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Research Article

A Fractional-Order Epidemic Model for Bovine Babesiosis Disease and Tick Populations

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This paper shows that the epidemic model, previously proposed under ordinary differential equation theory, can be generalized to fractional order on a consistent framework of biological behavior. The domain set for the model in which all variables are restricted is established. Moreover, the existence and stability of equilibrium points are studied. We present the proof that endemic equilibrium point when reproduction number $R_0 > 1$ is locally asymptotically stable. This result is achieved using the linearization theorem for fractional differential equations. The global asymptotic stability of disease-free point, when $R_0 < 1$, is also proven by comparison theory for fractional differential equations. The numeric simulations for different scenarios are carried out and data obtained are in good agreement with theoretical results, showing important insight about the use of the fractional coupled differential equations set to model babesiosis disease and tick populations.

1. Introduction

Bovine babesiosis is transmitted by the sting of ticks and is the most important disease to attack bovine populations in tropical regions. In warm and hot regions there is great economic loss due to bovine death by bovine babesiosis, with decrease of bovine products and by-products. Moreover, the climate conditions in those regions favor the survival and reproduction of ticks and, consequently, bovines have a permanent contact with these vectors [1]. Furthermore, the vertical transmission in bovines and ticks is possible provided that the ovaries of the female ticks are infected by parasites [1]. The behavior dynamics of diseases has been studied for a long time and is an important issue in the real world. The most important model that can be used to interpret the disease characteristic of epidemics is a susceptible-infected-recuperated model (SIR) that was developed by Kermack and McKendrick [2], and various types of diseases are studied by this type of ordinary differential equation system. Aranda et al. [3] introduced the epidemiological model for bovine babesiosis and tick populations disease. In this work the qualitative dynamics behavior is determined by the basic reproduction number, R_0. If the threshold parameter, $R_0 < 1$, is proved by LaSalle-Lyapunov theorem then the solution converges to the disease free equilibrium point. However, if $R_0 > 1$, the convergence is to the endemic equilibrium point by numerical simulations. In recent years, the theory of networks in epidemiological model has been introduced in the literature. The purpose of this modification is to have better understanding and prediction of epidemic patterns and intervention measures. For more details see [4–6].

The notion of fractional calculus was introduced by Leibniz, one of the founders of standard calculus, in a letter written in 1695. In recent decades, fractional differential equations are one of the most important topics in mathematics and have received attention due to the possibility of describing nonlinear systems, thus attracting much attention and increasing interest due to its potential applications in physics, control theory, and engineering (see [7–15]). The advantage of fractional-order differential equation systems is

that they allow greater degrees of freedom and incorporate the memory effect in the model. Due to this fact, they have been introduced in epidemiological modeling systems. In [16], a fractional order for the dynamics of A(H1N1) influenza disease is studied by numerical simulations. Pooseh et al. [17] and Diethelm [18] have introduced fractional dengue models. In this paper the parameters of the equations obtained in the field research do not reproduce well the evolution of the disease in the case of entire order model. However, when we consider the fractional system, with the same parameters obtained in the field, the data are better adjusted which shows an advantage of the fractional system. In [11] the parameter θ is associated with a memory effect. In [19], the authors attribute to θ the memory information of the dengue disease's. In this paper, we consider the fractional-order system associated with the evolution of bovine babesiosis disease and tick populations. We introduce a generalization of the classical model presented by Aranda et al. [3]. The generalization is obtained by changing the ordinary derivative by fractional Caputo derivative. It is easy to see that when $\theta = 1$ we return to the classical model. For the construction of this model by Aranda et al. [3], the compartments of populations and the biological hypothesis are used. This argument is well established in the disease transmission theory. In Aranda et al., theorems well established in the literature for ordinary differential systems are used. To prove our results, it is necessary to use different tools to those used for the integer order. This is due to the fact that the versions of La-Salle invariance theorem used by Aranda et al. are not found in the literature for fractional-order systems. Therefore, we emphasize that the work presents a collaboration in this direction as when using the comparison theory for fractional-order systems to prove the global stability of the equilibrium free point of the disease by introducing a new type of results in the literature. On the other hand, we also have a test on the local asymptotic stability of endemic equilibrium point, a result that is just enunciated in Aranda et al. [3]. We obtain a generalization of all results in [3]. Our simulation shows that the fractional model has great potential to describe the real problem without the need for adjustment of parameters obtained in field research. This is due to a greater flexibility of adjustment obtained with the introduction of the new parameter. This paper is organized in four sections. Introduction is the first section. In Section 2, we mention a few results and notations related to the theory of fractional differential equations; in Section 3, we consider the fractional-order model associated with the dynamics of bovine babesiosis and ticks populations. Qualitative dynamics of the model is determined by the basic reproduction number. We give a detailed analysis for the global asymptotical stability of disease-free equilibrium point and the local asymptotical stability of the endemic equilibrium point. Finally, in Section 4, numerical simulations are presented to verify the main results.

2. Preliminaries

For many years, there have been several definitions that fit the concept of fractional derivatives [10, 20]. In this paper the Riemann-Liouville fractional derivative and Caputo fractional derivative definitions are presented. Firstly, we introduce the definition of Riemann-Liouville fractional integral

$$J^{\theta} f(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s)\, ds, \tag{1}$$

where $\theta > 0$, $f \in L^1(\mathbb{R}^+)$, and $\Gamma(\cdot)$ is the Gamma function.

The Riemann-Liouville derivative is given by

$$\begin{aligned} D_R^{\theta} f(t) &= \frac{d^n}{dt^n}\left[J^{n-\theta} f(t)\right] \\ &= \frac{1}{\Gamma(n-\theta)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\theta-1} f(s)\, ds, \\ &\qquad n-1 \le \theta < n. \end{aligned} \tag{2}$$

The Caputo fractional derivative is given as follows:

$$\begin{aligned} D_C^{\theta} f(t) &= J^{n-\theta}\left[f^{(n)}(t)\right] \\ &= \frac{1}{\Gamma(n-\theta)} \int_0^t (t-s)^{n-\theta-1} f^{(n)}(s)\, ds, \end{aligned} \tag{3}$$

where n is the first integer which is not less than θ.

