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Research Article

Fixed Point Theorems for an Elastic Nonlinear Mapping in Banach Spaces

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Let E be a smooth Banach space with a norm $\|\cdot\|$. Let $V(x,y)=\|x\|^2+\|y\|^2-2\langle x,Jy\rangle$ for any $x,y\in E$, where $\langle\cdot,\cdot\rangle$ stands for the duality pair and J is the normalized duality mapping. We define a V-strongly nonexpansive mapping by $V(\cdot,\cdot)$. This nonlinear mapping is nonexpansive in a Hilbert space. However, we show that there exists a V-strongly nonexpansive mapping with fixed points which is not nonexpansive in a Banach space. In this paper, we show a weak convergence theorem and strong convergence theorems for fixed points of this elastic nonlinear mapping and give the existence theorem.

1. Introduction

Let E be a smooth Banach space with a norm $\|\cdot\|$ and let E^* be the dual space of E. We denote by $\langle\cdot,\cdot\rangle$ a duality pair on $E\times E^*$ and let J be the normalized duality mapping on E. It is well known that J is a continuous single-valued mapping in a smooth Banach space and a one-to-one mapping in a strictly convex Banach space (cf. [1]). We define a mapping $V: E\times E\to\mathbb{R}$ by $V(x,y)=\|x\|^2+\|y\|^2-2\langle x,Jy\rangle$ for all $x,y\in E$, where $\mathbb{R}$ is a set of real numbers. It is obvious that $V(x,y)\geq(\|x\|-\|y\|)^2\geq 0$. Let any $y\in E$ be fixed, and then $V(\cdot,y)$ is a convex function because of convexity of $\|\cdot\|^2$. Many nonlinear mappings which are defined by using $V(\cdot,\cdot)$ are studied (see [2–4]). We also defined a nonlinear mapping which is called a V-strongly nonexpansive mapping in [5] as follows.

Definition 1. Let C be a nonempty subset of a smooth Banach space E. A mapping $T: C\to E$ is called V-strongly nonexpansive if there exists a constant $\lambda>0$ such that for all $x,y\in C$

$$V(Tx,Ty)\leq V(x,y)-\lambda V((I-T)x,(I-T)y), \quad (1)$$

where I is the identity mapping on E.

From this definition, it is obvious that the identity mapping I is also a V-strongly nonexpansive mapping. In a Hilbert space, it is trivial that this mapping is nonexpansive since $V(x,y)=\|x-y\|^2$ and that any firmly nonexpansive mapping is a V-strongly nonexpansive mapping with $\lambda=1$ (see [5]). Moreover, we showed that if there exists a fixed point of a V-strongly nonexpansive mapping T, then T is strongly nonexpansive with a Bregman distance in [5]. However, in Banach spaces, as we give an example in the later section, we find that there exists a V-strongly nonexpansive mapping with fixed points which is not nonexpansive. We should point out that a guarantee of continuity of the V-strongly nonexpansive mappings has not been given in a generalized Banach space yet.

In this paper, we prove a weak convergence theorem and strong convergence theorems for finding fixed points of a V-strongly nonexpansive mapping in Banach spaces and show the existence theorem of fixed point with a dissipative property.

2. Preliminaries

In this section, at first we show the relationship between a V-strongly nonexpansive mapping and other nonlinear mappings, in a Hilbert space. Secondly, we state some properties of V-strongly nonexpansive mappings in a Banach space and give an example of a V-strongly nonexpansive mapping which is not a quasinonexpansive mapping in a Banach space although T has fixed points. We finally show some lemmas which are necessary in order to prove our theorems.

Let C be a subset of a Banach space E and let $T : C \to E$ be a mapping. Then a point p in the closure of C is said to be an asymptotically fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p and the sequence $\{x_n - Tx_n\}$ converges strongly to 0. $\widehat{F}(T)$ denotes the set of asymptotically fixed points of T. In [6], Reich introduced a strongly nonexpansive mapping which is defined by using the Bregman distance $D(\cdot,\cdot)$.

Definition 2. Let E be a Banach space. The Bregman distance corresponding to a function $f : E \to \mathbb{R}$ is defined by

$$D(x,y) = f(x) - f(y) - f'(y)(x-y), \tag{2}$$

where f is Gâteaux differentiable and $f'(x)$ stands for the derivative of f at the point x. Let C be a nonempty subset of E. We say that the mapping $T : C \to E$ is strongly nonexpansive if $\widehat{F}(T) \neq \emptyset$ and

$$D(p,Tx) \le D(p,x) \quad \forall p \in \widehat{F}(T) \;\; x \in C, \tag{3}$$

and if it holds that $\lim_{n\to\infty} D(Tx_n, x_n) = 0$ for a bounded sequence $\{x_n\}$ such that $\lim_{n\to\infty}(D(p,x_n) - D(p,Tx_n)) = 0$ for any $p \in \widehat{F}(T)$.

Taking the function $\|\cdot\|^2$ as the convex, continuous, and Gâteaux differentiable function f, we obtain the fact that the Bregman distance $D(\cdot,\cdot)$ coincides with $V(\cdot,\cdot)$. In particular, in a Hilbert space, it is trivial that $D(x,y) = V(x,y) = \|x - y\|^2$.

Proposition 3 (see [5]). *In a Hilbert space, a V-strongly nonexpansive mapping with $\widehat{F}(T) \neq \emptyset$ is strongly nonexpansive.*

Next we recall two mappings of other nonlinear mappings (cf. [6–9]). A firmly nonexpansive mapping and an α-inverse strongly monotone mapping are defined as follows.

Definition 4. Let C be a nonempty, closed, and convex subset of a Banach space E. A mapping $T : C \to E$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \langle x - y, j\rangle \tag{4}$$

for all $x, y \in C$ and some $j \in J(Tx - Ty)$.

It is trivial that a firmly nonexpansive mapping is nonexpansive.

Definition 5. Let H be a Hilbert space. A mapping $T : C \to H$ is said to be α-inverse strongly monotone if

$$\alpha \|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle \tag{5}$$

for all $x, y \in C$.

The relation among firmly nonexpansive mappings, α-inverse strongly monotone mappings and V-strongly nonexpansive mappings is shown in the following proposition.

Proposition 6 (see [5]). *In a Hilbert space, the following hold.*

(a) *A firmly nonexpansive mapping is V-strongly nonexpansive with $\lambda = 1$.*

(b) *Let A be an α-inverse strongly monotone mapping for $\alpha > 1/2$; then $S = (I - A)$ is V-strongly nonexpansive with $(2\alpha - 1)$.*

The above (b) is obvious by showing that, for all $x, y \in H$,

$$\langle Sx - Sy, x - y\rangle \le \|x - y\|^2 - \alpha \|(I - S)x - (I - S)y\|^2. \tag{6}$$

We will introduce some properties of V-strongly nonexpansive mappings in [5].

