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Research Article

Nonlinear Fuzzy Differential Equation with Time Delay and Optimal Control Problem

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The existence and uniqueness of a mild solution to nonlinear fuzzy differential equation constrained by initial value were proven. Initial value constraint was then replaced by delay function constraint and the existence of a solution to this type of problem was also proven. Furthermore, the existence of a solution to optimal control problem of the latter type of equation was proven.

1. Introduction

Fuzzy logic is originated by Zadeh in 1965. It is primarily based on the fact that "all things happening in real world are unstable and unpredictable." This idea was put forward and successfully applied to many fields of research—such as medicine, computer science, engineering, and economics—owing to its remarkable effectiveness at solving problems that could not be solved by traditional logic; see [1–3] and references therein. In particular, fuzzy logic has long been applied to dynamic systems expressed in differential equations; see [4–15] and references therein. Moreover, dynamic system with time delay can be advantageously applied to many important problems such as determining the current position of a particle from the history of its past movement; see [16–19] and references therein. In this study, fuzzy differential equation of dynamic system constrained by time delay was investigated. The objectives of this investigation were to delineate the definitions of and theorems on fuzzy control system with time delay and to find the necessary conditions for the existence of a solution to this type of system by functional analysis.

2. Preliminaries

This section discusses the definitions and theorems pertaining to this research.

Definition 1. Let $\mathbb{R}_F$ be a family of fuzzy subset of $\mathbb{R}$, called a *fuzzy number space*. It satisfies the following conditions, for each $u \in \mathbb{R}_F$:

(1) u is normal; that is, there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(2) u is a convex fuzzy set; that is, $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$ for all $t \in [0,1]$ and $x, y \in \mathbb{R}$.

(3) u is upper semicontinuous on $\mathbb{R}$; that is, for each $x \in \mathbb{R}$ and for all sequences $x_n \in \mathbb{R}$, if $x_n \to x$ then $\lim_{n\to\infty} \sup u(x_n) \leq u(x)$.

(4) $\overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$ is compact; that is, for all sequences $x_n \in \overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$, there is a subsequence x_{k_n} such that $x_{k_n} \to y \in \overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$.

Notice that $\mathbb{R} = \{\chi_{\{x\}} \mid x \text{ is real}\}$ and $\mathbb{R} \subset \mathbb{R}_F$.

Definition 2. Let $u \in \mathbb{R}_F$ and $0 \leq r \leq 1$. The set of r-cut of u, denoted as $[u]_r$, is defined by

$$[u]_r = \{x \in \mathbb{R} \mid u(x) \geq r\} \quad \text{for } 0 < r \leq 1,$$
$$[u]_0 = \overline{\{x \in \mathbb{R} \mid u(x) > 0\}}. \tag{1}$$

According to Definition 1, conditions (1)–(4) imply that $[u]_r$ for all $0 \leq r \leq 1$ is a compact set. Hence, we can denote

$[u]_r$ by a closed interval $[\underline{u}(r), \overline{u}(r)]$, where $\underline{u}, \overline{u} : [0,1] \to \mathbb{R}$ is a function satisfying the following conditions:

(1) $\underline{u}$ is a bounded, left continuous, and nondecreasing function on $[0,1]$.

(2) $\overline{u}$ is a bounded, right continuous, and nonincreasing function on $[0,1]$.

(3) $\underline{u}(r) \le \overline{u}(r)$ for all $r \in [0,1]$.

Next, we define addition and scalar multiplication for the set in the sense of Minkowski.

Definition 3. Let A and B be any nonempty subsets of $\mathbb{R}$ and $\lambda \in \mathbb{R}$; addition between A and B denoted by $A+B$ is defined by

$$A + B = \{a + b \mid a \in A,\ b \in B\}. \tag{2}$$

Multiplication of A by a scalar λ denoted by λA is defined by

$$\lambda A = \{\lambda a \mid a \in A\}, \tag{3}$$

where, for $\lambda > 0$, summation of $A + (-\lambda B)$ is denoted by $A - \lambda B$; that is,

$$A - \lambda B = A + (-\lambda B). \tag{4}$$

Definition 4 (see [18]). Let A and B be any nonempty subsets of $\mathbb{R}$. A nonempty subset C is called a Hukuhara difference between A and B if $A = B + C$. A Hukuhara difference between A and B is denoted by $A \ominus B$.

Note the following: (1) $A \ominus B$ may not exist even when $A - B$ definitely exists, so, for any A and B, $A \ominus B \ne A - B$ and (2) $A \ominus A = \{0\}$.

Definition 5. Let $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. Zadeh's extension of g is the function $g : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_F$ (again, labeled as g) defined by

$$g(u, v)(z) = \sup_{g(x,y)=z} \min\{u(x), v(y)\} \quad \forall z \in \mathbb{R}. \tag{5}$$

Theorem 6. *Let $g : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_F$ be Zadeh's extension of g. Then, the set of r-cut of $g(u,v)$ is of the form*

$$[g(u, v)]_r = g([u]_r, [v]_r) \tag{6}$$

for all $u, v \in \mathbb{R}_F$ and $0 \le r \le 1$.

Definition 5 together with Theorem 6 is called Zadeh's extension principle. Following Zadeh's extension principle and Minkowski's definition, addition and scalar multiplication can be defined by the next definition.

Definition 7. Let $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$; addition between u and v, denoted by $u \oplus v$, is defined by

$$(u \oplus v)(z) = \sup_{x+y=z} \min\{u(x), v(y)\} \quad \forall z \in \mathbb{R}, \tag{7}$$

and multiplication of u by a scalar λ, denoted by $\lambda \odot u$, is defined by

$$(\lambda \odot u)(z) = \begin{cases} u\left(\dfrac{z}{\lambda}\right), & \lambda \ne 0 \\ \overline{0}, & \lambda = 0. \end{cases} \tag{8}$$

Theorem 8. *Let $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$. Then, one has*

$$\begin{aligned} [u \oplus v]_r &= [u]_r + [v]_r = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)], \\ [\lambda \odot u]_r &= \lambda [u]_r = \begin{cases} [\lambda \underline{u}(r), \lambda \overline{u}(r)], & \lambda > 0 \\ [\lambda \overline{u}(r), \lambda \underline{u}(r)], & \lambda < 0 \\ \{0\}, & \lambda = 0, \end{cases} \end{aligned} \tag{9}$$

where $\overline{0} = \chi_{\{0\}}$ is the addition identity on $\mathbb{R}_F$.

Theorem 9 (see [20]). *Under addition $\oplus$ and multiplication $\odot$, one has the following:*

(1) *No element of $\mathbb{R}_F \setminus \mathbb{R}$, except $\overline{0}$, has an inverse under $\oplus$.*

(2) *For all $a, b \in \mathbb{R}$ such that both a and $b \le 0$ or ≥ 0 and, for all $u \in \mathbb{R}_F$,*

$$(a + b) \odot u = a \odot u \oplus b \odot u. \tag{10}$$

(3) *For all $\lambda \in \mathbb{R}$ and for all $u, v \in \mathbb{R}_F$,*

$$\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v. \tag{11}$$

(4) *For all $\lambda, \beta \in \mathbb{R}$ and for all $u \in \mathbb{R}_F$,*

$$(\lambda + \beta) \odot u = \lambda \odot u \oplus \beta \odot u. \tag{12}$$

Definition 10. Let $u, v \in \mathbb{R}_F$. If there exists $w \in \mathbb{R}_F$ such that $u = v \oplus w$, then w is called a *Hukuhara difference (fuzzy)* between u and v, denoted by $w = u \ominus v$.

Next, we define the distance between any two elements in $\mathbb{R}_F$. $\mathbb{R}_0^+$ denotes $[0, \infty)$.

Definition 11 (see [15]). Let $u, v \in \mathbb{R}_F$. The distance (*Hausdorff distance*) between u and v is defined by $d_H : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_0^+$, where

$$d_H(u, v) = \sup_{r\in[0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}. \tag{13}$$

Hence, according to the property of the distance d_H, $(\mathbb{R}_F, d_H)$ is a complete metric space.

Definition 12. A function $f : [a,b] \to \mathbb{R}_F$ is called *fuzzy function* and the r-cut of $f(t)$ for all $t \in [a,b]$ is denoted by $[f(t)]_r = [\underline{f}(t)(r), \overline{f}(t)(r)]$ for all $r \in [0,1]$.

Definition 13. Let $f : [a,b] \to \mathbb{R}_F$ be a fuzzy function. f is called *fuzzy continuous* on $[a,b]$ if for all $t_0 \in [a,b]$ and for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $t \in [a,b]$ if $|t - t_0| < \delta$, then $d_H(f(t), f(t_0)) < \epsilon$.

Definition 14. Let $f : [a,b] \to \mathbb{R}_F$ be a fuzzy function. f is called *fuzzy uniform continuous* on $[a,b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $s, t \in [a,b]$ if $|s - t| < \delta$, then $d_H(f(s), f(t)) < \epsilon$.

Definition 15. Let $f : [a,b] \to \mathbb{R}_F$ be a fuzzy function. One says that f is *bounded* on $[a,b]$ if there is $M > 0$ such that $d_H(f(t), \overline{0}) \le M$ for all $t \in [a,b]$.