The Laplace transform of the Caputo fractional derivative is given by

$$\mathscr{L}\left[D_C^{\theta} f(t)\right] = s^{\theta} F(s) - \sum_{k=0}^{n-1} f^{(k)}(0)\, s^{\theta-k-1}. \tag{4}$$

The Mittag-Leffler function is defined by the following infinite power series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{5}$$

The Laplace transform of the functions is

$$\mathscr{L}\left[t^{\beta-1} E_{\alpha,\beta}\left(\pm a t^{\alpha}\right)\right] = \frac{s^{\alpha-\beta}}{s^{\alpha} \mp a}. \tag{6}$$

Let $\alpha, \beta > 0$ and $z \in \mathbb{C}$, and the Mittag-Leffler functions satisfy the equality given by Theorem 4.2 in [10]

$$E_{\alpha,\beta}(z) = z E_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}. \tag{7}$$

Definition 1. A function f is Hölder-continuous if there are nonnegative constants C, ν such that

$$\left\| f(x) - f(y) \right\| \le C \left\| x - y \right\|^{\nu}, \tag{8}$$

for all x, y in the domain of f and ν is the Hölder exponent. We represent the space of Hölder-continuous functions by $C^{0,\nu}$.

We develop a generalized inequality, wherein the underlying comparison system is a vector fractional-order system.

A nonnegative (resp., positive) vector v means that every component of v is nonnegative (resp., positive). We denote a nonnegative (resp., positive) vector by $0 \leq\leq v$ (resp., $0 << v$).

Consider the fractional-order system:

$$D_C^\theta u(t) = f(t,u), \quad u(0) = u_0, \tag{9}$$

where $D_C^\theta u(t) = (D_C^\theta u_1(t), D_C^\theta u_2(t), \ldots, D_C^\theta u_m(t))^T$, $0 < \theta < 1$, $u(t) \in \mathscr{M} \subset \mathbb{R}^m$, $t \in [0,T)$ $(T \leq +\infty)$, $\mathscr{M}$ is an open set, $0 \in \mathscr{M}$, and $f : [0,T) \times \mathscr{M} \to \mathbb{R}^m$ is continuous in t and satisfies the Lipschitz condition:

$$\left\| f(t,u') - f(t,u'') \right\| \leq L \left\| u' - u'' \right\|, \quad t \in [0,T), \tag{10}$$

for all $u', u'' \in \Omega \subset \mathscr{M}$, where $L > 0$ is a Lipschitz constant.

Theorem 2 (see [15]). *Let $u(t)$, $t \in [0,T)$, be the solution of system (9). If there exists a vector function $v = (v_1, v_2, \ldots, v_m)^T : [0,T) \to \mathscr{M}$ such that $v_i \in C^{0,\nu}$, $\theta < \nu < 1$, $i = 1, \ldots, m$ and*

$$D_C^\theta v(t) \leq\leq f(t, v(t)), \quad t \in [0,T). \tag{11}$$

If $v(0) \leq\leq u_0$, $u_0 \in \mathscr{M}$, then $v(t) \leq\leq u(t)$, $t \in [0,T)$.

Now, we will introduce a Theorem of stability for linear systems of fractional order. Let $A \in M_{m\times m}(\mathbb{R})$, and we define the linear system homogeneous equation:

$$D_C^\theta x(t) = Ax(t), \quad x(0) = x_0. \tag{12}$$

Definition 3. We say that linear system (12) is stable if for all $\epsilon > 0$, $\delta > 0$ exists such that $\|x_0\| < \delta$; then $\|x(t)\| < \epsilon$, for all $t \geq 0$; linear system (12) is asymptotically stable if $\lim_{t\to\infty} x(t) = 0$.

The next result establishes the stability of the fractional linear system similarly to the theory of ordinary differential equation.

Theorem 4 (see [21]). *System (12) origin is asymptotically stable if and only if $|\arg(\lambda_i)| > \theta\pi/2$ is satisfied for all eigenvalues of the matrix A. Moreover, this system is stable if and only if $|\arg(\lambda_i)| \geq \theta\pi/2$ is satisfied for all eigenvalues of the matrix A, and the eigenvalues satisfying $|\arg(\lambda_i)| = \theta\pi/2$ have geometric multiplicity equal to one.*

Let $f : \mathscr{M} \to \mathbb{R}^m$, $\mathscr{M} \in \mathbb{R}^m$; we consider the following system of fractional order:

$$D_C^\theta x(t) = f(x), \quad x(0) = x_0. \tag{13}$$

Definition 5. We say that E is an equilibrium point for (13), if and only if $f(E) = 0$.

Remark 6. When $\theta \in (0,1)$, the fractional system $D_C^\theta x(t) = f(x)$ has the same equilibrium points as the system $x'(t) = f(x)$.

Definition 7. The equilibrium point E of autonomous system (13) is said to be stable if for all $\epsilon > 0$, $\delta > 0$ exists such that if $\|x_0 - E\| < \delta$, then $\|x(t) - E\| < \epsilon$, $t \geq 0$; the equilibrium point E of autonomous system (13) is said to be asymptotically stable if $\lim_{t\to\infty} x(t) = E$.

Theorem 8 (see [12]). *The equilibrium points of system (13) are locally asymptotically stable if all eigenvalues λ_i of Jacobian matrix J, calculated in the equilibrium points, satisfy $|\arg(\lambda_i)| > \theta\pi/2$.*

3. Mathematical Model

In this section, we introduce the fractional model for the babesiosis disease in bovine and tick populations. We use the assumptions in Aranda et al. [3] and introduce the following hypotheses.

(i) The total of bovine population $\overline{N}_B(t)$ is divided into three subpopulations:

  (a) bovines that may become infected (susceptible $\overline{S}_B(t)$);
  (b) bovines infected by Babesia parasite (infected $\overline{I}_B(t)$);
  (c) bovines that have been treated for the babesiosis (controlled $\overline{C}_B(t)$).

(ii) The parameter μ_B is the birth rate of bovine. The birth rate μ_B is assumed to be equal to the natural death.

(iii) The total population of ticks $\overline{N}_T(t)$ is divided into two subpopulations:

  (a) ticks which may become infected by the disease $\overline{S}_T(t)$;
  (b) ticks infected by the Babesia parasite $\overline{I}_T(t)$.

(iv) The parameter μ_T is the birth rate of the ticks and it is assumed to be equal to the death rate.

(v) A susceptible bovine can transit to the infected subpopulation $\overline{I}_B(t)$ because of an effective transmission due to a sting of an infected tick at a rate β_B.

(vi) A susceptible tick can be infected if there exists an effective transmission when it stings an infected bovine, at rate β_T.

(vii) We assumed a hundred percent vertical transmission in the bovine populations μ_B. In the tick populations it occurs with probability $1 - p$, where p is the probability that a susceptible tick was born from an infected one.