Proposition 7 (see [5]). *In a smooth Banach space E, the following hold.*

(a) *For $c \in (-1,1]$, $T = cI$ is V-strongly nonexpansive. For $c = 1$, $T = I$ is V-strongly nonexpansive for any $\lambda > 0$. For $c \in (-1,1)$, $T = cI$ is V-strongly nonexpansive for any $\lambda \in (0, (1+c)/(1-c)]$.*

(b) *If T is V-strongly nonexpansive with λ, then, for any $\alpha \in [-1,1]$ with $\alpha \neq 0$, αT is also V-strongly nonexpansive with $\alpha^2\lambda$.*

(c) *If T is V-strongly nonexpansive with $\lambda \ge 1$, then $A = I - T$ is V-strongly nonexpansive with λ^{-1}.*

(d) *Suppose that T is V-strongly nonexpansive with λ and that $\alpha \in [-1,1]$ satisfies $\alpha^2\lambda \ge 1$. Then $(I - \alpha T)$ is V-strongly nonexpansive with $(\alpha^2\lambda)^{-1}$. Moreover, if $T_\alpha = I - \alpha T$, then*

$$V(T_\alpha x, T_\alpha y) \le V(x,y) - \lambda^{-1} V(Tx, Ty). \tag{7}$$

Now we give an example of a V-strongly nonexpansive mapping in a Banach space.

Example 8 (see [10]). Let $1 < p, q < \infty$ such that $1/p + 1/q = 1$. Let $E = \mathbb{R} \times \mathbb{R}$ be a real Banach space with a norm $\|\cdot\|_p$ defined by

$$\|x\|_p = \left\{|x_1|^p + |x_2|^p\right\}^{1/p} \quad \forall x = (x_1, x_2) \in E. \tag{8}$$

Then E is smooth, and the normalized duality mapping J is single-valued. J is given by

$$Jx = \|x\|_p^{2-p}\left(x_1|x_1|^{p-2}, x_2|x_2|^{p-2}\right) \in l^q(\mathbb{R}\times\mathbb{R}) \qquad \forall x = (x_1, x_2) \in E. \tag{9}$$

Hence, we have for $x, y \in E$ that

$$\begin{aligned} V(x,y) &= \|x\|_p^2 + \|y\|_p^2 - 2\langle x, Jy\rangle \\ &= \|x\|_p^2 + \|y\|_p^2 - 2\|y\|_p^{2-p} \\ &\quad \cdot \left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\}. \end{aligned} \tag{10}$$

We define a mapping $T : E \to E$ as follows:

$$Tx = \begin{cases} x & \text{if } \|x\|_p \le 1, \\ \dfrac{1}{\|x\|_p} x & \text{if } \|x\|_p > 1. \end{cases} \quad (11)$$

In a case of $p = 1$, we have shown that the mapping T defined by (11) is a V-strongly nonexpansive mapping (see [5]). We will show that T is V-strongly nonexpansive with any $\lambda \le 1$, for $p > 1$.

Proposition 9. *Suppose that T is defined by the formula (11) under the above situation. Then, T is a V-strongly nonexpansive mapping with any $\lambda \le 1$.*

Proof. Case (a): suppose that $x, y \in E$ with $\|x\|_p \le 1$ and $\|y\|_p > 1$.

Since $Ty = ((Ty)_1, (Ty)_2) = (y_1\|y\|_p^{-1}, y_2\|y\|_p^{-1})$, we have that

$$\begin{aligned} V(Tx, Ty) = V(x, Ty) &= \|x\|_p^2 + \|Ty\|_p^2 - 2\|Ty\|_p^{2-p} \\ &\quad \cdot \left\{x_1 (Ty)_1 |(Ty)_1|^{p-2} + x_2 (Ty)_2 |(Ty)_2|^{p-2}\right\} \\ &= \|x\|_p^2 + 1 - 2\|y\|_p^{1-p} \\ &\quad \cdot \left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\}. \end{aligned} \quad (12)$$

Since

$$y - Ty = \left(\frac{\|y\|_p - 1}{\|y\|_p} y_1, \frac{\|y\|_p - 1}{\|y\|_p} y_2\right), \quad (13)$$

we have that

$$\begin{aligned} V(x - Tx, y - Ty) = V(0, y - Ty) &= \|y - Ty\|_p^2 \\ &= \left\{\frac{(\|y\|_p - 1)}{\|y\|_p} \|y\|_p\right\}^2 \\ &= (\|y\|_p - 1)^2. \end{aligned} \quad (14)$$

Hence, we obtain that

$$\begin{aligned} &V(x, y) - V(Tx, Ty) - \lambda V(x - Tx, y - Ty) \\ &\quad = \|x\|_p^2 + \|y\|_p^2 - 2\|y\|_p^{2-p} \left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\} \\ &\qquad - \|x\|_p^2 - 1 + 2\|y\|_p^{1-p} \left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\} \\ &\qquad - \lambda (\|y\|_p - 1)^2 \\ &\quad = \|y\|_p^2 - 1 - 2\|y\|_p^{1-p} (\|y\|_p - 1) \\ &\qquad \cdot \left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\} - \lambda (\|y\|_p - 1)^2 \\ &\quad \ge (\|y\|_p - 1) \left\{(\|y\|_p + 1) - 2\|y\|_p^{1-p} \right. \\ &\qquad \cdot \left(|x_1| |y_1|^{p-1} + |x_2| |y_2|^{p-1}\right) \\ &\qquad \left. - \lambda (\|y\|_p - 1)\right\}. \end{aligned} \quad (15)$$

Hölder's inequality implies that

$$\begin{aligned} |x_1| |y_1|^{p-1} + |x_2| |y_2|^{p-1} &\le \|x\|_p \left\{\left(|y_1|^{p-1}\right)^q + \left(|y_2|^{p-1}\right)^q\right\}^{1/q} \\ &= \|x\|_p \left(|y_1|^p + |y_2|^p\right)^{1/q} \\ &= \|x\|_p \|y\|_p^{p-1}. \end{aligned} \quad (16)$$

Therefore, we obtain that

$$\begin{aligned} &V(x, y) - V(Tx, Ty) - \lambda V(x - Tx, y - Ty) \\ &\quad \ge (\|y\|_p - 1) \\ &\qquad \cdot \left\{\|y\|_p + 1 - 2\|y\|_p^{1-p} \|x\|_p \|y\|_p^{p-1} - \lambda \|y\|_p + \lambda\right\} \\ &\quad = (\|y\|_p - 1) \left\{\|y\|_p + 1 - 2\|x\|_p - \lambda \|y\|_p + \lambda\right\} \\ &\quad \ge (\|y\|_p - 1) \left\{(1 - \lambda) \|y\|_p + 1 - 2 + \lambda\right\} \\ &\quad = (\|y\|_p - 1) \left\{(1 - \lambda) (\|y\|_p - 1)\right\} \\ &\quad = (1 - \lambda) (\|y\|_p - 1)^2 \ge 0, \quad \text{for any } \lambda \in [0, 1]. \end{aligned} \quad (17)$$

That is, the inequality (1) holds.

