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Research Article

Common Fixed Point Theorems for Probabilistic Nearly Densifying Mappings

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The aim of this paper is to prove some coincidence and common fixed point theorems for probabilistic nearly densifying mappings in complete Menger spaces. Our results improve the results of Chamola et al. (1991), Dimri and Pant (2002), and Pant et al. (2004) and extend the results of Khan and Liu (1997) in the framework of probabilistic settings.

1. Introduction and Preliminaries

Banach contraction mapping principle is one of the most interesting and useful tools in applied mathematics. In recent years many generalizations of Banach contraction mapping principle have appeared. The notion of probabilistic metric spaces (in short PM-spaces) is a probabilistic generalization of metric spaces which are appropriate to carry out the study of those situations wherein distances are measured in the sense of distribution functions rather than nonnegative real numbers. The study of PM-spaces was initiated by Menger [1]. Since then, Schweizer and Sklar [2] enriched this concept and provided a new impetus by proving some fundamental results on this theme. The first result on fixed point theory in PM-spaces was given by Sehgal and Bharucha-Reid [3] wherein the notion of probabilistic contraction was introduced as a generalization of the classical Banach fixed point principle in terms of probabilistic settings. Some recent fixed point results can be studied in [4–7].

Kuratowski [8] introduced the notion of measure of noncompactness of a bounded subset of a metric space. Further, this study was carried on by Furi and Vignoli [9]. They introduced the notion of densifying (also called condensing) mapping in terms of Kuratowski's measure of noncompactness and obtained some fixed point theorems. Following Furi and Vignoli [9], a number of mathematicians worked on densifying mappings and proved some metrical fixed point theorem (cf. [10–14]). As a generalization of Kuratowski's measure of noncompactness, Bocsan and Constantin [15] introduced the notion of Kuratowski's measure of noncompactness in PM-spaces. Subsequently, Bocşan [16] studied the notion of probabilistic densifying mappings. Later, Hadžić [17], Tan [18], Chamola et al. [19], Dimri and Pant [20], Pant et al. [21], Pant et al. [22], and Singh and Pant [23] proved some results for such mappings. In [24], Ganguly et al. introduced the notion of probabilistic nearly densifying mappings and proved some interesting results in this setting.

The aim of this paper is to prove some coincidence and common fixed point theorems for certain classes of nearly densifying mappings in complete Menger spaces. First, we give some topological definitions and terminology defined in [8, 15–17].

Definition 1. A semigroup G is said to be left reversible if for any $r, s \in G$ there exist $a, b \in G$ such that $ra = sb$.

It is easy to see that the notion of left reversibility is equivalent to the statement that any two right ideals of G have nonempty intersection.

Definition 2. Let G be a family of self-mappings in X. A subset Y of X is called G-invariant if $gY \subseteq Y$ for all $g \in G$.

Definition 3. Let G^* be the semigroup generated by G under composition $*$. Clearly, $G^* \supseteq \{g^n : n \geq 0\}$ for any $g \in G$ and $G^*(u) = \{u\} \cup \{gu : g \in G^*\}$ for $u \in X$.

We restate the notion of probabilistic diameter for the sake of quick reference.

Definition 4. Let A be a nonempty subset of X. A function $D_A(\cdot)$ defined by

$$D_A(x) = \sup_{y<x} \left\{ \inf_{u,v\in A} F_{u,v}(y) \right\} \tag{1}$$

is called probabilistic diameter of A. A is said to be bounded if

$$\sup_{x\in R} D_A(x) = 1. \tag{2}$$

The following definition is due to Bocsan and Constantin [15].

Definition 5. For a probabilistic bounded subset A of X, $\alpha_A(x)$ defined by $\alpha_A(x) = \sup\{\varepsilon \geq 0 : \exists$ a finite cover $\mathscr{A}$ of A such that $D_S(x) \geq \varepsilon$ for all $S \in \mathscr{A}\}$ is called Kuratowski's function.