(viii) A fraction λ_B of the infected bovine is controlled, that is, treated against Babesia parasite.

(ix) A fraction α_B of the controlled bovine may return to the susceptible state.

(x) Homogeneous mixing is assumed; that is, all susceptible bovines have the same probability to be infected and all susceptible ticks have the same probability to be infected.

Under the above assumptions, the transmission dynamics of babesiosis disease to bovine and tick population can be modeled by the following system nonlinear ordinary differential equations [3]:

$$\begin{aligned}
\overline{S}_B'(t) &= \mu_B\left(\overline{S}_B(t)+\overline{C}_B(t)\right)+\alpha_B\overline{C}_B(t)-\mu_B\overline{S}_B(t)\\
&\quad-\beta_B\overline{S}_B(t)\frac{\overline{I}_T(t)}{\overline{N}_T(t)},\\
\overline{I}_B'(t) &= \mu_B\overline{I}_B(t)+\beta_B\overline{S}_B(t)\frac{\overline{I}_T(t)}{\overline{N}_T(t)}-\mu_B\overline{I}_B(t)\\
&\quad-\lambda_B\overline{I}_B(t),\\
\overline{C}_B'(t) &= \lambda_B\overline{I}_B(t)-\left[\mu_B+\alpha_B\right]\overline{C}_B(t),\\
\overline{S}_T'(t) &= \mu_T\left(\overline{S}_T(t)+p\overline{I}_T(t)\right)-\beta_T\overline{S}_T(t)\frac{\overline{I}_B(t)}{\overline{N}_B(t)}\\
&\quad-\mu_T\overline{S}_T(t),\\
\overline{I}_T'(t) &= \beta_T\overline{S}_T\frac{\overline{I}_B(t)}{\overline{N}_B(t)}+(1-p)\mu_T\overline{I}_T(t)-\mu_T\overline{I}_T(t).
\end{aligned} \tag{14}$$

In recent years, a considerable interest in the fractional calculus has been shown, which allows us to consider integration and differentiation of any order. To a large extent this is due to the applications of the fractional calculus to problems in different areas of research. The advantage of fractional-order differential equation systems is that they allow greater degrees of freedom and incorporate memory effect in the model. Now we describe the new system of fractional differential equations to model the babesiosis disease in bovine and tick populations, and in this system, $\theta \in (0,1)$:

$$\begin{aligned}
D_C^\theta\overline{S}_B(t) &= \mu_B\left(\overline{S}_B(t)+\overline{C}_B(t)\right)+\alpha_B\overline{C}_B(t)-\mu_B\overline{S}_B(t)\\
&\quad-\beta_B\overline{S}_B(t)\frac{\overline{I}_T(t)}{\overline{N}_T(t)},\\
D_C^\theta\overline{I}_B(t) &= \mu_B\overline{I}_B(t)+\beta_B\overline{S}_B(t)\frac{\overline{I}_T(t)}{\overline{N}_T(t)}-\mu_B\overline{I}_B(t)\\
&\quad-\lambda_B\overline{I}_B(t),\\
D_C^\theta\overline{C}_B(t) &= \lambda_B\overline{I}_B(t)-\left[\mu_B+\alpha_B\right]\overline{C}_B(t),\\
D_C^\theta\overline{S}_T(t) &= \mu_T\left(\overline{S}_T(t)+p\overline{I}_T(t)\right)-\beta_T\overline{S}_T(t)\frac{\overline{I}_B(t)}{\overline{N}_B(t)}\\
&\quad-\mu_T\overline{S}_T(t),\\
D_C^\theta\overline{I}_T(t) &= \beta_T\overline{S}_T\frac{\overline{I}_B(t)}{\overline{N}_B(t)}+(1-p)\mu_T\overline{I}_T(t)\\
&\quad-\mu_T\overline{I}_T(t).
\end{aligned} \tag{15}$$

Simplifying the system (15) and using the bovine populations constant equal $\overline{N}_B$ and tick populations is $\overline{N}_T$ and introducing the proportions

$$\begin{aligned}
S_B(t) &= \frac{\overline{S}_B(t)}{\overline{N}_B(t)},\\
I_B(t) &= \frac{\overline{I}_B(t)}{\overline{N}_B(t)},\\
C_B(t) &= \frac{\overline{C}_B(t)}{\overline{N}_B(t)},\\
S_T(t) &= \frac{\overline{S}_T(t)}{\overline{N}_T(t)},\\
I_T(t) &= \frac{\overline{I}_T(t)}{\overline{N}_T(t)},
\end{aligned} \tag{16}$$

we obtain the following fractional system that describes the dynamics of the proportion of bovines in each class:

$$\begin{aligned}
D_C^\theta S_B(t) &= \left(\mu_B+\alpha_B\right)\left(1-S_B(t)-I_B(t)\right)\\
&\quad-\beta_B S_B(t)I_T(t),\\
D_C^\theta I_B(t) &= \beta_B S_B(t)I_T(t)-\lambda_B I_B(t),\\
D_C^\theta I_T(t) &= \beta_T\left(1-I_T(t)\right)I_B(t)-\mu_T pI_T(t),
\end{aligned} \tag{17}$$

defined in the region $\Omega = \{(S_B, I_B, I_T) : 0 \le S_B + I_B \le 1,\ 0 \le I_T \le 1\}$. Next, we show all variables of the babesiosis model living in Ω for all time $t \ge 0$. To establish our first result we introduce the following lemma.

Lemma 9 (see [22]). *Let the function $f \in C[t_0, t_1]$ and its fractional derivative $D_C^\theta f(t) \in C(t_0, t_1]$ for $0 \le \theta < 1$, and $t_0, t_1 \in \mathbb{R}$; then one has*

$$f(t) = f(t_0) + \frac{1}{\Gamma(\alpha)} D_C^\theta f(\tau)\left(t-t_0\right)^\alpha, \tag{18}$$

for all $t \in (t_0, t_1]$, where $t_0 \le \tau < t$.

Thus, considering the interval $[0, t_1]$ for any $t_1 > 0$, this theorem implies that the function $f : [0, t_1] \to \mathbb{R}^+$ is nonincreasing on $(0, t_1)$ if $D_C^\theta f(t) \le 0$ for all $t \in (0, t_0)$ and nondecreasing on $[0, t_0]$ if $D_C^\theta f(t) \ge 0$ for all $t \in (0, t_0)$.

Proposition 10. *The region $\Omega = \{(S_B, I_B, I_T) : 0 \le S_B + I_B \le 1,\ 0 \le I_T \le 1\}$ is a positive invariant set for system (17).*

Proof. By Theorem 3.1 and Remark 3.2 in [23] we obtain the global existence and uniqueness of the solutions of (17).