Case (b): suppose that $x, y \in E$ with $\|x\|_p \ge 1$ and $\|y\|_p \le 1$.

Then we have that

$$\begin{aligned} V(Tx, Ty) &= V(Tx, y) \\ &= 1 + \|y\|_p^2 - 2\|x\|_p^{-1} \|y\|_p^{2-p} \\ &\quad \cdot \left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\}, \end{aligned} \quad (18)$$

$$V(x - Tx, y - Ty) = V\left(\frac{(\|x\|_p - 1)}{\|x\|_p} x, 0\right) = (\|x\|_p - 1)^2. \quad (19)$$

Hence, we have that

$$\begin{aligned} &V(x, y) - V(Tx, Ty) - \lambda V(x - Tx, y - Ty) \\ &\quad = \|x\|_p^2 + \|y\|_p^2 - 2\|y\|_p^{2-p} \\ &\qquad \cdot \left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\} - 1 - \|y\|_p^2 \\ &\qquad + 2\|y\|_p^{2-p} \|x\|_p^{-1} \left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\} \\ &\qquad - \lambda (\|x\|_p - 1)^2 \end{aligned}$$

$$
\begin{aligned}
&\geq \|x\|_p^2 - 1 - 2\|y\|_p^{2-p} \\
&\quad \cdot \left\{|x_1||y_1|^{p-1} + |x_2||y_2|^{p-1}\right\}\left(1 - \|x\|_p^{-1}\right) \\
&\quad - \lambda\left(1 - \|x\|_p\right)^2 .
\end{aligned} \tag{20}
$$

As (a), we obtain from Hölder's inequality that

$$
\begin{aligned}
&V(x, y) - V(Tx, Ty) - \lambda V(x - Tx, y - Ty) \\
&\quad \geq \|x\|_p^2 - 1 - 2\|x\|_p \|y\|_p^{2-p} \|y\|_p^{p-1} \\
&\qquad \cdot \left(1 - \|x\|_p^{-1}\right) - \lambda\left(\|x\|_p - 1\right)^2 \\
&\quad = \left(\|x\|_p - 1\right)\left(\|x\|_p + 1\right) - 2\|y\|_p\left(\|x\|_p - 1\right) \\
&\qquad - \lambda\left(\|x\|_p - 1\right)^2 \\
&\quad = \left(\|x\|_p - 1\right)\left\{\|x\|_p + 1 - 2\|y\|_p - \lambda\|x\|_p + \lambda\right\} \\
&\quad \geq \left(\|x\|_p - 1\right)(1 - \lambda)\left(\|x\|_p - 1\right) \\
&\quad = (1 - \lambda)\left(\|x\|_p - 1\right)^2 \geq 0, \quad \text{for any } \lambda \in [0, 1] .
\end{aligned} \tag{21}
$$

That is, the inequality (1) holds.

Case (c): suppose that $x, y \in E$ with $\|x\|_p, \|y\|_p \geq 1$. Then we have that

$$
\begin{aligned}
&V(Tx, Ty) \\
&\quad = 1 + 1 - 2\left\langle \|x\|_p^{-1}(x_1, x_2), \right. \\
&\qquad\qquad \left. \|y\|_p^{1-p}\left(y_1|y_1|^{p-2}, y_2|y_2|^{p-2}\right)\right\rangle \\
&\quad = 2 - 2\|x\|_p^{-1}\|y\|_p^{1-p}\left\{x_1y_1|y_1|^{p-2} + x_2y_2|y_2|^{p-2}\right\}, \\
&V(x - Tx, y - Ty) \\
&\quad = V\left(\frac{\|x\|_p - 1}{\|x\|_p}x, \frac{\|y\|_p - 1}{\|y\|_p}y\right) \\
&\quad = \left(\|x\|_p - 1\right)^2 + \left(\|y\|_p - 1\right)^2 \\
&\qquad - 2\left(\|x\|_p - 1\right)\left(\|y\|_p - 1\right)\|x\|_p^{-1}\|y\|_p^{-1}\|y\|_p^{2-p} \\
&\qquad \cdot \left\langle (x_1, x_2), \left(|y_1|^{p-2}y_1, |y_2|^{p-2}y_2\right)\right\rangle \\
&\quad = \left(\|x\|_p - 1\right)^2 + \left(\|y\|_p - 1\right)^2 \\
&\qquad - 2\left(\|x\|_p - 1\right)\left(\|y\|_p - 1\right)\|x\|_p^{-1}\|y\|_p^{1-p} \\
&\qquad \cdot \left\{x_1y_1|y_1|^{p-2} + x_2y_2|y_2|^{p-2}\right\}.
\end{aligned} \tag{22}
$$

Hence, we have that

$$
\begin{aligned}
&V(x, y) - V(Tx, Ty) - \lambda V(x - Tx, y - Ty) \\
&\quad = \|x\|_p^2 + \|y\|_p^2 - 2\|y\|_p^{2-p}\left\{x_1y_1|y_1|^{p-2} + x_2y_2|y_2|^{p-2}\right\} \\
&\qquad - 2 + 2\|x\|_p^{-1}\|y\|_p^{1-p}\left\{x_1y_1|y_1|^{p-2} + x_2y_2|y_2|^{p-2}\right\} \\
&\qquad - \lambda\left(\|x\|_p - 1\right)^2 - \lambda\left(\|y\|_p - 1\right)^2 \\
&\qquad + 2\lambda\left(\|x\|_p - 1\right)\left(\|y\|_p - 1\right)\|x\|_p^{-1}\|y\|_p^{1-p} \\
&\qquad \cdot \left\{x_1y_1|y_1|^{p-2} + x_2y_2|y_2|^{p-2}\right\} \\
&\quad = \|x\|_p^2 + \|y\|_p^2 - 2 - \lambda\left(\|x\|_p - 1\right)^2 - \lambda\left(\|y\|_p - 1\right)^2 \\
&\qquad - 2\|x\|_p^{-1}\|y\|_p^{1-p}\left\{x_1y_1|y_1|^{p-2} + x_2y_2|y_2|^{p-2}\right\} \\
&\qquad \cdot \left\{\|x\|_p\|y\|_p - 1 - \lambda\left(\|x\|_p - 1\right)\left(\|y\|_p - 1\right)\right\}.
\end{aligned} \tag{23}
$$