The following properties of Kuratowski's functions are proved in [8]:

(a) $\alpha_A \in \mathfrak{I}$, the set of distribution functions;

(b) $\alpha_A(x) \geq D_A(x)$;

(c) if $\phi \neq A \subset B \subset X$, then $\alpha_A(x) \geq \alpha_B(x)$;

(d) $\alpha_{A\cup B}(x) = \min\{\alpha_A(x), \alpha_B(x)\}$;

(e) let $\overline{A}$ be the closure of A in the (ε, λ)-topology on X; then

$$\alpha_{\overline{A}}(x) = \alpha_A(x); \tag{3}$$

(f) A is probabilistic precompact (totally bounded) if $\alpha_A = H$,

where H denotes the specific distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0; \\ 1, & x > 0. \end{cases} \tag{4}$$

Definition 6. Let $(X, \mathscr{F})$ be a PM-space. A continuous mapping f of X into X is called a probabilistic densifying mapping if and only if, for every subset A of X, $\alpha_A < H$ implies $\alpha_{f(A)} > \alpha_A$.

Definition 7. A self-mapping $f : X \to X$ is probabilistic nearly densifying if $\alpha_{f(A)} > \alpha_A$, whenever $\alpha_A < H$, $A \subset H$, and A is f-invariant.

Definition 8. Suppose $\phi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function with $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.

2. Main Results

First, we prove some fixed point theorems for probabilistic nearly densifying mappings in Menger spaces.

Theorem 9. *Let P, Q, and R be three continuous and nearly densifying self-mappings on a complete Menger space $(X, \mathscr{F}, *)$ such that $\sup x * x = 1$ and R commutes with P and Q. If, for all $x < 1$, $u, v \in X$, the following conditions are satisfied:*

$$\phi_1(Pu, Qv) > \min\left\{\phi_2(Ru, Rv), \phi_2(Ru, Pu), \phi_1(Rv, Qv), \frac{\phi_2(Ru, Pu)\,\phi_1(Rv, Qv)}{\phi_2(Ru, Rv)}\right\} \quad \text{for } Ru \neq Rv, \quad Pu \neq Qv; \tag{5}$$

$$\phi_2(Qu, Pv) > \min\left\{\phi_1(Ru, Rv), \phi_1(Ru, Qu), \phi_2(Rv, Pv), \frac{\phi_1(Ru, Qu)\,\phi_2(Rv, Pv)}{\phi_1(Ru, Rv)}\right\}, \quad \text{for } Ru \neq Rv, \quad Qu \neq Pv, \tag{6}$$

where ϕ_1 and ϕ_2 are real valued mappings from $X \times X$ to ς, the collection of all distribution functions, with either ϕ_1 or ϕ_2 being upper semicontinuous (u.s.c.) and $\phi_1(u, u) = \phi_2(u, u) = 1$ for all $u \in X$.

Further, if, for some $u_0 \in X$, $G(u_0) = \{P^i Q^j R^k u_0 : i = 0, 1, 2, \ldots; j = 0, 1, 2, \ldots; k = 0, 1, 2, \ldots\}$ is bounded, then P and R or Q and R have a coincidence point.

Proof. For $u_0 \in X$, let $A = G(u_0)$ and $S = \{PQR\}$.

Then $A = \{u_0\} \cup P(A) \cup Q(A) \cup R(A)$.

If $\alpha_A < H$, then

$$\begin{aligned} \alpha_A &= \alpha_{\{u_0\}\cup P(A)\cup Q(A)\cup R(A)} \\ &= \min\{\alpha_{P(A)}, \alpha_{Q(A)}, \alpha_{R(A)}\} > \alpha_A, \end{aligned} \tag{7}$$

a contradiction. It implies that $\overline{A}$ is precompact.

Let $B = \bigcap_{n=0}^{\infty} (PQR)^n(\overline{A})$.

Then it is easy to see that $SB = B$ and B is nonempty compact subset of A. By the continuity of P, Q, and R, it follows that $P\overline{A} \subset \overline{A}$, $Q\overline{A} \subset \overline{A}$, and $R\overline{A} \subset \overline{A}$. Further, it is clear that $P(\overline{B}) \subset \overline{B}$, $Q(\overline{B}) \subset \overline{B}$, and $R(\overline{B}) \subset \overline{B}$.

Note that

$$\begin{gathered} R(B) = \bigcap_{n=0}^{\infty} R(PQR)^n(\overline{A}) \subset \bigcap_{n=0}^{\infty} (PQR)^n R(\overline{A}) \subset B, \\ B = PQR(B) = RPQ(B) \subset RP(B) \subset R(B), \end{gathered} \tag{8}$$

which implies $R(B) = B$ or $R^2(B) = B$.