We denote by $\Omega_+ = \{(S_B, I_B, I_T) : S_B \geq 0,\ I_B \geq 0 \text{ and } I_T \geq 0\}$. If $(S_B(0), I_B(0), I_T(0)) \in S_B$-axis $= \{(S_B, 0, 0) : S_B \geq 0\}$ (with the same form we define I_B-axis and I_T-axis). The vector field from (17) confined in S_B-axis assumes the form $F(S_B, I_B, I_T) = ((\mu_B+\alpha_B)-(\mu_B+\alpha_B)S_B(t), 0, 0)$, by the Laplace transform properties (6), and we obtain the solution

$$\begin{aligned}&\left(S_B(t), I_B(t), I_T(t)\right)\\&= \left(t^\theta E_{\theta,\theta+1}\left(-\left(\mu_B+\alpha_B\right)t^\theta\right)\left(\mu_B+\alpha_B\right)\right.\\&\left.+ E_{\theta,1}\left(-\left(\mu_B+\alpha_B\right)t^\theta\right)S_B(0), 0, 0\right) \in S_B\text{-axis}.\end{aligned} \tag{19}$$

By the same argument, if $(S_B(0), I_B(0), I_T(0)) \in I_B$-axis we obtain

$$\begin{aligned}\left(S_B(t), I_B(t), I_T(t)\right) &= \left(0, E_{\theta,1}\left(-\lambda_B t^\theta\right)I_B(0), 0\right)\\&\in I_B\text{-axis}\end{aligned} \tag{20}$$

and if $(S_B(0), I_B(0), I_T(0)) \in I_T$-axis, we have

$$\begin{aligned}\left(S_B(t), I_B(t), I_T(t)\right) &= \left(0, 0, E_{\theta,1}\left(-\mu_T p t^\theta I_T(0)\right)\right)\\&\in I_T\text{-axis}.\end{aligned} \tag{21}$$

This proves that axes S_B, I_B, and I_T are solutions and positive invariants sets.

Now, we will prove that Ω_+ is a positive invariant set. By way of contradiction, suppose there exists a solution (S_B, I_B, I_T) such that $(S_B(0), I_B(0), I_T(0)) \in \Omega_+$ and the solution $(S_B(t), I_B(t), I_T(t))$ to escape of Ω_+. From the previous argument and by the unicity of solutions $(S_B(t), I_B(t), I_T(t))$ do not cross the axis. From the previous conclusion we have three possibilities.

(i) If the solution $(S_B(t), I_B(t), I_T(t))$ escapes by the plane $S_B = 0$, then there exists t_0 such that $S_B(t_0) = 0$, $I_B(t_0) > 0$ and $I_T(t_0) > 0$ and for all $t > t_0$ sufficiently near t_0 we have $S_B(t) < 0$. On the other hand, $D_C^\theta S_B(t)|_{t=t_0} = (\mu_B+\alpha_B)(1-I_B(t_0)) > (\mu_B+\alpha_B) > 0$. From Lemma 9, we obtain $S_B(t) \geq S_B(t_0) \geq 0$ for all t sufficiently near t_0, and this is absurd.

(ii) If the solution $(S_B(t), I_B(t), I_T(t))$ escape by $I_B = 0$, then there exists t_0 such that $S_B(t_0) > 0$, $I_B(t_0) = 0$, and $I_T(t_0) > 0$ and for all $t > t_0$ sufficiently near t_0 we have $I_B(t) < 0$. Again, $D_C^\theta I_B(t)|_{t=t_0} = \beta_B S_B(t_0) I_T(t_0) > 0$. From Lemma 9, we obtain $I_B(t) \geq I_B(t_0) \geq 0$ for all t sufficiently near t_0, and this is a contradiction.

(iii) If the solution $(S_B(t), I_B(t), I_T(t))$ escape by $I_T = 0$, then there exists t_0 such that $S_B(t_0) > 0$, $I_B(t_0) > 0$ and $I_T(t_0) = 0$ and for all $t > t_0$ sufficiently near t_0 we have $I_T(t) < 0$. We obtain $D_C^\theta I_T(t)|_{t=t_0} = \beta_T I_B(t_0) > 0$ and by Lemma 9, we have $I_T(t) \geq I_B(t_0) \geq 0$ for all t sufficiently near t_0, and this is false.

Therefore, we obtain $S_B(t) \geq 0$, $I_B(t) \geq 0$ and $I_T(t) \geq 0$, for all $t \geq 0$.

If $0 \leq S_B(0) + I_B(0) \leq 1$, from the two first equations of system (17), we get

$$\begin{aligned}D_C^\theta\left(S_B(t) + I_B(t)\right) &= \left(\mu_B+\alpha_B\right)\\&\quad-\left(\mu_B+\alpha_B\right)\left(S_B(t)+I_B(t)\right)\\&\quad-\lambda_B I_B(t)\\&\leq \left(\mu_B+\alpha_B\right)\\&\quad-\left(\mu_B+\alpha_B\right)\left(S_B(t)+I_B(t)\right).\end{aligned} \tag{22}$$

Applying the Laplace transform in the previous inequality, we have

$$\begin{aligned}&\lambda^\theta \mathscr{L}\left(S_B(t)+I_B(t)\right) - \lambda^{\theta-1}\left(S_B(0)+I_B(0)\right)\\&\quad\leq \left(\mu_B+\alpha_B\right)\frac{1}{\lambda} - \left(\mu_B+\alpha_B\right)\mathscr{L}\left(S_B(t)+I_B(t)\right),\end{aligned} \tag{23}$$

that can be written as

$$\begin{aligned}&\mathscr{L}\left(S_B(t)+I_B(t)\right)\\&\quad\leq \left(\mu_B+\alpha_B\right)\frac{\lambda^{\theta-(1+\theta)}}{\lambda^\theta+\mu_B+\alpha_B}\\&\qquad+\frac{\lambda^{\theta-1}}{\lambda^\theta+\mu_B+\alpha_B}\left(S_B(0)+I_B(0)\right).\end{aligned} \tag{24}$$

From the Laplace transform properties (6) and equality (7) we infer

$$\begin{aligned}&\left(S_B(t)+I_B(t)\right)\\&\quad\leq t^\theta E_{\theta,\theta+1}\left(-\left(\mu_B+\alpha_B\right)t^\theta\right)\left(\mu_B+\alpha_B\right)\\&\qquad+E_{\theta,1}\left(-\left(\mu_B+\alpha_B\right)t^\theta\right)\left(S_B(0)+I_B(0)\right)\\&\quad\leq t^\theta E_{\theta,\theta+1}\left(-\left(\mu_B+\alpha_B\right)t^\theta\right)\left(\mu_B+\alpha_B\right)\\&\qquad+E_{\theta,1}\left(-\left(\mu_B+\alpha_B\right)t^\theta\right) = 1.\end{aligned} \tag{25}$$

Therefore, we have that $0 \leq S_B(t) + I_B(t) \leq 1$.