It is obvious that

$$
\|x\|_p\|y\|_p - 1 - \lambda\left(\|x\|_p - 1\right)\left(\|y\|_p - 1\right) \geq 0 \tag{24}
$$

for any $\lambda \in [0, 1]$ and $\|x\|_p, \|y\|_p \geq 1$. Thus, we have from Hölder's inequality that

$$
\begin{aligned}
&V(x, y) - V(Tx, Ty) - \lambda V(x - Tx, y - Ty) \\
&\quad \geq \|x\|_p^2 + \|y\|_p^2 - 2 - \lambda\left(\|x\|_p - 1\right)^2 - \lambda\left(\|y\|_p - 1\right)^2 \\
&\qquad - 2\|x\|_p^{-1}\|y\|_p^{1-p}\|x\|_p\|y\|_p^{p-1} \\
&\qquad \cdot \left\{\|x\|_p\|y\|_p - 1 - \lambda\left(\|x\|_p - 1\right)\left(\|y\|_p - 1\right)\right\} \\
&\quad = \|x\|_p^2 + \|y\|_p^2 - 2 - \lambda\left(\|x\|_p - 1\right)^2 - \lambda\left(\|y\|_p - 1\right)^2 \\
&\qquad - 2\left\{\|x\|_p\|y\|_p - 1 - \lambda\left(\|x\|_p - 1\right)\left(\|y\|_p - 1\right)\right\} \\
&\quad = \|x\|_p^2 + \|y\|_p^2 - 2 - \lambda \\
&\qquad \cdot \left\{\|x\|_p^2 - 2\|x\|_p + 1 + \|y\|_p^2 - 2\|y\|_p + 1\right\} \\
&\qquad - 2\|x\|_p\|y\|_p + 2 + 2\lambda\left\{\|x\|_p\|y\|_p - \|x\|_p - \|y\|_p + 1\right\} \\
&\quad = \left(\|x\|_p - \|y\|_p\right)^2 - \lambda\left(\|x\|_p - \|y\|_p\right)^2 \\
&\quad = (1 - \lambda)\left(\|x\|_p - \|y\|_p\right)^2 \geq 0, \quad \text{for any } \lambda \in [0, 1] .
\end{aligned} \tag{25}
$$

That is, the inequality (1) holds.

It is clear that if $\|x\|_p, \|y\|_p \leq 1$ then inequality (1) holds. Therefore, from Cases (a), (b), and (c), we obtain the conclusion that T is V-strongly nonexpansive for any $\lambda \in (0, 1]$. □

Remark 10. When $p = 1$, we have given the result in [5]. When $p = 2$, we already know that E is a Hilbert space and a V-strongly nonexpansive mapping T is nonexpansive.

Theorem 11. *There exists a V-strongly nonexpansive mapping T with a nonempty subset of fixed points such that T is not nonexpansive for some Banach space.*

Proof. It is enough to show that the V-strongly nonexpansive mapping which is given in the previous proposition is not nonexpansive.

Let $x = (0,1) \in E$. Suppose that $y = (y_1, y_2)$ satisfies that $\|y\|_p^p = |y_1|^p + |y_2|^p > 1$ and $0 < y_1, y_2 < 1$. Then $Ty = \|y\|_p^{-1} y$. Let $h = (y_2/y_1)$ and $t = \|y\|_p^{-1} y_1 - y_1$. We have that $t < 0$ and $\|y\|_p^{-1} y_2 - y_2 = ht < 0$. Then we obtain that $Ty = (\|y\|_p^{-1} y_1, \|y\|_p^{-1} h y_1)$. Then, we have that

$$\begin{aligned} \|Tx - Ty\|_p^p &= \left\| \left(-\|y\|_p^{-1} y_1, 1 - \|y\|_p^{-1} h y_1 \right) \right\|^p \\ &= \left| -\|y\|_p^{-1} y_1 \right|^p + \left| 1 - \|y\|_p^{-1} h y_1 \right|^p \\ &= \left(\|y\|_p^{-1} y_1 \right)^p + \left(1 - \|y\|_p^{-1} h y_1 \right)^p \\ &= (y_1 + t)^p + (1 - h(y_1 + t))^p, \end{aligned} \tag{26}$$

and since $\|x - y\|_p^p = y_1^p + (1 - hy_1)^p$, we have that

$$\begin{aligned} &\|Tx - Ty\|_p^p - \|x - y\|_p^p \\ &\quad = (y_1 + t)^p - y_1^p + (1 - h(y_1 + t))^p - (1 - hy_1)^p. \end{aligned} \tag{27}$$

Therefore, we will show that

$$\begin{aligned} &\|Tx - Ty\|_p^p - \|x - y\|_p^p > 0 \\ &\Longleftrightarrow (y_1 + t)^p - y_1^p + (1 - h(y_1 + t))^p - (1 - hy_1)^p > 0 \\ &\Longleftrightarrow \left\{ (y_1 + t)^p - y_1^p \right\} t^{-1} \\ &\qquad + \left\{ (1 - h(y_1 + t))^p - (1 - hy_1)^p \right\} t^{-1} < 0, \end{aligned} \tag{28}$$

since $t < 0$. Let h be fixed. As $\|y\|_p^p = y_1^p + (hy_1)^p \to 1$, $t = \|y\|_p^{-1} y_1 - y_1 \to 0$. Thus, we have for a sufficiently small $|t|$ that

$$\begin{aligned} &\left\{ (y_1 + t)^p - y_1^p \right\} t^{-1} \\ &\quad + \left\{ (1 - h(y_1 + t))^p - (1 - hy_1)^p \right\} t^{-1} < 0 \\ &\Longleftrightarrow p y_1^{p-1} - ph(1 - hy_1)^{p-1} < 0. \end{aligned} \tag{29}$$

It is trivial that

$$\begin{aligned} p y_1^{p-1} - ph(1 - hy_1)^{p-1} < 0 &\Longleftrightarrow y_1^{p-1} < h(1 - hy_1)^{p-1} \\ &\Longleftrightarrow y_1^p < y_2 (1 - y_2)^{p-1}. \end{aligned} \tag{30}$$