Definition 16 (see [18]). Let $f : [a,b] \to \mathbb{R}_F$ be a fuzzy function. One says that f is *fuzzy differentiable* at $x_0 \in (a,b)$ if there is $y \in \mathbb{R}_F$ and $\delta > 0$ such that for all $h < \delta$ if $f(x_0+h) \ominus f(x_0)$ and $f(x_0) \ominus f(x_0-h)$ exist, then

$$\lim_{h\to 0^+} d_H\left(\frac{f(x+h)\ominus f(x_0)}{h}, y\right) = 0 = \lim_{h\to 0^+} d_H\left(\frac{f(x_0)\ominus f(x-h)}{h}, y\right). \tag{14}$$

The fuzzy number $y \in \mathbb{R}_F$ is called *fuzzy derivative* of f at x_0 and is denoted by $f'(x_0)$ or $(df/dx)(x_0)$; that is,

$$f'(x_0) = \lim_{h\to 0^+} \frac{f(x+h)\ominus f(x_0)}{h} = \lim_{h\to 0^+} \frac{f(x_0)\ominus f(x-h)}{h}. \tag{15}$$

For the extremes of the interval $[a,b]$, the fuzzy derivative of f at a is $f'(a) = \lim_{h\to 0^+}((f(x+h)\ominus f(a))/h)$ if $\lim_{h\to 0^+}((f(x+h)\ominus f(a))/h)$ exists, and the fuzzy derivative of f at b is $f'(b) = \lim_{h\to 0^+}((f(b)\ominus f(x-h))/h)$ if $\lim_{h\to 0^+}((f(x+h)\ominus f(a))/h)$ exists (the multiplier $1/h$ denotes a scalar fuzzy multiple).

Theorem 17 (see [15]). *Let the following be true: $\lambda \in \mathbb{R}$; $\alpha : [a,b] \to \mathbb{R}$ is differentiable; and $f, g : [a,b] \to \mathbb{R}_F$ is fuzzy differentiable on $[a,b]$; then the following is true:*

(1) $(f \oplus g)'(t) = f'(t) \oplus g'(t)$.

(2) $(\lambda \odot f)'(t) = \lambda \odot f'(t)$.

(3) $(\alpha \odot g)'(t) = \alpha(t) \odot f'(t) \oplus a'(t) \odot f(t)$.

Definition 18. For each $f : [a,b] \to \mathbb{R}_F$, one says that f is *integrable* if there exists a fuzzy function $F : [a,b] \to \mathbb{R}_F$ such that $F'(t) = f(t)$ for all $t \in [a,b]$. The fuzzy function F is called *fuzzy antiderivative* of f and is denoted by $F(t) = \int_a^t f(s)ds$.

Theorem 19 (see [15]). *Let $f : [a,b] \to \mathbb{R}_F$ be fuzzy differentiable; then*

$$f(t) = f(a) \oplus \int_a^t f'(s)\,ds. \tag{16}$$

Theorem 20 (see [15]). *Let $f : [a,b] \to \mathbb{R}_F$ be fuzzy integrable and $c \in [a,b]$; then*

$$\int_a^b f(s)\,ds = \int_a^c f(s)\,ds \oplus \int_c^b f(s)\,ds. \tag{17}$$

Theorem 21 (see [15]). *If $f : [a,b] \to \mathbb{R}_F$ is fuzzy differentiable, then f is fuzzy continuous.*

Definition 22. A fuzzy sequence is a function from $\mathbb{N}$ to $\mathbb{R}_F$. A fuzzy sequence f (where $f(n) = u_n \in \mathbb{R}$ for all $n \in \mathbb{R}$) is denoted by $\{u_n\}$ or more briefly by u_n.

Definition 23. Let $\{u_n\}$ be a fuzzy sequence and $u \in \mathbb{R}_F$. One says that $\{u_n\}$ converges to u if and only if for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d_H(u_n, u) < \epsilon$ for all $n \geq n_0$, denoted by $\lim_{n\to\infty} u_n = u$.

Definition 24. Let $\{u_n\}$ be a fuzzy sequence. One says that $\{u_n\}$ is bounded if there is $N > 0$ such that $d_H(u_n, \overline{0}) < N$ for all $n \in \mathbb{N}$.

3. Fuzzy Initial Value Problem

In this section, we discuss initial value problem of fuzzy differential equation, give the definition of a solution and sufficient conditions for its existence, and prove the relevant theorems and lemmas and then the existence of the solution by using the method of successive approximation.

In this paper, $C([0,T], \mathbb{R}_F)$ denotes $\{f : [0,T] \to \mathbb{R}_F \mid f \text{ is fuzzy continuous}\}$ with a weighted metric defined by $d_C(u,v) = \sup_{t\in[0,T]} e^{-\lambda t} d_H(u(t), v(t))$, where $\lambda \geq 0$ (which can be any given value). Since $(\mathbb{R}_F, d_H)$ is complete, the space $(C([0,T], \mathbb{R}_F), d_C)$ is also complete; see [15]. For convenience, we denote $C([0,T], \mathbb{R}_F)$ as C^0.

3.1. Fuzzy Differential Equation. Consider an initial value problem of a fuzzy differential equation:

$$\begin{aligned} x'(t) &= a(t) \odot x(t) \oplus f(t, x(t)), \quad t \in [0,T], \\ x(0) &= x^0, \end{aligned} \tag{18}$$

where x is a fuzzy state function of time variable t, $f(t,x)$ is a fuzzy input function of variable t and x, x' is the fuzzy derivative of x, $x(0) = x^0$ is a fuzzy number, and $a : [0,T] \to \mathbb{R}$ is a continuous function. Throughout this paper, we denote x by $[\underline{x}, \overline{x}]$ and its r-cut by $[x(t)]_r = [\underline{x}(t)(r), \overline{x}(t)(r)]$ for all $0 < r \leq 1$.

The fuzzy function $f(t,x)$ denotes $[\underline{f}(t,x), \overline{f}(t,x)]$ with

$$\begin{aligned} \underline{f}(t,x) &= \min\{f(t,u) \mid u \in [x(t)]_r\}, \\ \overline{f}(t,x) &= \max\{f(t,u) \mid u \in [x(t)]_r\}. \end{aligned} \tag{19}$$

The r-cut of $f(t,x)$ for $t \in [0,T]$ is given by

$$[f(t,x(t))]_r = [\underline{f}(t,x(t))(r), \overline{f}(t,x(t))(r)] \quad \forall 0 < r \leq 1. \tag{20}$$

Consider the fuzzy derivative of $S(s,t) \odot x(s)$ for all $s \in [0,t]$, where $S(s,t) = e^{\int_s^t a(\tau)d\tau}$. If x is fuzzy differentiable, that is, a solution to (18), using Theorem 19, we get

$$\frac{d}{ds}(S(s,t) \odot x(s)) = S(s,t) \odot x'(s) \oplus \frac{dS(s,t)}{ds} \odot x(s)$$

$$\begin{aligned}
&= S(s,t) \\
&\quad \odot [a(s) \odot x(s) \oplus f(s, x(s))] \\
&\quad \oplus (-S(s,t))\, a(s) \odot x(s) \\
&= S(s,t)\, a(s) \odot x(s) \oplus S(s,t) \\
&\quad \odot f(s, x(s)) \oplus (-S(s,t))\, a(s) \\
&\quad \odot x(s) = S(s,t) \odot f(s, x(s)).
\end{aligned} \tag{21}$$

Using Theorem 20 and an initial value $x(0) = x^0$, we obtain

$$x(t) = S(0,t) \odot x^0 \oplus \int_0^t S(s,t) \odot f(s, x(s))\, ds. \tag{22}$$

Hence, (22) is a fuzzy integral that corresponds to the fuzzy differential equation (18). Solution to (22) is a type of solutions to (18) that we define next.

Definition 25. Let $x \in C([0,T], \mathbb{R}_F)$. x is called *fuzzy mild solution* of the fuzzy differential equation (18) if x satisfies the fuzzy integral equation

$$x(t) = S(0,t) \odot x^0 \oplus \int_0^t S(s,t) \odot f(s, x(s))\, ds, \tag{23}$$

where $S(s,t) = e^{\int_s^t a(\tau)d\tau}$.

In the next section, we prove the existence of a fuzzy mild solution of (18) under the following assumption.

Assumption H. It declares that if $f(t,x) = [\underline{f}(t,x), \overline{f}(t,x)]$ is a fuzzy function with $\underline{f}(t,x) = \min\{f(t,u) \mid u \in [x(t)]_r\} = F(t, \underline{x}, \overline{x})$, $\overline{f}(t,x) = \max\{f(t,u) \mid u \in [x(t)]_r\} = G(t, \underline{x}, \overline{x})$, where $F, G : [0,T] \times [C([0,T], C([0,1], \mathbb{R}_F))]^2 \to C([0,1], \mathbb{R}_F)$, then there is $l > 0$ such that $lMT < 1$ and $|F(t_1, x_1(t_1)(r), y_1(t_1)(r)) - F(t_2, x_2(t_2)(r), y_2(t_2)(r))(r)|$, $|G(t_1, x_1(t_1)(r), y_1(t_1)(r)) - G(t_2, x_2(t_2)(r), y_2(t_2)(r))(r)| \le l(|t_1 - t_2| + \max\{|x_1(t_1)(r) - x_2(t_2)(r)|, |y_1(t_1)(r) - y_2(t_2)(r)|\})$, for all $(t_1, x_1, y_1), (t_2, x_2, y_2) \in [0,T] \times C([0,T], C([0,1], \mathbb{R})) \times C([0,T], C([0,1], \mathbb{R}))$, where $M = \sup_{s,t \in [0,T]} |S(s,t)|$ and $S(s,t) = e^{\int_s^t a(\tau)d\tau}$.

3.2. Existence of a Solution. In this subsection, we prove the existence of a mild fuzzy solution to system (18) under Assumption H by using the method of successive approximation. Let us begin by defining a sequence of function $\{x_n\}$ for an initial value $x^0 \in \mathbb{R}_F$ as

$$x_n(t) = S(0,t) \odot x^0 \oplus \int_0^t S(s,t) \odot f(s, x_{n-1}(s))\, ds \quad \forall t \in [0,T], \tag{24}$$

where $x_0 \in C^0$ is a given initial function. For any $x^0 \in \mathbb{R}_F$ and $t \in [0,T]$, we have $x_n : [0,T] \to \mathbb{R}_F$. Next, we show that the sequence $\{x_n\}$ has the following properties:

(1) $x_n \in C^0$ for all $n \in \mathbb{N}$.