Now, assume that ϕ_1 is upper semicontinuous. Then the function $T : B \to \mathfrak{I}$, defined by $T(u) = \phi_1(Ru, Qu)$, is u.s.c. So T assumes its maximal value at some point p in B. Clearly,

$p \in R^2(B)$, so there is a $w \in B$ such that $p = R^2(w)$. Suppose that neither P and R nor Q and R have a coincidence point. Then

$$
\begin{aligned}
&T(PQ(w)) \\
&\quad = \phi_1(RPQ(w), QPQ(w)) \\
&\quad = \phi_1(PRQ(w), QPQ(w)) \text{ by } (5), \\
&\quad > \min\left\{\phi_2\left(R^2Q(w), RPQ(w)\right),\right. \\
&\qquad \phi_2\left(R^2Q(w), PRQ(w)\right), \phi_1(RPQ(w), QPQ(w)), \\
&\qquad \left.\frac{\phi_2\left(R^2Q(w), PRQ(w)\right)\phi_1(RPQ(w), QPQ(w))}{\phi_2(R^2Q(w), RPQ(w))}\right\} \\
&\quad = \phi_2\left(QR^2(w), PRQ(w)\right), \text{by } (6), \\
&\quad > \min\left\{\phi_1\left(RR^2(w), R^2Q(w)\right), \phi_1\left(RR^2(w), QR^2(w)\right),\right. \\
&\qquad \phi_2\left(R^2Q(w), PRQ(w)\right), \\
&\qquad \left.\frac{\phi_1\left(RR^2(w), QR^2(w)\right)\phi_2\left(R^2Q(w), PRQ(w)\right)}{\phi_1(RR^2(w), R^2Q(w))}\right\} \\
&\quad = \phi_1\left(RR^2(w), QR^2(w)\right) = \phi_1(Rp, Qp) = T(p),
\end{aligned} \tag{9}
$$

a contradiction to the selection of p. Hence, P and R or Q and R must have a coincidence point.

The same result holds good if ϕ_2 is upper semicontinuous. This completes the proof of the theorem. □

Remark 10. The above theorem extends the results of Khan and Liu [25, Theorem 3.1 and Corollary 3.3] to PM-spaces.

Theorem 11. *Let X, P, Q, and R be as in Theorem 9. Further, let P, Q, and R satisfying (5) and (6) have a coincidence point w; then Rw is a unique common fixed point of P, Q, and R.*

Proof. We have $Pw = Qw = Rw$. By commutativity of R with P and Q, $PR(w) = RP(w) = RR(w)$ and $QR(w) = RQ(w) = RR(w)$, or $PR(w) = RR(w) = QR(w)$.

Now let $R^2w \neq Rw$; then by (5) and (6), we have

$$
\begin{aligned}
&\phi_1\left(R^2w, Rw\right) \\
&\quad = \phi_1(PRw, Qw) \\
&\quad > \min\left\{\phi_2\left(R^2w, Rw\right), \phi_2\left(R^2w, PRw\right),\right. \\
&\qquad \left.\phi_1(Rw, Qw), \frac{\phi_2\left(R^2w, PRw\right)\phi_1(Rw, Qw)}{\phi_2(R^2w, Rw)}\right\} \\
&\quad = \phi_2\left(R^2w, PRw\right) = \phi_2(QRw, Pw) \\
&\quad > \min\left\{\phi_1\left(R^2w, Rw\right), \phi_1\left(R^2w, QRw\right),\right. \\
&\qquad \left.\phi_2(Rw, Pw), \frac{\phi_1\left(R^2w, QRw\right)\phi_2(Rw, Pw)}{\phi_1(R^2w, Rw)}\right\} \\
&\quad = \phi_1\left(R^2w, Rw\right),
\end{aligned} \tag{10}
$$

which is a contradiction. Hence, $R^2w = Rw$. Thus, Rw is a fixed point of R. Thus, $Rw = R(Rw) = P(Rw) = Q(Rw)$. Therefore, Rw is a common fixed point of P, Q, and R.

The uniqueness of Rw as a common fixed point of P, Q, and R follows from (5) and (6). □

Theorem 12. *Let f and g be commuting, continuous, and nearly densifying self-mappings on a complete Menger space X satisfying*

$$
\phi(gu, gv) > \min\{\phi(fu, fv), \phi(fu, gu), \phi(fv, gv)\} \tag{11}
$$

for $fu \neq fv$, $gu \neq gv$, and $u, v \in X$, where $\phi : X \times X \to \zeta$ is u.s.c. and $\phi(u, u,) = 1$, $u \in X$. If, for some u_0 in X, $G(u_0) = \{f^i g^j u_0 : i = 0, 1, 2, \ldots; j = 0, 1, 2, \ldots\}$ is bounded, then f and g have a unique common fixed point.