Let $p = 3/2$. For $y = (0.2, 0.95)$, we have that

$$y_1^p = (0.2)^{3/2} < 0.95\,(0.05)^{1/2} = y_2 (1 - y_2)^{p-1}. \tag{31}$$

We obtain that $\|y\|_p^p = (0.2)^{3/2} + (0.95)^{3/2} > 1$ and that

$$\begin{aligned} \|Tx - Ty\|_p^p &= \|y\|_p^{-p} \left\{ (0.2)^{3/2} + \left(\|y\|_p - 0.95 \right)^{3/2} \right\} \\ &> (0.2)^{3/2} + (0.05)^{3/2} = \|x - y\|_p^p. \end{aligned} \tag{32}$$

Therefore, we obtain the conclusion. □

We remark that the symbols $x_n \to u$ and $x_n \rightharpoonup u$ mean that $\{x_n\}$ converges strongly and weakly to u, respectively. We will introduce the following important lemmas for proofs of our theorems.

Lemma 12. *(a) For all $x, y, z \in E$,*

$$\begin{aligned} V(x, y) &\le V(x, y) + V(y, z) \\ &= V(x, z) - 2\langle x - y, Jy - Jz \rangle. \end{aligned} \tag{33}$$

(b) Let $\{x_n\}$ be a sequence in E such that there exists $\lim_{n\to\infty} V(x_n, p) < \infty$ for some $p \in E$; then $\{x_n\}$ is bounded.

Lemma 13 (see [3]). *Let E be a smooth and uniformly convex Banach space and C a nonempty, convex, and closed subset of E. Suppose that $T : C \to E$ satisfies*

$$V(Tx, Ty) \le V(x, y) \quad \forall x, y \in C. \tag{34}$$

If a weakly convergent sequence $\{z_n\}_{n\ge1} \subset C$ satisfies that $\lim_{n\to\infty} V(Tz_n, z_n) = 0$, it holds that $z_n \rightharpoonup z \in F(T)$.

Theorem 14 (see [1, 11]). *Let Y be a compact subset of a topological vector space E and let X be a convex subset of Y. Let $A : X \to 2^Y$ be an operator such that, for each $y \in Y$, $A^{-1}y$ is convex. Suppose that $B : X \to 2^Y$ satisfies the following:*

(1) $Bx \subset Ax$ for each $x \in X$,

(2) $B^{-1}y \ne \emptyset$ for each $y \in Y$,

(3) Bx is open for each $x \in X$.

Then there exists a point $x_0 \in X$ such that $x_0 \in Ax_0$.

Lemma 15 (see [12]). *Let $s > 0$ and let E be a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, such that*

$$\|x + y\|^2 \ge \|x\|^2 + 2\langle y, j \rangle + g(\|y\|) \tag{35}$$

for all $x, y \in \{z \in E : \|z\| \le s\}$ and $j \in Jx$.

Lemma 16 (see [13]). *Let E be a smooth and uniformly convex Banach space. Then, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and, for each real number $r > 0$,*

$$0 \le g(\|x - y\|) \le V(x, y) \tag{36}$$

for all $x, y \in B_r = \{z \in E : \|z\| \le r\}$.

Lemma 17 (see [13]). *Let E be a smooth and uniformly convex Banach space and $\{y_n\}$ and $\{z_n\}$ in E. If $\lim_{n\to\infty} V(y_n, z_n) = 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $\{y_n - z_n\} \to 0$.*

3. Main Results

In this section, we prove a weak convergence theorem and strong convergence theorems for finding fixed points of a V-strongly nonexpansive mapping T in Banach spaces, and then we show the existence theorem for fixed points of T with a dissipative property (cf. [10]).

Theorem 18. *Let E be a smooth and uniformly convex Banach space and C a nonempty, closed, and convex subset of E. Suppose that a mapping $T : C \to C$ is V-strongly nonexpansive with λ and that $F(T) \neq \emptyset$. One defines a Mann iterative sequence $\{x_n\}$ as follows: for any $x_1 \in C$ and $n \geq 1$,*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) T x_n, \tag{37}$$

where $\{\beta_n\} \subset (0,1)$ and $\lim_{n\to\infty} \beta_n = 0$. Then $x_n \rightharpoonup p_0$ for some $p_0 \in F(T)$.

Proof. Suppose that $p \in F(T)$. Then we have from the convexity of V that

$$\begin{aligned} V(x_{n+1}, p) &= V(\beta_n x_n + (1-\beta_n) T x_n, p) \\ &\leq \beta_n V(x_n, p) + (1-\beta_n) V(T x_n, p) \\ &= \beta_n V(x_n, p) + (1-\beta_n) V(T x_n, T p). \end{aligned} \tag{38}$$

Since T is V-strongly nonexpansive with λ, we have that

$$\begin{aligned} & V(x_{n+1}, p) \\ &\quad \leq \beta_n V(x_n, p) + (1-\beta_n) \\ &\qquad \cdot \{V(x_n, p) - \lambda V((I-T) x_n, (I-T) p)\} \\ &\quad = V(x_n, p) - (1-\beta_n) \lambda V(x_n - T x_n, 0) \\ &\quad \leq V(x_n, p). \end{aligned} \tag{39}$$

Hence, we have $\lim_{n\to\infty} V(x_n, p) = \alpha < \infty$. From Lemma 12 (b), $\{x_n\}$ is bounded. Furthermore, we have that

$$(1-\beta_n) \lambda V(x_n - T x_n, 0) \leq V(x_n, p) - V(x_{n+1}, p). \tag{40}$$

Since $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \{V(x_n, p) - V(x_{n+1}, p)\} = 0$, we obtain that

$$\lim_{n\to\infty} V(x_n - T x_n, 0) = \lim_{n\to\infty} \|x_n - T x_n\|^2 = 0. \tag{41}$$

This means that $\{x_n - T x_n\}$ converges strongly to 0. Hence, $\{T x_n\}$ is also bounded, and there exists $M > 0$ such that $\|x_n\|, \|T x_n\| \leq M - \|p\|$ for all $n \geq 1$.

