{m-2}(A_\nu)^{k_\nu}\right] \\ & \cdot\left\{\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}(\lambda;t,t_0)\right. \\ & - \lambda\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\alpha-\beta+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}(\lambda;t,t_0) \\ & \left. - \sum_{k=0}^{m-2}A_k\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\alpha-\alpha_k+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}(\lambda;t,t_0)\right\} \end{aligned}\tag{204}$$

for $j = 0, \ldots, l_{m-2} - 1$; *by*

$$y_j(t) = \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_{m-2}=n} \right) \frac{1}{k_0! \cdots k_{m-2}!} \left[\prod_{\nu=0}^{m-2} (A_\nu)^{k_\nu} \right] \cdot \left\{ \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta, j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu} (\lambda; t, t_0) - \lambda \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta, \alpha-\beta+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu} (\lambda; t, t_0) \right\} \tag{205}$$

for $j = l_{m-2}, \ldots, l_{m-1} - 1$; *and by*

$$y_j(t) = \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_{m-2}=n} \right) \frac{1}{k_0! \cdots k_{m-2}!} \left[\prod_{\nu=0}^{m-2} (A_\nu)^{k_\nu} \right] \times \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta, j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu} (\lambda; t, t_0) \tag{206}$$

for $j = l_{m-1}, \ldots, l - 1$.

Proof. Applying the Laplace transform to (202) and using (158), we obtain

$$\mathscr{L}_{\nabla, t_0} \{y(t)\}(z) = \sum_{j=0}^{l-1} d_j \frac{z^{\alpha-j-1}}{z^\alpha - \lambda z^\beta - \sum_{k=0}^{m-2} A_k z^{\alpha_k}} - \lambda \sum_{j=0}^{l_{m-1}-1} d_j \frac{z^{\beta-j-1}}{z^\alpha - \lambda z^\beta - \sum_{k=0}^{m-2} A_k z^{\alpha_k}} - \sum_{k=0}^{m-2} A_k \sum_{j=0}^{l_k-1} d_j \frac{z^{\alpha_k-j-1}}{z^\alpha - \lambda z^\beta - \sum_{k=0}^{m-2} A_k z^{\alpha_k}}, \tag{207}$$

where d_j $(j = 0, \ldots, l-1)$ are given by (159).

For $z \in \mathbb{C}$ and $|\sum_{k=0}^{m-2} A_k z^{\alpha_k-\beta}/(z^{\alpha-\beta} - \lambda)| < 1$, we have

$$\begin{aligned} \frac{1}{z^\alpha - \lambda z^\beta - \sum_{k=0}^{m-2} A_k z^{\alpha_k}} &= \frac{z^{-\beta}}{z^{\alpha-\beta} - \lambda} \cdot \frac{1}{\left(1 - \left(\sum_{k=0}^{m-2} A_k z^{\alpha_k-\beta} / \left(z^{\alpha-\beta} - \lambda\right)\right)\right)} \\ &= \sum_{n=0}^{\infty} \frac{z^{-\beta}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \left(\sum_{k=0}^{m-2} A_k z^{\alpha_k-\beta} \right)^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_{m-2}=n} \right) \frac{n!}{k_0! \cdots k_{m-2}!} \times \left[\prod_{\nu=0}^{m-2} (A_\nu)^{k_\nu} \right] \frac{z^{-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}}, \end{aligned} \tag{208}$$

if we also take into account the following relation:

$$\left(x_0 + \cdots + x_{m-2}\right)^n = \left(\sum_{k_0+\cdots+k_{m-2}=n} \right) \frac{n!}{k_0! \cdots k_{m-2}!} \prod_{\nu=0}^{m-2} x_\nu^{k_\nu}, \tag{209}$$

where the summation is taken over all $k_0, \ldots, k_{m-2} \in \mathbb{N}_0$ such that $k_0 + \cdots + k_{m-2} = n$.

In addition, for $z \in \mathbb{C}$ and $|\lambda z^{\beta-\alpha}| < 1$, we have

$$\begin{aligned} \frac{z^{\alpha-j-1-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} &= \frac{z^{(\alpha-\beta)-(j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu)}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\ &= \frac{1}{n!} \mathscr{L}_{\nabla, t_0} \left\{ \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta, j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu} (\lambda; t, t_0) \right\}(z), \end{aligned} \tag{210}$$

$$\begin{aligned} \frac{z^{\beta-j-1-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} &= \frac{z^{(\alpha-\beta)-(\alpha-\beta+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu)}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\ &= \frac{1}{n!} \mathscr{L}_{\nabla, t_0} \left\{ \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta, \alpha-\beta+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu} (\lambda; t, t_0) \right\}(z), \end{aligned} \tag{211}$$

$$\begin{aligned} \frac{z^{\alpha_k-j-1-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} &= \frac{z^{(\alpha-\beta)-(\alpha-\alpha_k+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu)}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\ &= \frac{1}{n!} \mathscr{L}_{\nabla, t_0} \left\{ \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta, \alpha-\alpha_k+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu} (\lambda; t, t_0) \right\}(z). \end{aligned} \tag{212}$$