On the other hand, if $0 \leq I_T(0) \leq 1$, from system (17), we obtain

$$\begin{aligned}D_C^\theta I_T(t) &= \beta_T\left(1-I_T(t)\right)I_B(t) - \mu_T p I_T(t)\\&\leq \left(\beta_T+\mu_T p\right) - \left(\beta_T+\mu_T p\right)I_T(t).\end{aligned} \tag{26}$$

The proof of $0 \leq I_T(t) \leq 1$ is similar to the previous case. Finally, we conclude that Ω is a positive invariant set. □

In the following result we study the existence and stability of the equilibrium points of system (17). Motivated by Aranda et al. [3], we will use the following threshold parameter. For more details on the threshold parameter, see [24, 25]:

$$R_0 = \frac{\beta_B\beta_T}{\lambda_B\mu_T p}. \tag{27}$$

The next result is similar to Proposition 1 in [3], and so we omit its proofs.

Theorem 11. *System (17) has the disease-free equilibrium point:*

$$E_1 = \left(S_{B_1}, I_{B_1}, I_{T_1}\right) = (1, 0, 0), \tag{28}$$

for all the values of the parameters in this system, whereas only if $R_0 > 1$, there is (unique) endemic equilibrium point:

$$E_2 = \left(S_{B_2}, I_{B_2}, I_{T_2}\right), \tag{29}$$

where

$$\begin{aligned} S_{B_2} &= \frac{\lambda_B(\alpha_B+\mu_B)\beta_T + (\alpha_B+\mu_B+\lambda_B)\lambda_B}{\beta_T[\alpha_B(\beta_B+\lambda_B)+\lambda_B\mu_B+\beta_B(\lambda_B+\mu_B)]}, \\ I_{B_2} &= \frac{(\mu_B+\alpha_B)(\beta_T\beta_B-\lambda_B\mu_T p)}{\beta_T[\alpha_B(\beta_B+\lambda_B)+\mu_B\lambda_B+\beta_B(\mu_B+\lambda_B)]}, \\ I_{T_2} &= \frac{(\alpha_B+\mu_B)(\beta_B\beta_T-\lambda_B\mu_T p)}{(\alpha_B+\mu_B)\beta_B\beta_T+(\alpha_B+\mu_B+\lambda_B)\beta_B\mu_T p}, \end{aligned} \tag{30}$$

in the interior of Ω.

Computing the Jacobian matrix of system (17) evaluated at the disease-free point, one gets

$$J(E_1) = \begin{vmatrix} -(\mu_B+\alpha_B) & -(\mu_B+\alpha_B) & -\beta_B \\ 0 & -\lambda_B & \beta_B \\ 0 & \beta_T & -\mu_T p \end{vmatrix}, \tag{31}$$

and consequently, the eigenvalues of $J(E_1)$ are

$$\begin{aligned} \lambda_1 &= -(\mu_B+\alpha_B), \\ \lambda_2 &= \frac{-(\lambda_B+\mu_T p)+\sqrt{\Delta}}{2}, \\ \lambda_3 &= \frac{-(\lambda_B+\mu_T p)-\sqrt{\Delta}}{2}, \end{aligned} \tag{32}$$

where $\Delta = (\lambda_B-\mu_T p)^2 + 4\beta_B\beta_T$. It is easy to see that λ_1 and λ_3 are negative numbers. If $R_0 < 1$ we observe

$$\begin{aligned} \Delta &= (\lambda_B-\mu_T p)^2 + 4\beta_B\beta_T \\ &= \lambda_B^2 + \mu_T^2 p^2 - 2\lambda_B\mu_T p + 4\beta_B\beta_T \\ &< \lambda_B^2 + \mu_T^2 p^2 + 2\lambda_B\mu_T p = (\lambda_B+\mu_T p)^2. \end{aligned} \tag{33}$$

We infer that

$$\begin{aligned} \lambda_2 &= \frac{-(\lambda_B+\mu_T p)+\sqrt{\Delta}}{2} \\ &< \frac{-(\lambda_B+\mu_T p)+(\lambda_B+\mu_T p)}{2} = 0. \end{aligned} \tag{34}$$

Therefore, $\lambda_2 < 0$; then we have that all eigenvalues of the Jacobian matrix at E_1 are negative: that is, $|\arg(\lambda_i)| = \pi$, $i = 1, 2, 3$, and from Theorem 8, we have that disease-free equilibrium point E_1 is locally asymptotically stable. Consequently, we have the following Theorem.

Theorem 12. *If $R_0 < 1$, then the disease-free point E_1 is locally asymptotically stable.*

In the next result we prove the global asymptotical stability of the disease-free equilibrium point.

Theorem 13. *If $R_0 < 1$, then the disease-free point E_1 is globally asymptotically stable.*

Proof. Suppose that $(S_B(t), I_B(t), I_T(t))$ is the solution of system (17). Making the change of variables $L_B = 1 - S_B$ we obtain the new system:

$$\begin{aligned} D_C^\theta L_B(t) &= -(\mu_B+\alpha_B)L_B(t) + (\mu_B+\alpha_B)I_B(t) \\ &\quad + \beta_B I_T(t) - \beta_B L_B(t) I_T(t), \\ D_C^\theta I_B(t) &= \beta_B(1-L_B(t))I_T(t) - \lambda_B I_B(t), \\ D_C^\theta I_T(t) &= \beta_T(1-I_T(t))I_B(t) - \mu_T p I_T(t). \end{aligned} \tag{35}$$

It is easy to see that

$$\begin{aligned} &-(\mu_B+\alpha_B)(L_B(t)-I_B(t)) \\ &+\beta_B(I_T(t)-L_B(t)I_T(t)) \\ &\le -(\mu_B+\alpha_B)(L_B(t)-I_B(t)) + \beta_B I_T(t), \\ &\beta_B(1-L_B(t))I_T(t) - \lambda_B I_B(t) \le \beta_B I_T(t) - \lambda_B I_B(t), \\ &\beta_T(1-I_T(t))I_B(t) - \mu_T p I_T(t) \le \beta_T I_B(t) \\ &-\mu_T p I_T(t). \end{aligned} \tag{36}$$