(2) $\{x_n\}$ is a Cauchy sequence in C^0.

Property 1. We show that $x_n \in C^0$ for all $n \in \mathbb{N}$ by referring to the following statements.

Lemma 26. *Let f be a fuzzy function that satisfies Assumption H. Then, for each t, where $t_0 \in [0,T]$, there exists $l > 0$ such that*

$$\begin{aligned}
&d_H(f(t, x(t)), f(t_0, x(t_0))) \\
&\quad \le l(|t - t_0| + d_H(x(t), x(t_0))).
\end{aligned} \tag{25}$$

Proof. Let $t, t_0 \in [0,T]$. Since

$$\begin{aligned}
d_H(f(t, x(t)), f(t_0, x(t_0))) = \sup_{r\in[0,1]} \max\{&|F(t, \underline{x}(t)(r), \overline{x}(t)(r))(r) - F(t_0, \underline{x}(t_0)(r), \overline{x}(t_0)(r))(r)|, \\
&|G(t, \underline{x}(t)(r), \overline{x}(t)(r))(r) - G(t_0, \underline{x}(t_0)(r), \overline{x}(t_0)(r))(r)|\},
\end{aligned} \tag{26}$$

by Assumption H, there is $l > 0$ such that

$$\begin{aligned}
&\max\{|F(t, \underline{x}(t)(r), \overline{x}(t)(r))(r) - F(t_0, \underline{x}(t_0)(r), \\
&\quad \overline{x}(t_0)(r))(r)|, |G(t, \underline{x}(t)(r), \overline{x}(t)(r))(r) - G(t_0, \\
&\quad \underline{x}(t_0)(r), \overline{x}(t_0)(r))(r)|\} \le l(|t - t_0| \\
&\quad + \max\{|\underline{x}(t)(r) - \underline{x}(t_0)(r)|, \\
&\quad |\overline{x}(t)(r) - \overline{x}(t_0)(r)|\}).
\end{aligned} \tag{27}$$

Hence,

$$\begin{aligned}
&d_H(f(t, x(t)), f(t_0, x(t_0))) \le l\Big(|t - t_0| \\
&\quad + \sup_{r\in[0,1]} \max\{(|\underline{x}(t)(r) - \underline{x}(t_0)(r)|, |\overline{x}(t)(r) - \overline{x}(t_0)(r)|)\}\Big) \\
&\quad = l(|t - t_0| + d_H(x(t), x(t_0))).
\end{aligned} \tag{28}$$

□

Lemma 27. *Let f be a fuzzy function that satisfies Assumption H. Then, for each $x \in C^0$, the map $t \mapsto f(t, x(t))$ is fuzzy continuous.*

Proof. Let $t, t_0 \in [0,T]$. By Lemma 26, there is $l > 0$ such that

$$d_H(f(t,x(t)), f(t_0,x(t_0))) \leq l(|t-t_0| + d_H(x(t),x(t_0))). \tag{29}$$

Given any $\varepsilon > 0$, by the fuzzy continuity of x, there exists $\delta_1 > 0$ such that, for all $t \in [0,T]$, if $|t-t_0| < \delta_1$, then $d_H(x(t),x(t_0)) < \varepsilon/2l$. Choose $\delta = \min\{\varepsilon/2l, \delta_1\}$. Then, for each $t \in [0,T]$ such that $|t-t_0| < \delta$, we have

$$\begin{aligned} &d_H(f(t,x(t)), f(t_0,x(t_0))) \\ &\quad \leq l(|t-t_0| + d_H(x(t),x(t_0))) < l\left(\frac{\varepsilon}{2l} + \frac{\varepsilon}{2l}\right) \\ &\quad = \varepsilon. \end{aligned} \tag{30}$$

Therefore, the map $t \mapsto f(t,x(t))$ is fuzzy continuous. □

Lemma 28. *If $y \in C^0$ and $\beta \in C([0,T],\mathbb{R})$, then $\beta \odot y \in C^0$.*

Proof. Let $t_0 \in [0,T]$. Because $\beta \in C([0,T],\mathbb{R})$ and $y \in C^0$, there exist M_1 and $M_2 > 0$ such that $|\beta(t)| \leq M_1$ and $d_H(y(t),\overline{0}) \leq M_2$ for all $t \in [0,T]$. Set $B = \max\{M_1, M_2\}$. Given any $\varepsilon > 0$, by the continuity of y and β, there is $\delta_1 > 0$ for each $t \in [0,T]$. If $|t-t_0| < \delta_1$, then

$$\begin{aligned} |\beta(t) - \beta(t_0)| &< \frac{\varepsilon}{2B}, \\ d_H(y(t),y(t_0)) &< \frac{\varepsilon}{2B}. \end{aligned} \tag{31}$$

Choose $\delta = \min\{\varepsilon/2B, \delta_1\}$. Then, for each $t \in [0,T]$ such that $|t-t_0| < \delta$, we have

$$\begin{aligned} &d_H(\beta(t) \odot y(t), \beta(t_0) \odot y(t_0)) = d_H(\beta(t) \odot y(t) \\ &\quad + \beta(t) \odot y(t_0), \beta(t_0) \odot y(t_0) + \beta(t) \odot y(t_0)) \\ &\quad \leq d_H(\beta(t) \odot y(t), \beta(t) \odot y(t_0)) + d_H(\beta(t) \\ &\quad \odot y(t_0), \beta(t_0) \odot y(t_0)) \\ &\quad \leq |\beta(t)| d_H(y(t),y(t_0)) \\ &\quad + |\beta(t) - \beta(t_0)| d_H(y(t_0),\overline{0}) \\ &\quad < B\left(\frac{\varepsilon}{2B}\right) + B\left(\frac{\varepsilon}{2B}\right) = \varepsilon. \end{aligned} \tag{32}$$

Therefore, we can conclude that the map $t \mapsto \beta(t) \odot y(t)$ is fuzzy continuous. □

Lemma 29. *If $y_1, y_2 \in C^0$, then $y_1 \oplus y_2 \in C^0$.*

Proof. Let $t_0 \in [0,T]$. Given any $\varepsilon > 0$, by the continuity of y_1 and y_2, there is $\delta_1 > 0$ for all $t \in [0,T]$ such that $|t-t_0| < \delta_1$, and so we obtain $d_H(y_1(t), y_1(t_0)), d_H(y_2(t), y_2(t_0)) < \varepsilon/2$.

Choose $\delta = \min\{\varepsilon/2, \delta_1\}$. Then, for each $t \in [0,T]$ such that $|t-t_0| < \delta$, we have

$$\begin{aligned} &d_H(y_1(t) \oplus y_2(t), y_1(t_0) \oplus y_2(t_0)) \\ &\quad \leq d_H(y_1(t), y_1(t_0)) + d_H(y_2(t), y_2(t_0)) \\ &\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{33}$$

Hence, $y_1 \oplus y_2 \in C^0$. □

Lemma 30. *If $y \in C^0$, then the map $t \mapsto \int_0^t y(s)ds$ is fuzzy continuous.*

Proof. Since $y \in C^0$, there is $M > 0$ such that $d_H(y(s),\overline{0}) \leq M$ for all $s \in [0,T]$.

Let $t_0 \in [0,T]$. Given any $\varepsilon > 0$, choose $\delta = \varepsilon/M$. Then, for each $t \in [0,T]$ such that $|t-t_0| < \delta$ and $t > t_0$, by Theorem 20, we have

$$\begin{aligned} &d_H\left(\int_0^t y(s)\,ds, \int_0^{t_0} y(s)\,ds\right) \\ &\quad = d_H\left(\int_0^{t_0} y(s)\,ds \oplus \int_{t_0}^t y(s)\,ds, \int_0^{t_0} y(s)\,ds \oplus \overline{0}\right) \\ &\quad \leq d_H\left(\int_0^{t_0} y(s)\,ds, \int_0^{t_0} y(s)\,ds\right) \\ &\quad + d_H\left(\int_{t_0}^t y(s)\,ds, \overline{0}\right) \leq \int_{t_0}^t d_H(y(s),\overline{0})\,ds \\ &\quad \leq M(t-t_0) < M\left(\frac{\varepsilon}{M}\right) = \varepsilon. \end{aligned} \tag{34}$$

If $t < t_0$, by Theorem 21, we have

$$\begin{aligned} &d_H\left(\int_0^t y(s)\,ds, \int_0^{t_0} y(s)\,ds\right) \\ &\quad = d_H\left(\int_0^t y(s)\,ds \oplus \overline{0}, \int_0^t y(s)\,ds \oplus \int_t^{t_0} y(s)\,ds\right) \\ &\quad \leq d_H\left(\int_0^t y(s)\,ds, \int_0^t y(s)\,ds\right) \\ &\quad + d_H\left(\overline{0}, \int_t^{t_0} y(s)\,ds\right) \leq \int_t^{t_0} d_H(y(s),\overline{0})\,ds \\ &\quad \leq -M(t-t_0) < M\left(\frac{\varepsilon}{M}\right) = \varepsilon. \end{aligned} \tag{35}$$

Hence, the map $t \mapsto \int_0^t y(s)ds$ is fuzzy continuous. □

Lemma 31. *Assuming that f is a fuzzy function satisfying Assumption H, for a given initial function $x_0 \in C^0$, one has a sequence of fuzzy function $\{x_n\}$ as defined in (24).*

Proof. We show that x_n is fuzzy continuous for all $n \in \mathbb{N}$ by using mathematical induction.