From (210) to (212), we derive the solution to (202):

$$y(t) = \sum_{j=0}^{l-1} d_j y_j(t), \tag{213}$$

where $y_j(t)$ $(j = 0, \ldots, l-1)$ are given by (204) for $j = 0, \ldots, l_{m-2} - 1$, by (205) for $j = l_{m-2}, \ldots, l_{m-1} - 1$, and by (206) for $j = l_{m-1}, \dots, l-1$. For $k = 0, \dots, l-1$, the direct evaluation yields

$$\begin{aligned}
D_\nabla^k y_j(t) &= \sum_{n=0}^{\infty}\left(\sum_{k_0+\cdots+k_{m-2}=n}\right)\frac{1}{k_0!\cdots k_{m-2}!}\left[\prod_{\nu=0}^{m-2}(A_\nu)^{k_\nu}\right] \\
&\quad\cdot D_\nabla^k\left\{\frac{\partial^n}{\partial\lambda^n}\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu+j}(t,t_0)\right. \\
&\quad -\lambda\frac{\partial^n}{\partial\lambda^n}\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu+j+\alpha-\beta}(t,t_0) \\
&\quad -\sum_{k=0}^{m-2}A_k\frac{\partial^n}{\partial\lambda^n} \\
&\quad \left.\times\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu+j+\alpha-\alpha_k}(t,t_0)\right\} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k_0+\cdots+k_{m-2}=n}\right)\frac{1}{k_0!\cdots k_{m-2}!}\left[\prod_{\nu=0}^{m-2}(A_\nu)^{k_\nu}\right] \\
&\quad\cdot\left\{\frac{\partial^n}{\partial\lambda^n}\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu+j-k}(t,t_0)\right. \\
&\quad -\lambda\frac{\partial^n}{\partial\lambda^n}\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu+j-k+\alpha-\beta}(t,t_0) \\
&\quad -\sum_{k=0}^{m-2}A_k\frac{\partial^n}{\partial\lambda^n} \\
&\quad \left.\times\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu+j-k+\alpha-\alpha_k}(t,t_0)\right\}
\end{aligned} \tag{214}$$

for $j = 0, \dots, l_{m-2}-1$,

$$\begin{aligned}
D_\nabla^k y_j(t) &= \sum_{n=0}^{\infty}\left(\sum_{k_0+\cdots+k_{m-2}=n}\right)\frac{1}{k_0!\cdots k_{m-2}!}\left[\prod_{\nu=0}^{m-2}(A_\nu)^{k_\nu}\right] \\
&\quad\cdot\left\{\frac{\partial^n}{\partial\lambda^n}\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu+j-k}(t,t_0)-\lambda\frac{\partial^n}{\partial\lambda^n}\right. \\
&\quad\left.\times\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu+j-k+\alpha-\beta}(t,t_0)\right\}
\end{aligned} \tag{215}$$

for $j = l_{m-2}, \dots, l_{m-1}-1$, and

$$\begin{aligned}
D_\nabla^k y_j(t) &= \sum_{n=0}^{\infty}\left(\sum_{k_0+\cdots+k_{m-2}=n}\right)\frac{1}{k_0!\cdots k_{m-2}!}\left[\prod_{\nu=0}^{m-2}(A_\nu)^{k_\nu}\right] \\
&\quad\times\frac{\partial^n}{\partial\lambda^n}\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu+j-k}(t,t_0)
\end{aligned} \tag{216}$$

for $j = l_{m-1}, \dots, l-1$. For $j > k$, $D_\nabla^k y_j(t_0) = 0$, and for $j = k$, $D_\nabla^k y_j(t_0) = 1$. Thus we have $W(t_0) = 1$. Thus the functions $y_j(t)$ $(j = 0, \dots, l-1)$ in (204)–(206) are linearly independent solutions to (202). The result follows. □

5.2. Nonhomogeneous Equations with Constant Coefficients. In this section, we still use Laplace transform method to find general solutions to the corresponding nonhomogeneous equations

$$\begin{gathered}
\sum_{k=1}^{m}A_k\left[{}^CD_{\nabla,t_0}^{\alpha_k}y(t)\right]+A_0y(t)=f(t) \\
\left(m\in\mathbb{N};\ 0<\alpha_1<\cdots<\alpha_m;\right. \\
\left. l-1<\alpha_m<l,\ l\in\mathbb{N},\ t_0,t\in\Omega_{k^l},\ t>t_0\right)
\end{gathered} \tag{217}$$

with real coefficients $A_k \in \mathbb{R}$ $(k = 0, \dots, m)$ and a given function $f(t)$. The general solution to (217) is a sum of its particular solution and of the general solution to the corresponding homogeneous equation (157). It is sufficient to construct a particular solution to (217).