From the above, it follows that the solutions $(L_B(t), I_B(t), I_T(t))$ of system (35) satisfy the differential inequality:

$$\begin{aligned} D_C^\theta L_B(t) &\le -(\mu_B+\alpha_B)L_B(t) + (\mu_B+\alpha_B)I_B(t) \\ &\quad + \beta_B I_T(t), \\ D_C^\theta I_B(t) &\le \beta_B I_T(t) - \lambda_B I_B(t), \\ D_C^\theta I_T(t) &\le \beta_T I_B(t) - \mu_T p I_T(t). \end{aligned} \tag{37}$$

Moreover, motivated by (37), let $(X(t), Y(t), Z(t))$ be the solution of fractional linear system:

$$\begin{aligned} D_C^\theta X(t) &= -(\mu_B+\alpha_B)X(t) + (\mu_B+\alpha_B)Y(t) \\ &\quad + \beta_B Z(t), \\ D_C^\theta Y(t) &= \beta_B Z(t) - \lambda_B Y(t), \\ D_C^\theta Z(t) &= \beta_T Y(t) - \mu_T p Z(t), \end{aligned} \tag{38}$$

with initial conditions $(X(0), Y(0), Z(0)) = (X_0, Y_0, Z_0) \in \Omega$.

The eigenvalues of system (38) are given by

$$\begin{vmatrix} -(\mu_B+\alpha_B) & (\mu_B+\alpha_B) & \beta_B \\ 0 & -\lambda_B & \beta_B \\ 0 & \beta_T & -\mu_T p \end{vmatrix}. \quad (39)$$

Similar to the proof of Theorem 12, we infer that all the eigenvalues are negatives; thus, $|\arg(x_i)| = \pi$, $i = 1, 2, 3$, and from Theorem 4, we can conclude that $\lim_{t\to\infty} X(t) = 0$, $\lim_{t\to\infty} Y(t) = 0$, and $\lim_{t\to\infty} Z(t) = 0$.

From the previous discussion and the comparison principle, Theorem 2, we have

$$(L_B(t), I_B(t), I_T(t)) \leq\leq (X(t), Y(t), Z(t)). \quad (40)$$

This implies $\lim_{t\to\infty}(L_B(t), I_B(t), I_T(t)) = (0, 0, 0)$, and it follows that $(S_B(t), I_B(t), I_T(t))$ converge to the disease-free equilibrium point $E_1 = (1, 0, 0)$, when $R_0 < 1$. This ends the proof. □

Now we show the local stability of the endemic equilibrium point E_2, and we give the definition of an additive compound matrix. For more details see [26, 27].

Definition 14. Let A be any $n \times m$ matrix of real and complex numbers, and let $a_{i1,\ldots,j_k}$ be the minor of A determined by the rows $(i_1, \ldots, i_k)$ and the columns $(j_1, \ldots, j_k)$, $1 \leq i_1 < i_2, \ldots, < i_k \leq n, 1 \leq j_1 < j_2, \ldots, < j_k \leq m$. The kth multiplicative compound matrix of A^k of A is the $\binom{n}{k} \times \binom{n}{k}$ matrix whose entries, written in a lexicographic order, are $a_{i1,\ldots,j_k}$. When A is a $n \times m$ matrix with columns $a_1, a_2, \ldots, a_k$, A^k is the exterior product $a_1 \wedge a_2 \wedge \cdots \wedge a_k$.

Definition 15. If $A = a_{ij}$ is a $n \times n$ matrix, its kth additive compound $A^{[k]}$ of the A is the $\binom{n}{k} \times \binom{n}{k}$ matrix given by $A^{[k]} = |D(I + hA)^{(k)}| = 0$, where D is a differentiation with respect to h. For any integers $i = 1, \ldots, \binom{n}{k}$, let $(i) = (i_1, \ldots, i_k)$ be the ith member in the lexicographic ordering of all k-tuples of integers such that $1 \leq i_1 < i_2 < \cdots < i_k \leq i_n$. Then

$$b_{ij} = \begin{cases} a_{i_1 i_1} + \cdots + a_{i_k i_k} & \text{if } (i) = (j), \\ (-1)^{r+s} a_{i_s i_r}, & \text{if one entry of } i_s \text{ of } (i) \text{ does not occur in } (j) \text{ and } j_s \text{ does not occur in } (i), \\ 0, & \text{if } (i) \text{ differs from } (j) \text{ in two or more entries.} \end{cases} \quad (41)$$

Remark 16. For $n = 3$, the matrices $A^{[k]}$ are as follows:

$$A^{[1]} = A,$$

$$A^{[2]} = \begin{bmatrix} a_{11}+a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11}+a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22}+a_{33} \end{bmatrix}, \quad (42)$$

$$A^{[3]} = a_{11} + a_{22} + a_{33}.$$

The next lemma is stated and proved in [28].

Lemma 17. *Let M be a 3×3 real matrix. If $\operatorname{tr}(M) < 0$, $\det(M) < 0$, and $\det(M^{[2]}) < 0$ are all negative, then all eigenvalues of M have negative real part.*

Theorem 18. *If $R_0 > 1$, $\mu_B + \alpha_B > \beta_T$, and $\mu_B + \alpha_B > \beta_B$, then endemic equilibrium point E_2 is locally asymptotically stable.*

Proof. The Jacobian matrix of systems (17) in the endemic equilibrium point is given by

$$J(E_2) = \begin{pmatrix} -(\mu_B+\alpha_B+\beta_B I_T) & -(\mu_B+\alpha_B) & -\beta_B S_B \\ \beta_B I_T & -\lambda_B & \beta_B S_B \\ 0 & \beta_T(1-I_T) & -I_B\beta_T - \mu_T p \end{pmatrix}. \quad (43)$$

From $J(E_2)$, we have $\operatorname{tr}(J(E_2)) = -(\mu_B + \alpha_B + \beta_B I_T) - \lambda_B - I_B\beta_T - \mu_T p < 0$.