Basis Step. Because $x_0 \in C^0$ and x_1 is defined as

$$x_1(t) = S(0,t) \odot x^0 \oplus \int_0^t S(s,t) \odot f(s, x_0(s))\, ds \quad \forall t \in [0,T], \tag{36}$$

by Lemmas 26–30, x_1 is fuzzy continuous.

Induction Step. For $k > 1$, assuming that $x_k \in C^0$, since $x_{k+1}(t) = S(0,t) \odot x^0 \oplus \int_0^t S(s,t) \odot f(s, x_k(s)) ds$, by Lemmas 26–30, we have $x_{k+1} \in C^0$.

Therefore, by mathematical induction, $x_n \in C^0$ for all $n \in \mathbb{N}$. □

Property 2. We show that $\{x_n\}$ is a Cauchy sequence in C^0.

Lemma 32. *Let f be a fuzzy function satisfying Assumption H and let $x_0 \in C^0$ be a given initial function. Then, $d_C(x_n, x_{n-1}) \le AP^{n-1}$ for all $n \in \mathbb{N}$ with $P = lMT$ and $A = d_C(x_1, x_0)$.*

Proof. We show that $d_C(x_n, x_{n-1}) \le AP^{n-1}$ for all $n \in \mathbb{N}$ by mathematical induction.

Basis Step. For $k = 1$, $d_C(x_1, x_0) = AP^0$.

Induction Step. For $k > 1$, assuming that $d_C(x_k, x_{k-1}) \le AP^{k-1}$, we have

$$\begin{aligned}
d_H(x_{k+1}(t), x_k(t)) &= d_H\left(S(0,t) \odot x^0 \oplus \int_0^t S(s,t) \odot f(s, x_k(s))\, ds, S(0,t) \odot x^0 \oplus \int_0^t S(s,t) \odot f(s, x_{k-1}(s))\, ds\right) \\
&= d_H\left(\int_0^t S(s,t) \odot f(s, x_k(s))\, ds, \int_0^t S(s,t) \odot f(s, x_{k-1}(s))\, ds\right) \\
&= \sup_{r\in[0,1]} \max\left\{\left|\int_0^t S(s,t) \odot \underline{f}(s, x_k(s))(r)\, ds - \int_0^t S(s,t) \odot \underline{f}(s, x_{k-1}(s))(r)\, ds\right|,\right. \\
&\left.\left|\int_0^t S(s,t) \odot \overline{f}(s, x_k(s))(r)\, ds - \int_0^t S(s,t) \odot \overline{f}(s, x_{k-1}(s))(r)\, ds\right|\right\} \\
&\le \sup_{r\in[0,1]} \max\left\{\int_0^t |S(s,t)| \left|F(s, \underline{x}_k(s)(r), \overline{x}_k(s)(r)) - F(s, \underline{x}_{k-1}(s)(r), \overline{x}_{k-1}(s)(r))\right| ds,\right. \\
&\left.\int_0^t |S(s,t)| \left|G(s, \underline{x}_k(s)(r), \overline{x}_k(s)(r)) - G(s, \underline{x}_{k-1}(s)(r), \overline{x}_{k-1}(s)(r))\right| ds\right\} \\
&\le Ml\left\{\int_0^t \sup_{r\in[0,1]} \max\left\{\left|\underline{x}_k(s)(r) - \underline{x}_{k-1}(s)(r)\right|, \left|\overline{x}_k(s)(r) - \overline{x}_{k-1}(s)(r)\right|\right\} ds\right\} \le Ml\left(\int_0^t d_H(x_{k-1}(s), x_k(s))\, ds\right) \\
&\le MlTd_C(x_k, x_{k-1}) \le PAP^{k-1} = AP^k.
\end{aligned} \tag{37}$$

Thus, by mathematical induction, $d_C(x_n, x_{n-1}) \le AP^{n-1}$ for all $n \in \mathbb{N}$. □

Lemma 33. *Let f be a fuzzy function satisfying Assumption H and let $x_0 \in C^0$ be a given initial function; one has $d_C(x_n, x_0) \le A(1-P^n)/(1-P)$ for all $n \in \mathbb{N}$ with $P = lMT$ and $A = d_C(x_1, x_0)$.*

Proof. We show that $d_C(x_n, x_0) \le A(1-P^n)/(1-P)$ for all $n \in \mathbb{N}$ by using mathematical induction.

Basis Step. For $k = 1$, the above statement is true since $d_C(x_1, x_0) = A(1-P)/(1-P)$.

Induction Step. For $m > 1$, assuming that $d_C(x_j, x_0) \le A(1 - P^j)/(1-P)$ for all $j \in \{1, 2, \ldots, m\}$, by Lemma 32, for $t \in [0,T]$, we have

$$\begin{aligned}
d_H(x_{m+1}(t), x_0(t)) &= d_H(x_{m+1}(t) \oplus x_m(t) \oplus \cdots \\
&\quad \oplus x_1(t), x_0(t) \oplus x_m(t) \oplus \cdots \oplus x_1(t)) \\
&\le d_H(x_{m+1}(t), x_m(t)) + d_H(x_m(t), x_{m-1}(t)) \\
&\quad + \cdots + d_H(x_1(t), x_0(t)) \le AP^m + AP^{m-1} + \cdots \\
&\quad + A = \frac{A(1-P^{m+1})}{1-P},
\end{aligned} \tag{38}$$

implying that $d_C(x_{m+1}, x_0) \le A(1-P^{m+1})/(1-P)$.

Therefore, by the principle of mathematical induction, $d_C(x_n, x_0) \le A(1-P^n)/(1-P)$ for all $n \in \mathbb{N}$. □

Lemma 34. *Assume that Assumption H holds. Given an initial function $x_0 \in C^0$, $\{x_n\}$ is a bounded sequence.*

Proof. Assumption H implies that there is $l > 0$ such that $P = lMT < 1$.

Therefore, there is $B > 0$ such that $A(1-P^n)/(1-P) \le B$ for all $n \in \mathbb{N}$. By Lemma 33, we have

$$d_C(x_n, 0) - d_C(x_0, 0) \le d_C(x_n, x_0) \le \frac{A(1-P^n)}{1-P} \le B. \tag{39}$$

Hence, $d_C(x_n, 0) \le B + d_C(x_0, 0)$ for all $n \in \mathbb{N}$. □

Lemma 35. *Assume that Assumption H holds. Let $R_n = A(1-P^n)/(1-P)$ and $0 < c < 1$. Then, for a given initial function $x_0 \in C^0$ and for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $2R_n c^{N+1} < \varepsilon$ for all $n \ge N$ with $P = lMT$ and $A = d_C(x_1, x_0)$.*

Proof. By Assumption H, there is $l > 0$ such that $P = lMT < 1$. Hence, there is $B > 0$ such that $R_n = A(1-P^n)/(1-P) \le B$ for all $n \in \mathbb{N}$. Given any $\varepsilon > 0$, choose $N > \log_c(\varepsilon/2Bc)$.

Then, $2R_n c^{N+1} \le 2Bc^N c < 2Bcc^{\log_c(\varepsilon/2Bc)} = 2Bc(\varepsilon/2Bc) = \varepsilon$. □

Next, let us define a mapping g with an initial value $x^0 \in \mathbb{R}_F$ to be

$$g(u)(t) := S(0,t) \odot x^0 \oplus \int_0^t S(s,t) \odot f(s, u(s))\, ds \tag{40}$$

for all $u \in C([0,T], \mathbb{R}_F)$.

Lemma 36. *Suppose that Assumption H holds. Then, g is a mapping from C^0 to C^0.*

Proof. Let $u \in C([0,T], \mathbb{R}_F)$. Similar to the proof of Lemma 31, we have $g(u) \in C([0,T], \mathbb{R}_F)$. Let $u_1, u_2 \in C([0,T], \mathbb{R}_F)$ such that $u_1 = u_2$.

For any $t \in [0,T]$, we have

$$\begin{aligned} g(u_1)(t) &= S(0,t) \odot x^0 \oplus \int_0^t S(s,t) \odot f(s, u_1(s))\, ds \\ &= S(0,t) \odot x^0 \oplus \int_0^t S(s,t) \odot f(s, u_2(s))\, ds \\ &= g(u_2)(t). \end{aligned} \tag{41}$$

Therefore, $g : C([0,T], \mathbb{R}_F) \to C([0,T], \mathbb{R}_F)$. □

Lemma 37. *Suppose that Assumption H holds. Then, g is a contraction mapping.*

Proof. By Assumption H, there is $l > 0$ such that $P = lMT < 1$ and

$$\begin{aligned} \max \{ &|F(s, \underline{u}(s)(r), \overline{u}(s)(r))(r) \\ &- F(s, \underline{v}(s)(r), \overline{v}(s)(r))(r)|, \\ &|G(s, \underline{u}(s)(r), \overline{u}(s)(r))(r) \\ &- G(s, \underline{v}(s)(r), \overline{v}(s)(r))(r)| \} \le l \max \{ |\underline{u}(s)(r) \\ &- \underline{v}(s)(r)|, |\overline{u}(s)(r) - \overline{v}(s)(r)| \}. \end{aligned} \tag{42}$$