By (158) and (159), for suitable functions y, the Laplace transform of ${}^CD_{\nabla,t_0}^{\alpha}y$ is given by

$$\mathscr{L}_{\nabla,t_0}\left\{{}^CD_{\nabla,t_0}^{\alpha}y(t)\right\}(z)=z^{\alpha}\mathscr{L}_{\nabla,t_0}\{y(t)\}(z). \tag{218}$$

Applying the Laplace transform to (217) and taking (218) into account, we have

$$\left[A_0+\sum_{k=1}^{m}A_kz^{\alpha_k}\right]\mathscr{L}_{\nabla,t_0}\{y(t)\}(z)=\mathscr{L}_{\nabla,t_0}\{f(t)\}(z). \tag{219}$$

Using the inverse Laplace transform $\mathscr{L}_\nabla^{-1}$ from here we obtain a particular solution to (217) in the form

$$y(t)=\mathscr{L}_{\nabla,t_0}^{-1}\left[\frac{\mathscr{L}_{\nabla,t_0}\{f(t)\}(z)}{A_0+\sum_{k=1}^{m}A_kz^{\alpha_k}}\right](t). \tag{220}$$

Using the Laplace convolution formula

$$\mathscr{L}_{\nabla,t_0}\{f*g\}(z)=\mathscr{L}_{\nabla,t_0}\{f\}(z)\,\mathscr{L}_{\nabla,t_0}\{g\}(z), \tag{221}$$

we can introduce the *Laplace fractional analog* of the Green function as follows:

$$\begin{gathered}
G_{\alpha_1,\dots,\alpha_m}(t)=\mathscr{L}_{\nabla,t_0}^{-1}\left\{\frac{1}{P_{\alpha_1,\dots,\alpha_m}(z)}\right\}(t), \\
P_{\alpha_1,\dots,\alpha_m}(z)=A_0+\sum_{k=1}^{m}A_kz^{\alpha_k},
\end{gathered} \tag{222}$$

and express a particular solution of (217) in the form of the Laplace convolution $G_{\alpha_1,\dots,\alpha_m}(t)$ and $f(t)$

$$y(t)=G_{\alpha_1,\dots,\alpha_m}(t)*f(t). \tag{223}$$

Theorem 47. *Let $l-1<\alpha\le l$ $(l\in\mathbb{N})$, $\lambda\in\mathbb{R}$, and $f(t)$ be a given function. Then the equation*

$$ {}^{C}D^{\alpha}_{\nabla,t_0}y(t)-\lambda y(t)=f(t) \quad (224)$$

is solvable, and its general solution is given by

$$y(t)={}_{\nabla}F_{\alpha,\alpha}(\lambda;t,t_0)*f(t)+\sum_{j=0}^{l-1}c_j\,{}_{\nabla}F_{\alpha,j+1}(\lambda;t,t_0), \quad (225)$$

where c_j $(j=0,\dots,l-1)$ are arbitrary real constants.

In particular, the general solutions to (224) with $0<\alpha\le 1$ and $1<\alpha\le 2$ have the forms

$$y(t)={}_{\nabla}F_{\alpha,\alpha}(\lambda;t,t_0)*f(t)+c_0{}_{\nabla}F_{\alpha,1}(\lambda;t,t_0), \quad (226)$$

$$y(t)={}_{\nabla}F_{\alpha,\alpha}(\lambda;t,t_0)*f(t)+c_0{}_{\nabla}F_{\alpha,1}(\lambda,t,t_0)+c_1{}_{\nabla}F_{\alpha,2}(\lambda;t,t_0), \quad (227)$$

respectively, where c_0 and c_1 are arbitrary real constants.