To show the $\det J(E_2) < 0$, we will make a simplification into system (17), where it comes from

$$-(\mu_B+\alpha_B) = -\frac{\beta_B S_B I_T}{1 - S_B - I_B}, \qquad \lambda_B = \frac{\beta_B S_B I_T}{I_B}, \qquad \mu_T p = \frac{\beta_T(1-I_T) I_B}{I_T}. \quad (44)$$

Substituting (44) in the matrix (43), we obtain

$$\det(J(E_2)) = \begin{vmatrix} -\dfrac{\beta_B I_T(1-I_B)}{1-S_B-I_B} & -\dfrac{\beta_B S_B I_T}{1-S_B-I_B} & -\beta_B S_B \\ \beta_B I_T & -\dfrac{\beta_B S_B I_T}{I_B} & \beta_B S_B \\ 0 & \beta_T(1-I_T) & -\dfrac{\beta_T I_B}{I_T} \end{vmatrix}. \quad (45)$$

Then

$$\begin{aligned}\det\left(J\left(E_2\right)\right) &= -\frac{\left(1-S_B-I_B\right)I_BI_T\left(1-I_B\right)\beta_BS_B\beta_TI_T}{\left(1-S_B-I_B\right)I_BI_T\left(1-S_B-I_B\right)}\\ &\quad -\beta_BS_B\beta_BI_T\beta_T\left(1-I_T\right)-\frac{\beta_BS_BI_T\beta_BI_T\beta_TI_B}{\left(1-S_B-I_B\right)I_T}\\ &= -\beta_BS_B\beta_TI_T\left[\frac{\left(1-S_B-I_B\right)I_BI_T\left(1-I_B\right)}{\left(1-S_B-I_B\right)I_BI_T\left(1-S_B-I_B\right)}\right.\\ &\quad \left.+\beta_B\left(1-I_T\right)\right]-\frac{\beta_BS_B\beta_TI_T\beta_BI_TI_B}{\left(1-S_B-I_B\right)I_T}.\end{aligned} \tag{46}$$

Therefore, as all are constant positive parameters, it follows that $\det(J(E_2)) < 0$.

Let $J^{[2]}(E_2)$ be the additive compound matrix:

$$J^{[2]}\left(E_2\right)=\begin{pmatrix} M-\lambda_B & \beta_BS_B & \beta_BS_B\\ \beta_T-\beta_TI_T & M+K & -\left(\mu_B+\alpha_B\right)\\ 0 & \beta_BI_T & -\left(\lambda_B+K\right)\end{pmatrix}, \tag{47}$$

where $M = -(\mu_B + \alpha_B + \beta_BI_T)$ and $K = -(I_B\beta_T + \mu_Tp)$. From the hypothesis $0 \le 1 - I_T \le 1$, we get

$$\begin{aligned}\det\left(J^{[2]}\left(E_2\right)\right) &= -\left[\left(\mu_B+\alpha_B+\beta_BI_T+\lambda_B\right)\right.\\ &\quad \cdot\left(\mu_B+\alpha_B+\beta_BI_T+I_B\beta_T+\mu_Tp\right)\\ &\quad \left.\cdot\left(\lambda_B+I_B\beta_T+\mu_Tp\right)\right]+\beta_BS_B\beta_T\left(1-I_T\right)\left[I_T\beta_B\right.\\ &\quad \left.+\lambda_B+I_B\beta_T+\mu_Tp\right]-\left(\mu_B+\alpha_B+\beta_BI_T+\lambda_B\right)\left(\mu_B\right.\\ &\quad \left.+\alpha_B\right)\beta_BI_T\le-\left[\left(\mu_B+\alpha_B+\beta_BI_T+\lambda_B\right)\right.\\ &\quad \cdot\left(\mu_B+\alpha_B+\beta_BI_T+I_B\beta_T+\mu_Tp\right)\\ &\quad \left.\cdot\left(\lambda_B+I_B\beta_T+\mu_Tp\right)\right]+\beta_B\beta_T\left(I_T\beta_B+\lambda_B+I_B\beta_T\right.\\ &\quad \left.+\mu_Tp\right)-\left(\mu_B+\alpha_B+\beta_BI_T+\lambda_B\right)\left(\mu_B+\alpha_B\right)\beta_BI_T\\ &= -\left(\lambda_B+I_B\beta_T+\mu_Tp\right)\left[\left(\mu_B+\alpha_B+\beta_BI_T+\lambda_B\right)\right.\\ &\quad \left.\cdot\left(\mu_B+\alpha_B+\beta_BI_T+I_B\beta_T+\mu_Tp\right)-\beta_B\beta_T\right]\\ &\quad -\beta_BI_T\left[\left(\mu_B+\alpha_B+\beta_BI_T+\lambda_B\right)\left(\mu_B+\alpha_B\right)-\beta_B\beta_T\right].\end{aligned} \tag{48}$$

Analyzing the terms of equality above, we have

$$\begin{aligned}&\left(\mu_B+\alpha_B+\beta_BI_T+\lambda_B\right)\\ &\quad\cdot\left(\mu_B+\alpha_B+\beta_BI_T+I_B\beta_T+\mu_Tp\right)>\beta_B\beta_T,\\ &\left(\mu_B+\alpha_B+\beta_BI_T+\lambda_B\right)\left(\mu_B+\alpha_B\right)>\beta_B\beta_T.\end{aligned} \tag{49}$$

Then $\det(J^{[2]}(E_2)) < 0$ and from Lemma 17, the endemic equilibrium point (E_2) is locally asymptotically stable. This concludes the proof. □

4. Numerical Simulations

In this section, we simulate different possible scenarios to check the effect that some values of fractional exponent θ have on the dynamics of bovine babesiosis disease and tick populations. For comparison purposes, we will use the same parameters as Aranda et al. [3]. To solve a nonlinear differential equation set with fractional order, a method based on the classical Adams-Bashforth-Moulton approach was used, as presented in [29]:

$$f_{j,m} = f_{j,1} + J, \tag{50}$$

in which $j = 1, 2$, and 3 represents population number: S_B, I_B, and I_T, respectively. The time is defined as $t_m = (m-1)H$ in which $2 < m < N+1$ and $N = T/H$, with T equal to the final time. The fractional integral is determined by modified trapezoidal rule as

$$J=\sum_{k=0}^{n-1}f_{i,k}w_\mu+\frac{\left(f_{i,k}-f_{i,k+1}\right)g_\mu}{h}, \tag{51}$$

in which $\mu = n - k$, $h = t_m/n$, and $t_k = kh$,

$$\begin{aligned}w_\mu &= \left(\frac{h^\theta}{\gamma\left(\theta+1\right)}\right)\left(\mu^\theta-\left(\mu-1\right)^\theta\right),\\ g_\mu &= \left(\frac{h^{(\theta+1)}}{\gamma\left(\theta+2\right)}\right)\left(\mu^{(\theta+1)}-\left(\mu+\theta\right)\left(\mu-1\right)^\theta\right).\end{aligned} \tag{52}$$

In this work $H = 40$ and $n = 400$. More details about the numerical integration algorithm can be found in [29, 30].