For all $t \in [0,T]$, choose $c = lMT$. Then,

$$\begin{aligned} d_H(g(u)(t), g(v)(t)) &= d_H\left(\int_0^t S(s,t) \odot f(s, u(s))\, ds, \int_0^t S(s,t) \odot f(s, v(s))\, ds \right) \\ &= \sup_{r \in [0,1]} \max \left\{ \left| \int_0^t S(s,t) \odot \underline{f}(s, u(s))(r)\, ds - \int_0^t S(s,t) \odot \underline{f}(s, v(s))(r)\, ds \right|, \right. \\ &\quad \left. \left| \int_0^t S(s,t) \odot \overline{f}(s, u(s))(r)\, ds - \int_0^t S(s,t) \odot \overline{f}(s, v(s))(r)\, ds \right| \right\} \\ &\le \sup_{r \in [0,1]} \max \left\{ \int_0^t |S(s,t)| \left| F(s, \underline{u}(s)(r), \overline{u}(s)(r)) - F(s, \underline{v}(s)(r), \overline{v}(s)(r)) \right| ds, \right. \\ &\quad \left. \int_0^t |S(s,t)| \left| G(s, \underline{u}(s)(r), \overline{u}(s)(r)) - G(s, \underline{v}(s)(r), \overline{v}(s)(r)) \right| ds \right\} \\ &\le Ml \int_0^t \sup_{r \in [0,1]} \max \{ |\underline{u}(s)(r) - \underline{v}(s)(r)|, |\overline{u}(s)(r) - \overline{v}(s)(r)| \}\, ds \le c d_C(u, v). \end{aligned} \tag{43}$$

Therefore, g is a contraction mapping. □

Theorem 38. *Suppose that Assumption H holds. Then, $\{x_n\}$ is a Cauchy sequence in C^0.*

Proof. Given any $\varepsilon > 0$. By Lemma 35, for all $0 < c < 1$, there is $N > 0$ such that $2R_n c^{N+1} < \varepsilon$ for all $n \geq N$, where $R_n = A(1 - P^n)/(1 - P)$ with $P = lMT$ and $A = d_C(x_1, x_0)$.

Let $m, n \in \mathbb{N}$ be such that $m, n > N$. WLOG, assume that $n > m$. By Lemma 33, we have

$$\begin{aligned} &d_C(x_{m-N-1}, x_{n-N-1}) \\ &\quad = d_C(x_{m-N-1} \oplus x_0, x_{n-N-1} \oplus x_0) \\ &\quad \leq d_C(x_{m-N-1}, x_0) + d_C(x_{n-N-1}, x_0) \\ &\quad \leq \frac{A(1 - P^{m-N-1})}{1 - P} + \frac{A(1 - P^{n-N-1})}{1 - P} \\ &\quad < \frac{A(1 - P^{n-N-1})}{1 - P} + \frac{A(1 - P^{n-N-1})}{1 - P} \\ &\quad = 2\left[\frac{A(1 - P^{n-N-1})}{1 - P}\right] < 2\left[\frac{A(1 - P^{n})}{1 - P}\right] = 2R_n. \end{aligned} \tag{44}$$

By definition of sequence $\{x_n\}$ and mapping g, we can write x_n and x_m as a composition of g,

$$\begin{aligned} x_m &= \underbrace{g \circ g \circ \cdots \circ g}_{N+1}(x_{m-N-1}), \\ x_n &= \underbrace{g \circ g \circ \cdots \circ g}_{N+1}(x_{n-N-1}). \end{aligned} \tag{45}$$

Since g is a contraction mapping, there is some c, where $0 < c < 1$

$$\begin{aligned} d_C(x_m, x_n) &= d_C(g \circ g \circ \cdots \circ g(x_{m-N-1}), g \circ g \circ \cdots \\ &\quad \circ g(x_{n-N-1})) \leq c^{N+1} d_C(x_{m-N-1}, x_{n-N-1}) \\ &\quad < 2R_n c^{N+1} < \varepsilon. \end{aligned} \tag{46}$$

Hence, $\{x_n\}$ is a Cauchy sequence in $C([0,T], \mathbb{R}_F)$. □

By using Properties 1 and 2, we prove the existence of a mild fuzzy solution to system (18) in the following theorem.

Theorem 39. *If Assumption H holds, system (18) has a mild fuzzy solution; that is, there is* $x \in C([0,T], \mathbb{R}_F)$ *such that* $x(t) = S(0,t) \odot x^0 \oplus \int_0^t S(s,t) \odot f(s, x(s))ds$.

Proof. Given an initial function $x_0 \in C([0,T], \mathbb{R}_F)$ and a sequence $\{x_n\}$ defined by

$$x_n(t) = S(0,t) \odot x^0 \oplus \int_0^t S(s,t) \odot f(s, x_{n-1}(s))\, ds \qquad \forall t \in [0,T], \tag{47}$$

by Theorem 38, $\{x_n\}$ is a Cauchy sequence in $C([0,T], \mathbb{R}_F)$.

Since $C([0,T], \mathbb{R}_F)$ is complete, $\{x_n\}$ converges in $C([0,T], \mathbb{R}_F)$; that is, there is $x \in C([0,T], \mathbb{R}_F)$ such that $\lim_{n\to\infty} x_n = x$.

Given any $\varepsilon > 0$, since $\lim_{n\to\infty} x_n = x$, there is $N_1 \in \mathbb{N}$ such that $d_C(x_{n-1}, x) < \varepsilon/lMT$ for all $n \geq N_1$. Let $w(t) := S(0,t) \odot x_0 \oplus \int_0^t S(s,t) \odot f(s, x(s))ds$. We show that $\lim_{n\to\infty} x_n = w$ as follows. For each $t \in [0,T]$, choose $N = N_1$. Then,

$$\begin{aligned} d_H(x_n(t), w(t)) &= d_H\left(\int_0^t S(s,t) \odot f(s, x_{n-1}(s))\, ds, \int_0^t S(s,t) \odot f(s, x(s))\, ds\right) \\ &= \sup_{r\in[0,1]} \max\left\{\left|\int_0^t S(s,t) \odot \underline{f}(s, x_{n-1}(s))(r)\, ds - \int_0^t S(s,t) \odot \underline{f}(s, x(s))(r)\, ds\right|,\right. \\ &\quad \left.\left|\int_0^t S(s,t) \odot \overline{f}(s, x_{n-1}(s))(r)\, ds - \int_0^t S(s,t) \odot \overline{f}(s, x(s))(r)\, ds\right|\right\} \\ &\leq \sup_{r\in[0,1]} \max\left\{\int_0^t |S(s,t)| \left|F(s, \underline{x}_{n-1}(s)(r), \overline{x}_{n-1}(s)(r)) - F(s, \underline{x}(s)(r), \overline{x}(s)(r))\right| ds,\right. \\ &\quad \left.\int_0^t |S(s,t)| \left|G(s, \underline{x}_{n-1}(s)(r), \overline{x}_{n-1}(s)(r)) - G(s, \underline{x}(s)(r), \overline{x}(s)(r))\right| ds\right\} \\ &\leq Ml \int_0^t \sup_{r\in[0,1]} \max\left\{\left|\underline{x}_{n-1}(s)(r) - \underline{x}(s)(r)\right|, \left|\overline{x}_{n-1}(s)(r) - \overline{x}(s)(r)\right|\right\} ds \leq lMTd_C(x_{n-1}, x) < \varepsilon. \end{aligned} \tag{48}$$

Hence, $x(t) = \lim_{n\to\infty} x_n(t) = w(t) = S(0,t) \odot x_0 \oplus \int_0^t S(s,t) \odot f(s,x(s))ds$, implying that system (18) has a mild fuzzy solution. □

4. Fuzzy Delay System

In this section, we investigate a fuzzy system with delay:

$$\begin{aligned} x'(t) &= a(t) \odot x(t) \oplus f(t, x(t), x_t), \quad 0 \le t \le T, \\ x(t) &= \varphi(t), \quad -l \le t \le 0, \end{aligned} \tag{49}$$

where x is a fuzzy state function of variable t and $x_t = x(t+\theta)$, for $-l \le \theta \le 0$, is the state that is time-delayed. We may consider x_t as a state in the past, before time t. Here, φ is a given fuzzy function of past state, before or at $t = 0$. In this system, we assume that the fuzzy input function $f(t, x, x_t)$ depends on t, x, and x_t, and the scalar function $a : [0, T] \to \mathbb{R}$ is continuous. The fuzzy derivative of x with respect to t is denoted by x'. All functions are defined using the following notation.

The fuzzy functions x and φ are denoted by $[\underline{x}, \overline{x}]$ and $[\underline{\varphi}, \overline{\varphi}]$, respectively, with their r-cuts denoted by

$$\begin{aligned} [x(t)]_r &= [\underline{x}(t)(r), \overline{x}(t)(r)], \\ [\varphi(t)]_r &= [\underline{\varphi}(t)(r), \overline{\varphi}(t)(r)] \quad \forall 0 < r \le 1, \end{aligned} \tag{50}$$

respectively.

The fuzzy function $f(t, x(t), x_t)$ is denoted by $[\underline{f}(t, x, x_t), \overline{f}(t, x, x_t)]$ with

$$\begin{aligned} &\underline{f}(t, x, x_t) \\ &\quad = \min\{f(t,u,v) \mid u \in [x(t)]_r,\ v \in [x_t]_r\}, \\ &\overline{f}(t, x, x_t) \\ &\quad = \max\{f(t,u,v) \mid u \in [x(t)]_r,\ v \in [x_t]_r\}. \end{aligned} \tag{51}$$

The r-cut of $f(t, x(t), x_t)$ is denoted by $[f(t, x(t), x_t)]_r = [\underline{f}(t, x(t), x_t)(r), \overline{f}(t, x(t), x_t)(r)]$, for all $0 < r \le 1$.

Let $S(s,t) = e^{\int_s^t a(\tau)d\tau}$ and assume that x is fuzzy differentiable that satisfies the conditions of system (49). Consider the fuzzy derivative of $S(s,t) \odot x(s)$ for all $s \in [0,t]$.