On the other hand, we have from Lemma 12 (a) that

$$\begin{aligned} 0 &\leq V(x_n, T x_n) \\ &= V(x_n, p) - V(T x_n, p) - 2\langle x_n - T x_n, J T x_n - J p\rangle \\ &\leq V(x_n, p) - V(T x_n, p) + 2\|x_n - T x_n\|(\|T x_n\| + \|p\|) \\ &\leq V(x_n, p) - V(T x_n, p) + 2M\|x_n - T x_n\| \\ &= \|x_n\|^2 - \|T x_n\|^2 - 2\langle x_n - T x_n, J p\rangle + 2M\|x_n - T x_n\| \\ &= (\|x_n\| - \|T x_n\|)(\|x_n\| + \|T x_n\|) \\ &\quad - 2\langle x_n - T x_n, J p\rangle + 2M\|x_n - T x_n\| \\ &\leq \|x_n - T x_n\|(\|x_n\| + \|T x_n\| + 2M) - 2\langle x_n - T x_n, J p\rangle. \end{aligned} \tag{42}$$

Hence, we obtain that $\lim_{n\to\infty} V(x_n, T x_n) = \lim_{n\to\infty} V(T x_n, x_n) = 0$. From Lemma 13, there exists a point $p_0 \in F(T)$ such that $x_n \rightharpoonup p_0$ and $T x_n \rightharpoonup p_0$. □

The duality mapping J of a Banach space E with Gâteaux differentiable norm is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ in E implies that $\{J x_n\}$ converges weak star to $J x$ in E^* (cf. [14]). This happens, for example, if E is a Hilbert space, or finite-dimensional and smooth, or l^p if $1 < p < \infty$ (cf. [15]). Next we prove a strong convergence theorem.

Theorem 19. *Let E be a reflexive, smooth, and strictly convex Banach space. Suppose that the duality mapping J of E is weakly sequentially continuous. Suppose that C is a nonempty, closed, and convex subset of E, $T : C \to C$ is V-strongly nonexpansive with λ, and $F(T) \neq \emptyset$. One defines a Mann iterative sequence $\{x_n\}$ as follows: for any $x_1 \in C$ and $n \geq 1$,*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) T x_n, \tag{43}$$

where $\{\beta_n\} \subset (0,1)$ and $\lim_{n\to\infty} \beta_n = 0$. If T satisfies that

$$\langle x, J T x\rangle \leq 0 \quad \forall x \in C, \tag{44}$$

then $x_n \to p_0$ and $T x_n \to p_0$ for some $p_0 \in F(T)$.

Proof. As in the proof of Theorem 18, we obtain that $\lim_{n\to\infty} V(x_n, T x_n) = 0$ and $x_n \rightharpoonup p_0$ and $T x_n \rightharpoonup p_0$ for some $p_0 \in F(T)$. Furthermore, from Lemma 12 (a), we have that

$$\begin{aligned} 0 &\leq V(x_n, p_0) + V(p_0, T x_n) \\ &= V(x_n, T x_n) - 2\langle x_n - p_0, J p_0 - J T x_n\rangle \\ &= V(x_n, T x_n) - 2\langle x_n - p_0, J p_0\rangle \\ &\quad + 2\langle x_n, J T x_n\rangle - 2\langle p_0, J T x_n\rangle. \end{aligned} \tag{45}$$

Hence, the assumptions imply that

$$V(x_n, p_0) \longrightarrow 0, \quad V(p_0, T x_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{46}$$

From Lemma 17, we have the conclusion that $x_n \to p_0$ and $T x_n \to p_0$. □

Condition (44) is a definition of a linear dissipative mapping T (cf. [16]). Moreover, we give a definition of a J-dissipative mapping for nonlinear mappings in a Banach space.

Definition 20. Let J be a single-valued duality mapping on E and let C be a nonempty subset of E. Then a mapping $T : C \to E$ is called J-dissipative if it holds that

$$\langle x - y, JTx - JTy \rangle \le 0 \tag{47}$$

for all $x, y \in C$.

In a Hilbert space, such a mapping T is called dissipative. In Banach spaces, we remark that the J-dissipative mapping is not equal to the dissipative mapping (cf. [17]). Next we give a characterization of J-dissipative mappings by using $V(\cdot,\cdot)$.

Theorem 21. *Let E be a smooth Banach space, C a nonempty subset of E, and $T : C \to E$ a mapping. Then, the following are equivalent.*

(a) *T is J-dissipative.*

(b) *For all $x, y \in C$,*

$$V(x, Ty) + V(y, Tx) \le V(x, Tx) + V(y, Ty). \tag{48}$$

Proof. For any $x, y \in C$,

$$\langle x - y, JTx - JTy \rangle \le 0 \tag{49}$$

is equal to

$$\begin{aligned} &-2\langle x, JTy\rangle - 2\langle y, JTx\rangle \le -2\langle x, JTx\rangle - 2\langle y, JTy\rangle, \\ &-2\langle x, JTy\rangle - 2\langle y, JTx\rangle + \|x\|^2 + \|Ty\|^2 + \|y\|^2 + \|Tx\|^2 \\ &\quad \le -2\langle x, JTx\rangle - 2\langle y, JTy\rangle + \|x\|^2 + \|Tx\|^2 \\ &\qquad + \|y\|^2 + \|Ty\|^2. \end{aligned} \tag{50}$$

From the definition of V, this inequality is equivalent to

$$V(x, Ty) + V(y, Tx) \le V(x, Tx) + V(y, Ty). \tag{51}$$

□

Furthermore, we have the following result by this theorem.

Lemma 22. *Suppose that E is a smooth and strictly convex Banach space and that $C \subset E$ is a nonempty convex subset. Assume that a mapping $T : C \to E$ is J-dissipative. If there are fixed points of T, then $F(T)$ is singleton.*

Proof. Assume that there exist p_0 and q_0 such that $Tp_0 = p_0$ and $Tq_0 = q_0$. Since T is J-dissipative, we have by Theorem 21 that

$$\begin{aligned} 0 &\le V(p_0, Tq_0) + V(q_0, Tp_0) \\ &\le V(p_0, Tp_0) + V(q_0, Tq_0) \\ &= V(p_0, p_0) + V(q_0, q_0) = 0. \end{aligned} \tag{52}$$

Thus, we have that $V(p_0, q_0) = V(q_0, p_0) = 0$. This implies that

$$\begin{gathered} 0 \le (\|p_0\| - \|q_0\|)^2 \le V(p_0, q_0) = 0, \\ \|p_0\| = \|q_0\|. \end{gathered} \tag{53}$$

Furthermore, we have

$$\begin{aligned} V(p_0, q_0) &= \|p_0\|^2 + \|q_0\|^2 - 2\langle p_0, Jq_0\rangle \\ &= \|p_0\|^2 + \|p_0\|^2 - 2\langle p_0, Jq_0\rangle = 0, \end{aligned} \tag{54}$$

and we have $\|p_0\|^2 = \langle p_0, Jq_0\rangle$. Since E is strictly convex and J is one-to-one, we obtain that $p_0 = q_0$. □

We give a result before proving an existence theorem for fixed points.