Proof. Equation (224) is (217) with $m=1$, $\alpha_1=\alpha$, $A_1=1$, $A_0=-\lambda$ and (222) takes the form

$$G_{\alpha}(t)=\mathscr{L}^{-1}_{\nabla,t_0}\left\{\frac{1}{z^{\alpha}-\lambda}\right\}(t)={}_{\nabla}F_{\alpha,\alpha}(\lambda;t,t_0). \quad (228)$$

Thus (223), with $G_{\alpha_1,\dots,\alpha_m}(t)=G_{\alpha}(t)$, and Theorem 40 yield (225). Theorem is proved. □

Theorem 48. *Let $l-1<\alpha\le l$ $(l\in\mathbb{N})$, $0<\beta<\alpha$, $\lambda,\mu\in\mathbb{R}$, and let $f(x)$ be a given function. Then the equation*

$$ {}^{C}D^{\alpha}_{\nabla,t_0}y(t)-\lambda{}^{C}D^{\beta}_{\nabla,t_0}y(t)-\mu y(t)=f(t) \quad (229)$$

is solvable, and its general solution has the form

$$y(t)=\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\alpha+n\beta}(\lambda,t,t_0)*f(t)+\sum_{j=0}^{l-1}c_jy_j(t), \quad (230)$$

where $y_j(t)$ $(j=0,\dots,l-1)$ are given by (183) *and* (184) *and c_j $(j=0,\dots,l-1)$ are arbitrary real constants.*

Proof. Equation (229) is the same as (217) with $m=2$, $\alpha_2=\alpha$, $\alpha_1=\beta$, $A_2=1$, $A_1=-\lambda$, $A_0=-\mu$, and (222) is given by

$$G_{\alpha,\beta;\lambda,\mu}(t)=\mathscr{L}^{-1}_{\nabla,t_0}\left\{\frac{1}{z^{\alpha}-\lambda z^{\beta}-\mu}\right\}(t). \quad (231)$$

For $z\in\mathbb{C}$ and $|\mu z^{-\beta}/(z^{\alpha-\beta}-\lambda)|<1$, we have

$$\begin{aligned}\frac{1}{z^{\alpha}-\lambda z^{\beta}-\mu}&=\frac{z^{-\beta}}{z^{\alpha-\beta}-\lambda}\cdot\frac{1}{1-\left(\mu z^{-\beta}/\left(z^{\alpha-\beta}-\lambda\right)\right)}\\&=\sum_{n=0}^{\infty}\frac{\mu^n z^{-\beta-n\beta}}{\left(z^{\alpha-\beta}-\lambda\right)^{n+1}}\end{aligned} \quad (232)$$

and then we get

$$G_{\alpha,\beta;\lambda,\mu}(t)=\mathscr{L}^{-1}_{\nabla,t_0}\left\{\sum_{n=0}^{\infty}\mu^n\frac{z^{-\beta-n\beta}}{\left(z^{\alpha-\beta}-\lambda\right)^{n+1}}\right\}(t). \quad (233)$$

In addition, for $z\in\mathbb{C}$ and $|\lambda z^{\beta-\alpha}|<1$, we have

$$\frac{z^{-\beta-n\beta}}{\left(z^{\alpha-\beta}-\lambda\right)^{n+1}}=\frac{1}{n!}\mathscr{L}_{\nabla,t_0}\left\{\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\alpha+n\beta}(\lambda;t,t_0)\right\}(z) \quad (234)$$

and hence (233) takes the following form:

$$G_{\alpha,\beta;\lambda,\mu}(t)=\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\alpha+n\beta}(\lambda;t,t_0). \quad (235)$$

Thus the result in (230) follows from (223) with $G_{\alpha_1,\dots,\alpha_m}(t)=G_{\alpha,\beta;\lambda,\mu}(t)$ and Theorem 42. □

Theorem 49. *Let $m\in\mathbb{N}\setminus\{1,2\}$, $l-1<\alpha\le l$ $(l\in\mathbb{N})$, $\beta,\alpha_1,\dots,\alpha_{m-2}$ be such that $\alpha>\beta>\alpha_{m-2}>\cdots>\alpha_1>\alpha_0=0$, and let $\lambda,A_0,\dots,A_{m-2}\in\mathbb{R}$, and let $f(t)$ be a given function. The equation*

$$\begin{gathered}{}^{C}D^{\alpha}_{\nabla,t_0}y(t)-\lambda{}^{C}D^{\beta}_{\nabla,t_0}y(t)-\sum_{k=0}^{m-2}A_k{}^{C}D^{\alpha_k}_{\nabla,t_0}y(t)=f(t)\\(m\in\mathbb{N}\setminus\{1,2\};\ \alpha>\beta>\alpha_{m-2}>\cdots>\alpha_1>\alpha_0=0;\\\lambda,A_0,\dots,A_{m-2}\in\mathbb{R})\end{gathered} \quad (236)$$