By Theorem 17, we have

$$\begin{aligned} &\frac{d}{ds}(S(s,t) \odot x(s)) \\ &\quad = S(s,t) \odot x'(s) \oplus \frac{dS(s,t)}{ds} \odot x(s) \\ &\quad = S(s,t) \odot [a(s) \odot x(s) \oplus f(s, x(s), x_s)] \\ &\qquad \ominus S(s,t)\, a(s) \odot x(s) \\ &\quad = S(s,t)\, a(s) \odot x(s) \oplus S(s,t) \odot f(s, x(s), x_s) \\ &\qquad \ominus S(s,t)\, a(s) \odot x(s) = S(s,t) \odot f(s, x(s), x_s). \end{aligned} \tag{52}$$

By Theorem 19 and the initial value $x(0) = \varphi(0) = \varphi_0$, we get

$$x(s) = S(0,t) \odot \varphi_0 \oplus \int_0^t S(s,t) \odot f(s, x(s), x_s)\, ds \tag{53}$$

as a fuzzy integral equation satisfying system (49).

Definition 40. Let $x \in C([-l,T], \mathbb{R}_F)$. x is called a *mild fuzzy solution* to system (49), if x satisfies the fuzzy integral equation

$$x(t) = \begin{cases} S(0,t) \odot \varphi_0 \oplus \int_0^t S(s,t) \odot f(s, x(s), x_s)\, ds, & 0 \le t \le T \\ \varphi(t), & -l \le t \le 0. \end{cases} \tag{54}$$

Next, we prove the existence of a mild fuzzy solution to system (49) under the following assumption.

Assumption A. Let $M = \sup_{s,t\in[0,T]} |S(s,t)|$ and

$$\begin{aligned} B([0,T], \mathbb{R}_F) = \{&x_{(\cdot)} : [0,T] \longrightarrow \mathbb{R}_F \mid x \\ &\in C([-l,T], \mathbb{R}_F),\ x_t = x(t+\theta),\ -r \le \theta \le 0\}. \end{aligned} \tag{55}$$

Assume that f satisfies the following conditions:

(A-1) There is a constant $k > 0$ such that

$$\begin{aligned} &d_H(f(t, x(t), y_t), \overline{0}) \\ &\quad \le k[1 + d_H(x(t), \overline{0}) + d_H(y_t, \overline{0})] \end{aligned} \tag{56}$$

for all $t \in [0,T]$, $x \in C([-l,T], \mathbb{R}_F)$, $y_{(\cdot)} \in B([0,T], \mathbb{R}_F)$.

(A-2) There is a constant $L > 0$ such that

$$\begin{aligned} &d_H(f(t, x_1(t), y_1(t)), f(s, x_2(s), y_2(s))) \\ &\quad \le L[|t-s| + d_H(x_1(t), x_2(t)) + d_H(y_1(t), y_2(t))] \end{aligned} \tag{57}$$

for all $s, t \in [0,T]$, $x_1, x_2 \in C([-l,T], \mathbb{R}_F)$, $y_{1(\cdot)}, y_{2(\cdot)} \in B([-l,T], \mathbb{R}_F)$.

Definition 41. If there exists $\tau_0 > 0$ such that $x \in C([-l, \tau_0], \mathbb{R}_F)$ satisfies the fuzzy integral equation

$$x(t) = \begin{cases} S(0,t) \odot \varphi_0 \oplus \int_0^t S(s,t) \odot f(s, x(s), x_s)\, ds, & 0 \le t \le \tau_0 \\ \varphi(t), & -l \le t \le 0, \end{cases} \tag{58}$$

one says that system (49) is *mildly fuzzily solvable* in $[0, \tau_0]$ and x is called a mild fuzzy solution in $[0, \tau_0]$.

For each $\tau > 0$. Let us denote a weighted metric space $C([-l,\tau], \mathbb{R}_F)$ by C^τ. Its metric is defined by

$$d_C(u,v) = \sup_{t\in[0,\tau]} e^{-\lambda t} d_H(u(t), v(t)), \tag{59}$$

for some given $\lambda \geq 0$. The metric space $(C([-l,\tau],\mathbb{R}_F),d_C)$ is a complete metric space.

For each $\gamma > 0$, define $\Omega(\gamma,\tau)$ to be

$$\Omega(\gamma,\tau) = \Big\{y \in C^\tau \mid \max_{0\leq t\leq\tau} d_H(y(t),\varphi_0) \leq \gamma,\ y(t) = \varphi(t),\ -l \leq t \leq 0\Big\}. \tag{60}$$

Then, $\Omega(\gamma,\tau)$ is convex and closed.

Let $\tau > 0$. We define a mapping $Q : \Omega(\gamma,\tau) \to C^\tau$ to be

$$Qy(t) = \begin{cases} S(0,t)\odot\varphi_0 \oplus \int_0^t S(s,t)\odot f(s,y(s),y_s)\,ds, & 0 \leq t \leq \tau \\ \varphi(t), & -l \leq t \leq 0, \end{cases} \tag{61}$$

for all $y \in \Omega(\gamma,\tau)$. Then, Q is a bounded mapping. By assumption (A-1), there is a constant $k > 0$ such that

$$d_H(f(s,y(s),y_s),\bar{0}) \leq k\left[1 + d_H(y(s),\bar{0}) + d_H(y_s,\bar{0})\right]. \tag{62}$$

Since $y \in C([-l,\tau],\mathbb{R}_F)$, there is $N > 0$ such that $d_H(y(s),\bar{0})+d_H(y_s,\bar{0}) \leq N$ for all $s \in [-l,\tau]$. Hence, for each $t \in [0,\tau]$, we have

$$\begin{aligned} d_H(Qy(t),\bar{0}) &= d_H\Big(S(0,t)\odot\varphi_0 \oplus \int_0^t S(s,t)\odot f(s,y(s),y_s)\,ds,\bar{0}\Big) \leq M d_H(\varphi_0,\bar{0}) \\ &\quad + Mk\Big(\int_0^t 1 + d_H(y(s),\bar{0}) + d_H(y_s,\bar{0})\,ds\Big) \\ &\leq M\left[d_H(\varphi_0,\bar{0}) + k(1+N)\tau\right] < \infty. \end{aligned} \tag{63}$$

Lemma 42. *The mapping $Q : \Omega(\gamma,\tau) \to C^\tau$ is well-defined and there is $\tau_0 > 0$ such that $Q(\Omega(\gamma,\tau_0)) \subseteq \Omega(\gamma,\tau_0)$.*

Proof. Let $\gamma,\tau > 0$, $\{y_n\}$ be a sequence in $\Omega(\gamma,\tau)$, and let $y \in \Omega(\gamma,\tau)$ be such that $y_n \to y$.

By condition (A-2), there is $L > 0$ such that

$$\begin{aligned} &d_H(f(s,y_n(s),(y_n)_s), f(s,y(s),y_s)) \\ &\quad \leq L\left[d_H(y_n(s),y(s)) + d_H((y_n)_s,y_s)\right] \quad \forall s \in [0,\tau]. \end{aligned} \tag{64}$$

Given any $\epsilon > 0$, since $y_n \to y$, there is $n_0 \in \mathbb{N}$ such that $d_C(y_n,y) < \varepsilon/2ML$ for all $n \geq n_0$. Therefore, for $t \in [0,\tau]$, we obtain

$$\begin{aligned} d_H(Qy_n(t),Qy(t)) &= d_H\Big(\int_0^t S(s,t)\odot f(s,y_n(s),(y_n)_s)\,ds, \int_0^t S(s,t)\odot f(s,y(s),y_s)\,ds\Big) \\ &\leq \int_0^t |S(s,t)|\, d_H(f(s,y_n(s),(y_n)_s), f(s,y(s),y_s))\,ds \\ &\leq ML\int_0^t d_H(y_n(s),y(s)) + d_H((y_n)_s,y_s)\,ds \leq 2ML\tau d_C(y_n,y) < \varepsilon. \end{aligned} \tag{65}$$

This implies that the mapping $Q : \Omega(\gamma,\tau) \to C^\tau$ is well-defined.

Next, we show that there exists $\tau_0 > 0$ such that $Q(\Omega(\gamma,\tau_0)) \subseteq \Omega(\gamma,\tau_0)$.

By conditions (A-1) and (A-2), there exist L_1 and $L_2 > 0$ such that

$$\begin{aligned} &d_H(f(0,y(0),y_0),\bar{0}) \leq L_1(1 + d_C(\varphi,\bar{0})), \\ &d_H(f(s,y(s),y_s), f(0,y(0),y_0)) \\ &\quad \leq L_2 \max_{v\in[0,\tau]} d_H(y(v),\varphi_0) \quad \forall s \in [0,\tau]. \end{aligned} \tag{66}$$

Hence,

$$\begin{aligned} d_H(Qy(t),\varphi_0) &= d_H\Big(S(0,t)\odot\varphi_0 \oplus \int_0^t S(s,t) \odot f(s,y_n(s),(y_n)_s)\,ds,\varphi_0\Big) \leq d_H(S(0,t)\odot\varphi_0, \\ &\quad \varphi_0) + d_H\Big(\int_0^t S(s,t)\odot f(0,y(0),y_0),\bar{0}\Big) \\ &\quad + d_H\Big(\int_0^t S(s,t)\odot f(s,y(s),y_s)\,ds, \int_0^t S(s,t) \odot f(0,y(0),y_0)\,ds\Big) \end{aligned}$$

$$\leq \sup_{r\in[0,1]} \max \left\{ \left| S(0,t)\underline{\varphi}_0(r) - \underline{\varphi}_0(r) \right|, \right.$$
$$\left. \left| S(0,t)\overline{\varphi}_0(r) - \overline{\varphi}_0(r) \right| \right\} + ML_1 \left(\int_0^t 1 + d_C\left(\varphi_0, \right.\right.$$
$$\left.\left. \overline{0}\right) ds \right) + M \int_0^t d_H\left(f\left(s, y(s), y_s\right), f\left(0, y(0),\right.\right.$$
$$\left.\left. y_0\right)\right) ds \leq |S(0,\tau) - 1| \, d_C\left(\varphi_0, \overline{0}\right) + ML_1\tau\left(1\right.$$
$$\left. + d_C\left(\varphi_0, \overline{0}\right)\right) + ML_2\tau \max_{v\in[0,\tau]} d_H\left(y(v), \varphi_0\right) \equiv q(\tau). \tag{67}$$

Since $q(\tau) \to 0$ as $\tau \to 0^+$, there exists $\tau_0 > 0$ such that $0 < q(\tau_0) < 1$.