Proof. Let $A = G(u_0)$. Since f and g are commuting and continuous, we have $f(\overline{A}) \subseteq \overline{A}$ and $g(\overline{A}) \subseteq \overline{A}$ and then $A = \{u_0\} \cup f(A) \cup g(A)$.

If $\alpha_A < H$, then

$$
\begin{aligned}
\alpha_A &= \alpha_{\{u_0\} \cup f(A) \cup g(A)} \\
&= \min\left\{\alpha_{f(A)}, \alpha_{g(A)}\right\} > \alpha_A,
\end{aligned} \tag{12}
$$

which is a contradiction. It implies that $\overline{A}$ is precompact.

Now define $B = \bigcap_{n=0}^{\infty} (fg)^n(\overline{A})$. Since $\{(fg)^n A\}$ is a decreasing sequence of nonempty compact subset of A, it follows that B is nonempty set such that $f(\overline{B}) \subset \overline{B}$, $g(\overline{B}) \subset \overline{B}$.

Suppose that $u \in B$; then $u \in (fg)^{n+1}\overline{A}$ for all n. Hence, there exists $\{x_n\} \subseteq (fg)^n\overline{A}$. Since $(fg)^n\overline{A}$ is compact and closed for all n, f and g are continuous and nearly densifying; therefore, there exists a point $p \in (fg)^n\overline{A}$ for all n so that $fg(p) = u$. Hence, $u \in f(B)$ and $u \in g(B)$. Thus, we have

$$
f(B) = B = g(B). \tag{13}
$$

Let us define a real valued function ψ on B by $\psi(u) = \phi(fu, gu)$. It is u.s.c. and hence attains its maximum at some point $p \in B$. Then there exists a $w \in B$ such that $p = fw$.

Suppose that there is no point u in X such that $fu = gu$; then we have by (11)

$$\begin{aligned}\psi(gw) &= \phi(fgw, ggw) = \phi(gfw, ggw) \\ &> \min\left\{\phi\left(f^2w, fgw\right), \phi\left(f^2w, gfw\right), \phi(fgw, ggw)\right\} \\ &= \min\left\{\phi\left(f^2w, fgw\right), \phi(fgw, ggw)\right\} \\ &= \phi\left(f^2w, fgw\right) = \phi(fp, gp) = \psi(p),\end{aligned} \tag{14}$$

which is a contradiction to the selection of p. Hence, there exists a $w_0 \in B$ such that $fw_0 = gw_0$ or $f^2w_0 = fgw_0 = gfw_0$.

Suppose $f^2w_0 \neq fw_0$; then we have

$$\begin{aligned}\phi\left(f^2w_0, fw_0\right) &= \phi(gfw_0, gw_0) \\ &> \min\left\{\phi\left(f^2w_0, fw_0\right), \phi\left(f^2w_0, gfw_0\right), \phi(fw_0, gw_0)\right\} \\ &= \phi\left(f^2w_0, fw_0\right),\end{aligned} \tag{15}$$

which is a contradiction. Hence, $f^2w_0 = gfw_0 = fw_0$. Therefore, fw_0 is common fixed point of f and g. Now we will prove the uniqueness of fw_0. Let w be the other fixed point of f and g; then, by (11), we have

$$\begin{aligned}\phi(w, fw_0) &= \phi(gw, fgw_0) = \phi(gw, gfw_0) \\ &> \min\left\{\phi\left(fw, f^2w_0\right), \phi(fw, gw), \phi\left(f^2w_0, gfw_0\right)\right\} \\ &= \phi\left(fw, f^2w_0\right) = \phi(w, fw_0), \text{ a contradiction.}\end{aligned} \tag{16}$$

Hence, fw_0 is unique. This completes the proof of the theorem. □

Remark 13. Theorems 9, 11, and 12 improve the result of Chamola et al. [19], Dimri and Pant [20], Ganguly et al. [24], and Pant et al. [21] under more natural conditions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally to this paper. The guidance of Aeshah Hassan Zakri is very important and she helped in revising the paper according to reviewers reports.

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