is solvable, and its general solution is given by

$$\begin{aligned}y(t)&=\sum_{n=0}^{\infty}\left(\sum_{k_0+\cdots+k_{m-2}=n}\right)\frac{1}{k_0!\cdots k_{m-2}!}\\&\quad\times\left[\prod_{\nu=0}^{m-2}\left(A_{\nu}\right)^{k_{\nu}}\right]\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\alpha+\sum_{\nu=0}^{m-2}(\beta-\alpha_{\nu})k_{\nu}}(\lambda;t,t_0)*f(t)\\&\quad+\sum_{j=0}^{l-1}c_jy_j(t),\end{aligned} \quad (237)$$

where $y_j(t)$ $(j=0,\dots,l-1)$ are given by (204)–(206) *and c_j $(j=0,\dots,l-1)$ are arbitrary real constants.*

Proof. Equation (236) is the same equation as (217) with $\alpha_m=\alpha$, $\alpha_{m-1}=\beta$, $A_m=1$, $A_{m-1}=-\lambda$, and with $-A_k$ instead of A_k for $k=0,\dots,m-2$. Since $\alpha_0=0$, (222) takes the form

$$G_{\alpha_1,\dots,\alpha_{m-2},\beta,\alpha;\lambda}(t)=\mathscr{L}^{-1}_{\nabla,t_0}\left\{\frac{1}{z^{\alpha}-\lambda\alpha^{\beta}-\sum_{k=0}^{m-2}A_kz^{\alpha_k}}\right\}(t). \quad (238)$$

For $z \in \mathbb{C}$ and $| \sum_{k=0}^{m-2} A_k z^{\alpha_k - \beta} / (z^{\alpha-\beta} - \lambda) | < 1$, we have

$$\begin{aligned}
&\frac{1}{z^{\alpha} - \lambda z^{\beta} - \sum_{k=0}^{m-2} A_k z^{\alpha_k}} \\
&\quad = \frac{z^{-\beta}}{z^{\alpha-\beta} - \lambda} \cdot \frac{1}{\left(1 - \left(\sum_{k=0}^{m-2} A_k z^{\alpha_k-\beta} / \left(z^{\alpha-\beta} - \lambda\right)\right)\right)} \\
&\quad = \sum_{n=0}^{\infty} \frac{z^{-\beta}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \left(\sum_{k=0}^{m-2} A_k z^{\alpha_k - \beta}\right)^n \\
&\quad = \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_{m-2}=n}\right) \frac{n!}{k_0! \cdots k_{m-2}!} \\
&\qquad \times \left[\prod_{\nu=0}^{m-2} \left(A_\nu\right)^{k_\nu}\right] \frac{z^{-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}},
\end{aligned} \tag{239}$$

if we also take into account the following relation:

$$\left(x_0 + \cdots + x_{m-2}\right)^n = \left(\sum_{k_0+\cdots+k_{m-2}=n}\right) \frac{n!}{k_0! \cdots k_{m-2}!} \prod_{\nu=0}^{m-2} x_\nu^{k_\nu}, \tag{240}$$

where the summation is taken over all $k_0, \ldots, k_{m-2} \in \mathbb{N}_0$ such that $k_0 + \cdots + k_{m-2} = n$, and then we get

$$\begin{aligned}
&G_{\alpha_1,\ldots,\alpha_{m-2},\beta,\alpha;\lambda}(t) \\
&\quad = \mathscr{L}^{-1}_{\nabla,t_0} \left\{ \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_{m-2}=n}\right) \frac{n!}{k_0! \cdots k_{m-2}!} \right. \\
&\qquad \left. \times \left[\prod_{\nu=0}^{m-2} \left(A_\nu\right)^{k_\nu}\right] \frac{z^{-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \right\}(t).
\end{aligned} \tag{241}$$

For $z \in \mathbb{C}$ and $|\lambda z^{\beta-\alpha}| < 1$, we have

$$\begin{aligned}
&\frac{z^{-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\
&\quad = \frac{1}{n!} \mathscr{L}_{\nabla,t_0} \left\{ \frac{\partial^n}{\partial \lambda^n} \, {}_{\nabla}F_{\alpha-\beta,\alpha+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu} \left(\lambda; t, t_0\right) \right\}(z).
\end{aligned} \tag{242}$$

Thus the result in (237) follows from (223) with $G_{\alpha_1,\ldots,\alpha_m}(t) = G_{\alpha_1,\ldots,\alpha_{m-2},\beta,\alpha;\lambda}(t)$ and Theorem 46. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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