Theorem 23 (see [10]). *Let E be a smooth and uniformly convex Banach space, and let $T : E \to E$ be a V-strongly nonexpansive mapping with λ. Then, one has that*

$$\lim_{\|x-y\| \to 0} \|Tx - Ty\| = 0, \tag{55}$$

for $\|x\|, \|y\|, \|Tx\|, \|Ty\| \le r$, where $r > 0$.

Proof. Since T is a V-strongly nonexpansive with λ, we have

$$\begin{aligned} 0 &\le V(Tx, Ty) + \lambda V(x - Tx, y - Ty) \\ &\le V(x, y) \\ &= \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle \\ &= \|x\|^2 - \|y\|^2 - 2\langle x - y, Jy\rangle \\ &\le \|x - y\|(\|x\| + \|y\| + 2\|y\|) \\ &= \|x - y\|(\|x\| + 3\|y\|), \quad \text{for any } x, y \in E. \end{aligned} \tag{56}$$

Thus, we obtain, for x, y with $\|x\|, \|y\| \le r$,

$$\begin{gathered} V(Tx, Ty) \longrightarrow 0, \\ V(x - Tx, y - Ty) \longrightarrow 0 \quad \text{as } \|x - y\| \longrightarrow 0. \end{gathered} \tag{57}$$

From Lemma 16, we have that

$$0 \le g(\|Tx - Ty\|) \le V(Tx, Ty). \tag{58}$$

Therefore, we have from (57) that $\lim_{\|x-y\|\to 0} g(\|Tx - Ty\|) = 0$. From the definition of g, we obtain that

$$\lim_{\|x-y\| \to 0} \|Tx - Ty\| = 0. \tag{59}$$

□

Remark 24. If $x \in E$ satisfies that $\|Tx\| < r_0$ for $r_0 > 0$, the (57) implies that $\|Ty\| < r_0 + 1$ for y in the neighborhood of x.

We will prove the following existence theorem by using Theorem 14.

Theorem 25. *Let E be a reflexive, strictly convex, and smooth Banach space and C a nonempty, bounded, closed, and convex subset of E. Suppose $T : C \to C$ is a V-strongly nonexpansive and J-dissipative mapping. Then, there exists a unique fixed point of T.*

Proof. At first, we will show that there exists $y_0 \in C$ such that

$$\{x \in C : V(x, Tx) < V(y_0, Tx)\} = \emptyset. \tag{60}$$

Assume that, for all $y \in C$,

$$\{x \in C : V(x, Tx) < V(y, Tx)\} \neq \emptyset. \tag{61}$$

Let $Ax = \{y \in C : V(x, Ty) < V(y, Ty)\}$ and $Bx = \{y \in C : V(x, Tx) < V(y, Tx)\}$ for all $x \in C$. Then, from the assumption, $B^{-1}y$ is nonempty for all $y \in C$. Since T is J-dissipative, Theorem 21 implies that

$$V(x, Ty) - V(y, Ty) \leq V(x, Tx) - V(y, Tx) \tag{62}$$

for all $y \in Bx$. This means that $Bx \subset Ax$ for any $x \in C$. For any $y \in C$, let $v_j \in A^{-1}y$ with $j \in \{1, 2, \dots, n\}$, and suppose that $v = \sum_{j=1}^{n} \alpha_j v_j$ and $\sum_{j=1}^{n} \alpha_j = 1$ with $\alpha_j > 0$. From the convexity of V, we have

$$\begin{aligned} V(v, Ty) &= V\left(\sum_{j=1}^{n} \alpha_j v_j, Ty\right) \leq \sum_{j=1}^{n} \alpha_j V(v_j, Ty) \\ &\leq \sum_{j=1}^{n} \alpha_j V(y, Ty) = V(y, Ty). \end{aligned} \tag{63}$$

Thus, we obtain that $A^{-1}y$ is convex for all $y \in C$. Since it is obvious that Bx is open for each $x \in C$, Theorem 14 implies that there exists a point $x_0 \in C$ such that $x_0 \in Ax_0$. This means that

$$V(x_0, Tx_0) < V(x_0, Tx_0). \tag{64}$$

This is a contradiction. Thus, we have for some $y_0 \in C$ that

$$\{x \in C : V(x, Tx) < V(y_0, Tx)\} = \emptyset. \tag{65}$$

This means that there exists $y_0 \in C$ such that

$$V(y_0, Tx) \leq V(x, Tx) \tag{66}$$

for all $x \in C$.

Furthermore, we will show $V(y_0, Ty_0) \leq V(x, Ty_0)$ for all $x \in C$ if y_0 satisfies (66). Let $y_t = (1-t)y_0 + tx$ for any $t \in (0, 1)$ and $x \in C$. Since C is convex, then $y_t \in C$. Thus, we obtain that

$$\begin{aligned} V(y_0, Ty_t) &\leq V(y_t, Ty_t) \\ &= V((1-t)y_0 + tx, Ty_t). \end{aligned} \tag{67}$$

From the convexity of $V(\cdot, y)$ for $y \in C$,

$$V(y_0, Ty_t) \leq (1-t)V(y_0, Ty_t) + tV(x, Ty_t) \tag{68}$$

and we have $V(y_0, Ty_t) \leq V(x, Ty_t)$. From the definition of $V(\cdot, \cdot)$, we have that

$$\begin{aligned} &|V(x, Ty_t) - V(x, Ty_0)| \\ &\quad = \left|\|Ty_t\|^2 - \|Ty_0\|^2 - 2\langle x, JTy_t - JTy_0\rangle\right| \\ &\quad \leq (\|Ty_t\| + \|Ty_0\|)\|Ty_t - Ty_0\| + 2\|x\|\|JTy_t - JTy_0\|. \end{aligned} \tag{69}$$

Therefore, we have, by Theorem 23 and the continuity of J on a smooth Banach space, that $\lim_{t \to 0+} V(x, Ty_t) = V(x, Ty_0)$ and

$$\begin{aligned} V(y_0, Ty_0) &= \lim_{t \to 0+} V(y_0, Ty_t) \\ &\leq \lim_{t \to 0+} V(x, Ty_t) = V(x, Ty_0) \end{aligned} \tag{70}$$

for all $x \in C$. Letting $x = Ty_0$, we have that

$$V(y_0, Ty_0) \leq V(Ty_0, Ty_0) = 0. \tag{71}$$

Hence, $V(y_0, Ty_0) = 0$. This implies that

$$\|y_0\|^2 + \|Ty_0\|^2 = 2\langle y_0, JTy_0\rangle \leq 2\|y_0\|\|Ty_0\|, \tag{72}$$

and then we obtain that

$$(\|y_0\| - \|Ty_0\|)^2 \leq 0. \tag{73}$$

Thus, we have $\|y_0\| = \|Ty_0\|$ and we have by (72) that $\|y_0\|^2 = \langle y_0, JTy_0\rangle$. Since J is one-to-one on a strictly convex Banach space, $JTy_0 = Jy_0$ implies that $Ty_0 = y_0$. Therefore, we have the conclusion. □

Finally, we will prove a strong convergence theorem for finding fixed points of a V-strongly nonexpansive mapping T in a Banach space, without the assumption that $F(T) \neq \emptyset$.