Therefore, we can conclude that there is $\tau_0 > 0$ such that $Q(\Omega(\gamma, \tau_0)) \subseteq \Omega(\gamma, \tau_0)$. □

Lemma 43. *Assume that conditions (A-1) and (A-2) hold. Then, there exists $\tau_0 > 0$ such that system (49) is mildly fuzzily solvable in $[0, \tau_0]$ and its mild fuzzy solution is unique.*

Proof. Let $\tau > 0$. Define $\Omega(1,\tau) = \{y \in C^\tau \mid \max_{0\leq t\leq \tau} d_H(y(t), \varphi_0) \leq 1,\ y(t) = \varphi(t),\ t \in [-l, 0]\}$.

Then, $\Omega(1,\tau)$ is convex and closed. Define a mapping $Q : \Omega(1,\tau) \to C^\tau$ as that in (61).

By Lemma 42, the mapping Q is well-defined on $\Omega(1,\tau)$ and there is τ_0 such that $Q : \Omega(1,\tau_0) \to \Omega(1,\tau_0)$. Let $\tau_1 \leq \tau_0$. We show that Q is a strong contraction mapping on $\Omega(1,\tau_2)$ for some $\tau_2 > 0$. Let $y_1, y_2 \in \Omega(1,\tau_1)$. By condition (A-2), there is $a(\tau_1) > 0$ such that

$$\begin{aligned} & d_H\left(f\left(s, y_1(s), (y_1)_s\right), f\left(s, y_2(s), (y_2)_s\right)\right) \\ & \quad \leq a(\tau_1)\left[d_H\left(y_1(s), y_2(s)\right) + d_H\left((y_1)_s, (y_2)_s\right)\right] \\ & \quad \leq 2a(\tau_1)\, d_C\left(y_1, y_2\right) \quad \forall s \in [0, \tau_1]. \end{aligned} \tag{68}$$

Hence, $d_H(Qy_1(t), Qy_2(t)) \leq 2Ma(\tau_1)\tau_1 d_C(y_1, y_2) = p(\tau_1) d_C(y_1, y_2)$ for all $t \in [0, \tau_1]$ with $p(\tau_1) = 2Ma(\tau_1)\tau_1$. Since $p(\tau_1) = 2Ma(\tau_1)\tau_1 \to 0$ as $\tau_1 \to 0^+$, there is $\tau_2 > 0$ such that $p(\tau_2) < 1$. This implies that Q is a strong contraction mapping on $\Omega(1,\tau_2)$ for some $\tau_2 > 0$. By the contraction mapping principle, there exists a unique $x \in \Omega(1,\tau_2)$ such that $Qx = x$; that is,

$$x(t) = \begin{cases} S(0,t) \odot \varphi_0 \oplus \int_0^t S(s,t) \odot f\left(s, x(s), x_s\right) ds, & 0 \leq t \leq \tau_2 \\ \varphi(t), & -l \leq t \leq 0. \end{cases} \tag{69}$$

□

Theorem 44. *Assume that conditions (A-1) and (A-2) hold. Then, system (49) is mildly fuzzily solvable in $[0, T]$.*

Proof. Let $[-r, \tau_{\max})$ be the biggest interval where system (49) is mildly fuzzily solvable.

We show that $\tau_{\max} > T$ by contradiction. Suppose that $\tau_{\max} \leq T$. Then, $\lim_{t\to\tau_{\max}} d_H(x(t), \overline{0}) = \infty$ because if $\lim_{t\to\tau_{\max}} d_H(x(t), \overline{0}) < \infty$, then there exists a sequence $\{t_n\}$ and $\kappa > 0$ such that $t \to \tau_{\max}$ and $d_H(x(t), \overline{0}) \leq \kappa$ for all n. So x can extend beyond $[0, t_n + \delta]$ for some $\delta = \delta(t_n) > 0$. This implies that system (49) is mildly fuzzily solvable in $[-r, \tau_{\max} + \delta)$, which contradicts the definition of $[-r, \tau_{\max})$. However, the case that $\lim_{t\to\tau_{\max}} d_H(x(t), \overline{0}) = \infty$ also contradicts the a priori boundary property of solution x. Hence, $\tau_{\max} > T$; that is, system (49) is mildly fuzzily solvable on $[0, T]$. □

5. Fuzzy Control Problem

In this section, we study a fuzzy differential equation system with time delay and regulation:

$$\begin{aligned} x'(t) &= a(t) \odot x(t) \oplus f\left(t, x(t), x_t, u(t)\right), \\ & \qquad\qquad 0 \leq t \leq T, \\ x(t) &= \varphi(t), \quad -l \leq t \leq 0. \end{aligned} \tag{70}$$

In the above equations, x is a fuzzy function of time variable t; x_t (equaling $x(t+\theta)$ for some $-l \leq \theta \leq 0$) is a state of time delay (consider x_t as a state in the past before time t); φ is a fuzzy history function before start time $t = 0$; u is a fuzzy controller function of time variable t; and $a : [0, T] \to \mathbb{R}$ is a given continuous function. In this study, we assume that the input function $f(t, x, x_t, u)$ is a fuzzy function of time variable t, state variable x, delay variable x_t, and controller variable $u \in U_{\text{ad}}$ (U_{ad} is an admissible control set). The fuzzy derivative of x is denoted by x'.

From now on, we denote x, φ, and u by $[\underline{x}, \overline{x}]$, $[\underline{\varphi}, \overline{\varphi}]$, and $[\underline{u}, \overline{u}]$, respectively.

The fuzzy function $f(t, x, x_t, u)$ is denoted by $[\underline{f}(t, x, x_t, u), \overline{f}(t, x, x_t, u)]$ with

$$\begin{aligned} \underline{f}(t, x, x_t, u) &= \min\left\{f\left(t, x_1, x_2, x_3\right) \mid x_1 \right. \\ & \qquad \left. \in [x(t)]_r,\ x_2 \in [x_t]_r,\ x_3 \in [u]_r\right\}, \\ \overline{f}(t, x, x_t, u) &= \max\left\{f\left(t, x_1, x_2, x_3\right) \mid x_1 \right. \\ & \qquad \left. \in [x(t)]_r,\ x_2 \in [x_t]_r,\ x_3 \in [u]_r\right\}. \end{aligned} \tag{71}$$

In this research, we investigate equation system (70) under the following assumptions.

Assumption B. (B-1) Let $B([0,T], \mathbb{R}_F) = \{x_{(\cdot)} : [0,T] \to \mathbb{R}_F \mid x \in C([-l,T], \mathbb{R}_F),\ x_t = x(t+\theta),\ -r \leq \theta \leq 0\}$ and $U_{\text{ad}} = C([0,T], \mathbb{R}_F)$ and let f be a fuzzy function; there exists a constant $k > 0$ such that

$$\begin{aligned} & d_H\left(f\left(t, x(t), y_t, z(t)\right), \overline{0}\right) \\ & \quad \leq k\left[1 + d_H\left(x(t), \overline{0}\right) + d_H\left(y_t, \overline{0}\right) + d_H\left(z(t), \overline{0}\right)\right] \end{aligned} \tag{72}$$

for all $t \in [0, T]$, $x \in C([-l, T], \mathbb{R}_F)$, $y_{(\cdot)} \in B([0,T], \mathbb{R}_F)$, and $z \in U_{\text{ad}}$.

(B-2) There exists a constant $L > 0$ such that

$$\begin{aligned} d_H\left(f\left(s, x_1(s), y_{1(s)}, z_1(s)\right), f\left(t, x_2(t), y_{2(t)}, z_2(t)\right)\right) \\ \leq L\left[|s-t| + d_H\left(x_1(s), x_2(t)\right) + d_H\left(y_{1(s)}, y_{2(s)}\right) \right. \\ \left. + d_H\left(z_1(s), z_2(t)\right)\right] \end{aligned} \quad (73)$$

for all $s, t \in [0, T]$, $x_1, x_2 \in C([-l, T], \mathbb{R}_F)$, $y_{1(\cdot)}, y_{2(\cdot)} \in B([0, T], \mathbb{R}_F)$, and $z_1, z_2 \in U_{\mathrm{ad}}$.

Definition 45. Let $x \in C([-l, T], \mathbb{R}_F)$ and $u \in U_{\mathrm{ad}}$. x is called a *mild fuzzy solution* of system (70) with respect to control u in $[-l, T]$, if x satisfies this system of fuzzy integral equation:

$$x(t) = \begin{cases} S(0, t) \odot \varphi_0 \oplus \int_0^t S(s, t) \odot f\left(s, x(s), x_s, u(s)\right) ds, & 0 \leq t \leq T \\ \varphi(t), & -l \leq t \leq 0. \end{cases} \quad (74)$$

Theorem 46. *Assume that Assumption B holds. Then, for each $u \in U_{ad}$, system (70) has a mild fuzzy solution with respect to control u.*

Proof. Let $u \in U_{\mathrm{ad}}$. Define $f_u(t, x(t), x_t)$ to be $f(t, x(t), x_t, u(t))$. By Assumption B and the continuity of u, f_u satisfies Assumption A. Therefore, by Theorem 44, system (70) has a mild fuzzy solution with respect to u. □

Note that for each solution x with respect to a control u, we can denote x by x^u and call the ordered pair (x^u, u) a *pairwise control pair*, sometimes written shortly as (x, u).