Figure 1 shows the dynamics of the bovine babesiosis disease and tick populations, with initial condition of $S_B = 0.3756$, $I_B = 0.5184$, and $I_T = 0.6000$, and reproduction number $R_0 = 67.54$. As can be seen, following the course of the disease, the system evolves to the endemic equilibrium point with population number of $S_{B_2} = 0.04967$, $I_{B_2} = 0.7894$, and $I_{T_2} = 0.7019$, as determined by (30). The convergence to the equilibrium point, when $R_0 > 1$, is predicted by Theorem 18. The variables S_B, I_B, and I_T drop to less than 1% of the equilibrium values above 6280 years, when a veterinary intervention was simulated making R_0 less than 1 ($R_0 = 0.6754$). This new R_0 value was obtained with β_B equal to 1/10 of the initial value. Now the system gets out of endemic equilibrium point and evolves to the disease-free equilibrium point $(1, 0, 0)$, as predicted by Theorems 12 and 13. The control parameters of differential equation set are presented in Table 1.

A comparison between two different values of the fractional order is shown in Figure 2, with the same control parameter shown in Table 1. Figure 2 shows a different behavior for $\theta = 0.9$ and $\theta = 1$, with a maximum value of S_B and a minimum value of I_B, that does not appear when $\theta = 1$. For both cases, the disease evolves to the endemic equilibrium point; however, it is slower when $\theta = 0.9$.

Table 2 shows the time $\tau_{1\%}$ in which variables drop to less than 1% of the equilibrium values. These times were obtained with different values of θ. As we can see, the time $\tau_{1\%}$ increases when θ decreases. The time $\tau_{1\%}$ as function θ was adjusted by two linear equations, $\tau_{1\%} \times 1/\theta$ and $\ln(\tau_{1\%}) \times 1/\theta$. The first case is consistent with exponential behavior and the second case

Table 1: The control parameters.

Parameter	Value
μ_B	0.0002999
α_B	0.001
β_B	0.006
λ_B	0.000265
β_T	0.00048
μ_T	0.001609
p	0.1

Table 2: Relaxation time.

θ	$\tau_{1\%}$/years
1	5120
0.975	8640
0.95	15200
0.925	26720
0.9	42840
0.875	49560
0.85	65840
0.825	70880
0.8	96680

with power law $t^{-\theta}$. After the statistical analysis based on the correlation coefficient, 0.90013 against 0.98018, one concludes that the system decays to equilibrium condition like power law $t^{-\theta}$. This result was previously proven under theoretical assumptions [21].

5. Conclusions

We did not find global stability results for fractional differential order equations in the literature. This way, we obtain a new result for global asymptotical stability of disease-free equilibrium using comparison theory of fractional differential equations since $R_0 < 1$, and therefore the proof that endemic equilibrium point, when $R_0 > 1$, $\mu_B + \alpha_B > \beta_C$, and $\mu_B + \alpha_B > \beta_C$, is locally asymptotically stable was achieved using the linearization theorem for fractional differential equations. Therefore, if $R_0 < 1$ so the system evolves to endemic equilibrium point. To return to disease-free status, the R_0 value should be greater than 1. The $R_0 < 1$ is achieved when parameters β_B and β_C are very small or when parameters λ_B, μ_C, and p are very large. Therefore, biological strategy to combat babesiosis disease would have to focus on one of these parameters. These results were confirmed by numerical simulations using the extension of Adams-Bashforth-Moulton algorithm.

Numeric simulations of improved epidemic model with arbitrary order have shown that fractional order is related to relaxation time, in other words, the time taken to reach equilibrium. Numerical simulations with different order show that the system decays to equilibrium condition like power law $t^{-\theta}$, as previously established in [21]. This result provides an important insight about the use of fractional order to model the dynamics of babesiosis disease and tick population. The proof shown here should be used as a guide in the study of equilibrium conditions in similar problems, such as tuberculosis [28], malaria [31], or toxoplasmosis disease [32].

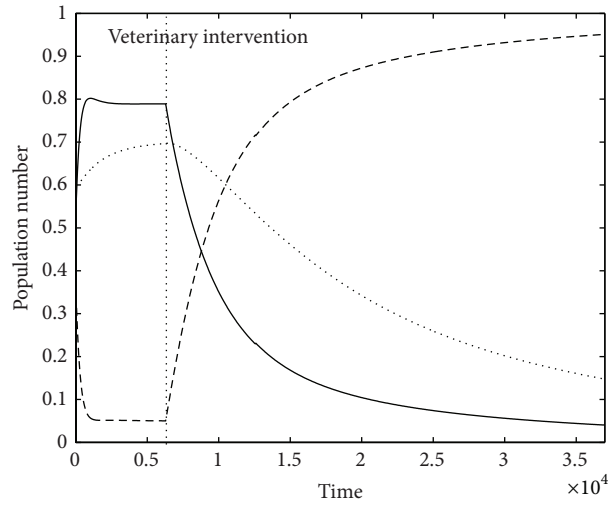

Figure 1: Dynamic of the bovine babesiosis disease. I_B (continuous line) together with S_B (dashed line) and I_T (dotted line) were shown as function of the time.

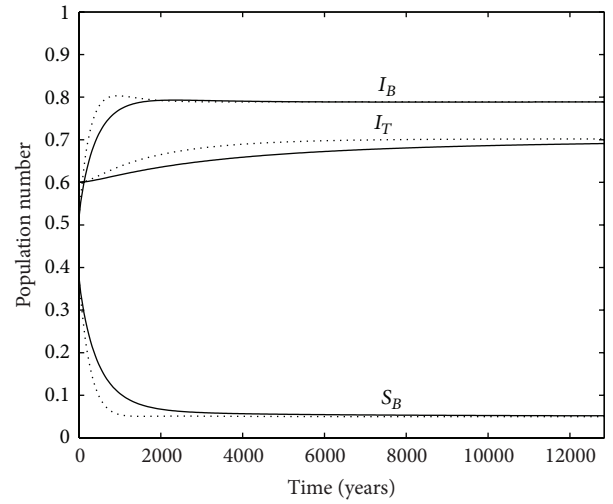

Figure 2: Dynamic of the bovine babesiosis disease, with $\theta = 1$ (continuous line) and $\theta = 0.9$ (dotted line).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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