Theorem 26. *Let E be a smooth and uniformly convex Banach space, and let C be a nonempty, compact, and convex subset of E. Suppose that $T : C \to C$ is J-dissipative and V-strongly nonexpansive with λ. One defines a Mann iterative sequence $\{x_n\}$ as follows: for any $x_1 \in C$ and $n \geq 1$,*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) Tx_n, \tag{74}$$

where $\{\beta_n\} \subset (0, 1)$ and $\lim_{n \to \infty} \beta_n = 0$. Then, there exists a unique fixed point $p_0 \in C$ such that $x_n \to p_0$ and $Tx_n \to p_0$.

Proof. From Theorem 25, we have that $F(T) \neq \emptyset$. As in the proof of Theorem 18, we obtain that $\lim_{n \to \infty} V(x_n, Tx_n) = 0$ and that there exists a point $p_0 \in F(T)$ such that $x_n \rightharpoonup p_0$ and $Tx_n \rightharpoonup p_0$. Since T is J-dissipative, Theorem 21 implies that

$$0 \leq V(x_n, Tp_0) + V(p_0, Tx_n) \leq V(x_n, Tx_n) + V(p_0, Tp_0). \tag{75}$$

From $Tp_0 = p_0$, we have for $n \geq 1$ that

$$\begin{aligned} 0 &\leq V(x_n, p_0) + V(p_0, Tx_n) \\ &\leq V(x_n, Tx_n) + V(p_0, p_0) = V(x_n, Tx_n). \end{aligned} \tag{76}$$

Since $\lim_{n\to\infty} V(x_n, Tx_n) = 0$, we have that

$$\lim_{n\to\infty} V(x_n, p_0) = \lim_{n\to\infty} V(p_0, Tx_n) = 0. \tag{77}$$

By Lemma 17, we obtain that $x_n \to p_0$ and $Tx_n \to p_0$. We have the conclusion. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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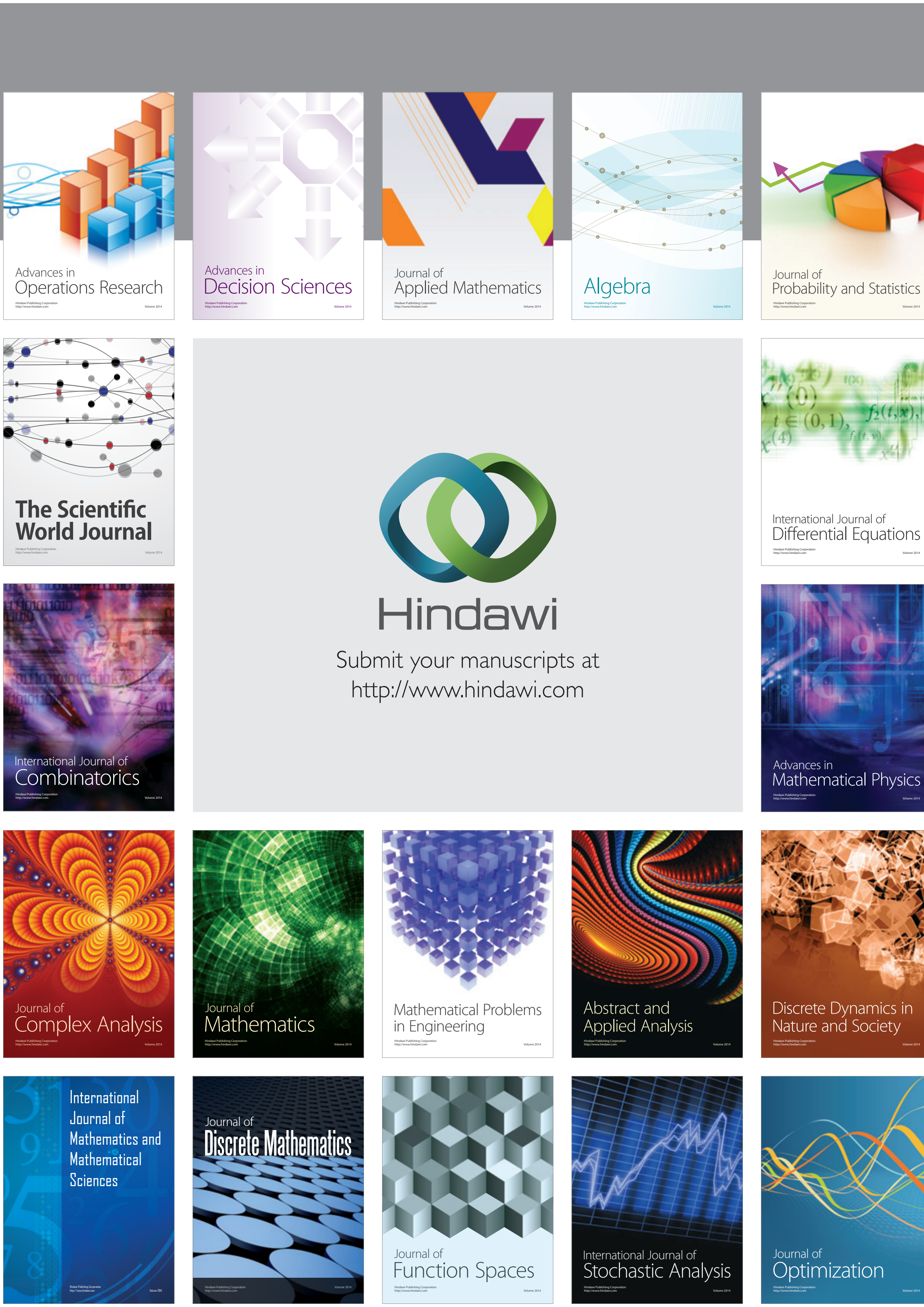

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