Next, we investigate an optimization control problem, problem (B), or *Bolza problem*.

Problem P. Problem P is to find the pairwise control pair $(x^0, u^0) \in C([-l, T], \mathbb{R}_F) \times U_{\mathrm{ad}}$ such that

$$J\left(x^0, u^0\right) \leq J\left(x^u, u\right) \quad \forall u \in U_{\mathrm{ad}}, \quad (75)$$

where $J(x^u, u) = \int_0^T r(t, x^u, x_t^u, u(t)) dt + g(x^u(T))$ is a Bolza cost functional. The multivariable function r is called a *running function* and the function g is called a *terminal function*. For convenience, $J(x^u, u)$ is written as $J(u)$. We prove the existence of a solution to Problem P constrained by system (70) under the following assumptions.

Assumption U. Assume that $U_{\mathrm{ad}} = C([0, T], \mathbb{R}_F)$:

(U-1) The running function $r : [0, T] \times C([-l, T], \mathbb{R}_F) \times B([0, T], \mathbb{R}_F) \times U_{\mathrm{ad}} \to (-\infty, \infty]$ is Borel measurable.

(U-2) The terminal function $g : C([-l, T], \mathbb{R}_F) \to \mathbb{R}$ is nonnegative and continuous.

(U-3) The running function $r(t, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous on $C([-l, T], \mathbb{R}_F) \times B([0, T], \mathbb{R}_F) \times U_{\mathrm{ad}}$ for almost every $t \in [0, T]$.

(U-4) The running function $r(t, x, y_{(\cdot)}, \cdot)$ is convex on $C([0, T], \mathbb{R}_F)$ for all $x \in C([-l, T], \mathbb{R}_F)$, $y_{(\cdot)} \in B([0, T], \mathbb{R}_F)$ and almost every $t \in [0, T]$.

(U-5) There are constants $a, b, c > 0$ and $\lambda \in L([0, T], \mathbb{R})$ such that

$$\begin{aligned} r\left(t, x, y_{(\cdot)}, u\right) \geq \lambda(t) + a d_H\left(x(t), \bar{0}\right) + b d_H\left(y_t, \bar{0}\right) \\ + c d_H\left(u(t), \bar{0}\right) \end{aligned} \quad (76)$$

for all $t \in [0, T]$, $x \in C([-l, T], \mathbb{R}_F)$, $y_{(\cdot)} \in B([0, T], \mathbb{R}_F)$, and $u \in U_{\mathrm{ad}}$.

Theorem 47. *Under Assumptions B and U, Problem P constrained by system (70) has at least one solution; that is, there exists a pairwise control pair $(x^0, u^0) \in C([-l, T], \mathbb{R}_F) \times U_{ad}$ such that $J(x^0, u^0) \leq J(x^u, u)$ for all $(x^u, u) \in C([-l, T], \mathbb{R}_F) \times U_{ad}$.*

Proof. Let $m = \inf\{J(x^u, u) \mid u \in U_{\mathrm{ad}}\}$. If $m = +\infty$, the theorem is already true. Assume that $m < +\infty$. By assumption (U-5), there are $a, b, c > 0$ and $\lambda \in L([0, T], \mathbb{R})$ such that

$$\begin{aligned} r\left(t, x^u, x_t^u, u\right) \geq \lambda(t) + a d_C\left(x^u, \bar{0}\right) + b d_C\left(x_t^u, \bar{0}\right) \\ + c d_C\left(u, \bar{0}\right) \end{aligned} \quad (77)$$

for all $t \in [0, T]$, $x^u, x_t^u \in C([-l, T], \mathbb{R}_F)$, and $u \in U_{\mathrm{ad}}$. Since g is a nonnegative function, we have

$$\begin{aligned} J\left(x^u, u\right) &= \int_0^T r\left(t, x^u, x_t^u, u(t)\right) dt + g\left(x^u(T)\right) \\ &\geq \int_0^T \lambda(t) dt + a \int_0^T d_H\left(x^u(t), \bar{0}\right) dt \\ &\quad + b \int_0^T d_H\left(x_t^u, \bar{0}\right) dt \\ &\quad + c \int_0^T d_H\left(u(t), \bar{0}\right) dt + g\left(x^u(T)\right) \geq -\omega \\ &> -\infty, \quad \text{for some } \omega > 0, \ \forall u \in U_{\mathrm{ad}}. \end{aligned} \quad (78)$$

So $m \geq \omega > -\infty$. By the definition of minimum, there is a sequence of the minimum point, say $\{u_n\}$, of the cost functional J such that $\lim_{n \to \infty} J(x^{u_n}, u_n) = m$ and

$$\begin{aligned} J\left(x^u, u\right) \geq \int_0^T \lambda(t) dt + a \int_0^T d_H\left(x^{u_n}(t), \bar{0}\right) dt \\ + b \int_0^T d_H\left(x_t^{u_n}, \bar{0}\right) dt \\ + c \int_0^T d_H\left(u_n(t), \bar{0}\right) dt + g\left(x^{u_n}(T)\right). \end{aligned} \quad (79)$$

Thus, there is $N_0 > 0$ and $m_1 > 0$ such that $m + m_1 \geq J(x^{u_n}, u_n) \geq c \int_0^T d_H(u_n(t), \bar{0})dt$ for all $n \geq N_0$. This implies that $(m + m_1)/c \geq \int_0^T d_H(u_n(t), \bar{0})dt$. Consequently, $d_C(u_n, \bar{0}) \leq (m + m_1)/Tc$ for all $n \geq N_0$. Hence, $\{u_n\}$ is a bounded sequence in $U_{\text{ad}} \subset L_2([0,T], \mathbb{R}_F)$ with respect to the norm $\|\cdot\|$ defined by $\|x\| = d_C(x, 0)$. Since $L_2([0,T], \mathbb{R}_F)$ is a reflexive Banach space, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \xrightarrow{w} u^0$ for some $u^0 \in U_{\text{ad}}$.

Let $x^0 \in C([0,T], \mathbb{R}_F)$ be a mild fuzzy solution with respect to a control u^0 and let $\{x_{n_k}\}$ be a sequence of mild fuzzy solution corresponding to the sequence of control $\{u_{n_k}\}$; that is,

$$x_{n_k}(t) = \begin{cases} S(0,t) \odot \varphi_0 \oplus \int_0^t S(s,t) \odot f\left(s, x_{n_k}(s), x_{n_{k(s)}}, u_{n_k}(s)\right) ds, & 0 \leq t \leq T \\ \varphi(t), & -l \leq t \leq 0. \end{cases} \tag{80}$$

By assumption (B-2), for all $0 \leq t \leq T$, there is a constant $a > 0$ such that

$$\begin{aligned} d_H\left(x_{n_k}(t), x^0(t)\right) &\leq 2Ma\left[\int_0^t d_H\left(x_{n_k}(s), x^0(s)\right) ds \right. \\ &\quad \left. + \int_0^t d_H\left(u_{n_k}(s), u^0(s)\right) ds\right] \\ &\leq 2Ma\left[\int_0^t d_H\left(x_{n_k}(s), x^0(s)\right) ds \right. \\ &\quad \left. + T d_C\left(u_{n_k}, u^0\right)\right]. \end{aligned} \tag{81}$$

By Gronwall lemma, there is $M_1 > 0$ such that

$$d_C\left(x_{n_k}, x^0\right) \leq M_1 d_C\left(u_{n_k}, u^0\right). \tag{82}$$

Since $u_{n_k} \xrightarrow{w} u^0$, $x_{n_k} \xrightarrow{w} x^0$. By using assumptions (U-2) and (U-3), we obtain

$$\begin{aligned} m &= \lim_{n_k \to \infty} J\left(x_{n_k}^{u_{n_k}}, u_{n_k}\right) \\ &= \lim_{n_k \to \infty} \int_0^T r\left(t, x_{n_k}^{u_{n_k}}, x_{n_k(t)}^{u_{n_k}}, u_{n_k}(t)\right) dt \\ &\quad + g\left(x_{n_k}^{u_{n_k}}(T)\right) \\ &\geq \int_0^T \lim_{n_k \to \infty} r\left(t, x_{n_k}^{u_{n_k}}, x_{n_k(t)}^{u_{n_k}}, u_{n_k}(t)\right) dt \\ &\quad + g\left(\lim_{n_k \to \infty} x_{n_k}^{u_{n_k}}(T)\right) \\ &\geq \int_0^T r\left(t, x^0, x_t^0, u^0(t)\right) dt + g\left(x^0(T)\right) \\ &= J\left(x^0, u^0\right). \end{aligned} \tag{83}$$

Thus, $J(x^0, u^0) = m$; that is, $J(x^0, u^0) \leq J(x^u, u)$ for all $(x^u, u) \in C([-l, T], \mathbb{R}_F) \times U_{\text{ad}}$. □

6. Conclusion

This paper is concerned with proving of the existence and uniqueness of a mild solution to nonlinear fuzzy differential equation constrained by initial value. Then, we already proved the existence of a solution to the system that the initial value constraint was then replaced by delay function constraint. Furthermore, we prove the existence of a solution to optimal control problem of the latter type of equation. Last but not least we should be interested in studying applications and numerical method of these problems. Even though it seems likely that efforts in this direction can be successful, there is no guarantee for that. Therefore, we can only hope for the best and prepare for the worst.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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