Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 951021, 6 pages
http://dx.doi.org/10.1155/2015/951021

Research Article

φ-Multipliers on Banach Algebras and Topological Modules

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Received 26 October 2014; Accepted 29 December 2014

Academic Editor: Antonio M. Peralta

We prove some results concerning Arens regularity and amenability of the Banach algebra $M_\varphi(A)$ of all φ-multipliers on a given Banach algebra A. We also consider φ-multipliers in the general topological module setting and investigate some of their properties. We discuss the φ-strict and φ-uniform topologies on $M_\varphi(A)$. A characterization of φ-multipliers on $L_1(G)$-module $L_p(G)$, where G is a compact group, is given.

1. Introduction

The concept of a multiplier was introduced by Helgason [1] as follows. Let A be a commutative and semisimple Banach algebra and let $\Delta(A)$ be its maximal ideal space. Let $\widehat{A}$ denote the Gelfand representation of A as a subalgebra of the algebra of continuous functions on $\Delta(A)$. A bounded continuous function g on $\Delta(A)$ is a multiplier on A if $g\widehat{A} \subseteq \widehat{A}$. The general theory of multipliers on faithful Banach algebras was developed by Wang [2] and Birtel [3].

Recall that a mapping $T : A \to A$ is called a left (resp., right) multiplier on A if

$$T(xy) = T(x)\,y \quad (\text{resp., } T(xy) = xT(y)) \tag{1}$$

for all $x, y \in A$. We say T is a multiplier on A if it is both a left multiplier and a right multiplier on A.

We denote with $M(A)$ the algebra of all multipliers on A.

A Banach algebra A is called left (resp., right) faithful if, for all $x \in A$, $xA = \{0\}$ (resp., $Ax = \{0\}$) implies that $x = 0$; A is called faithful if it is both left and right faithful.

In [4] we generalized the concept of multipliers on faithful Banach algebras to φ-multipliers as follows. Let A be a Banach algebra and let $\varphi : A \to A$ be an algebra homomorphism. A linear continuous mapping $T : A \to A$ is called a left (resp., right) φ-multiplier on A if

$$T(xy) = T(x)\,\varphi(y) \quad (\text{resp., } T(xy) = \varphi(x)\,T(y)) \tag{2}$$

for all $x, y \in A$. We say T is a φ-multiplier on A if it is both a left φ-multiplier and a right φ-multiplier on A. We denote by $M_\varphi(A)$ (resp., $M_\varphi^l(A)$, $M_\varphi^r(A)$) the collection of all φ-multipliers (resp., left φ-multipliers, right φ-multipliers) on A.

It turns out that this concept is considerably more general than the concept of multipliers on Banach algebras. Also by using some well-known homomorphisms like Jordan homomorphism, spectrum preserving homomorphism, and idempotent preserving homomorphism, we can transfer these useful properties from homomorphism φ to the algebra of φ-multipliers.

In [4], we studied various properties of φ-multipliers, for instance, the faithfulness of the Banach algebra $M_\varphi(A)$ and the existence of a bounded approximate identity in the range of a φ-multiplier. Finally, as an example, we have characterized φ-multipliers on $L_1(G)$.

In Section 2 we are concerned by Arens regularity and amenability of the Banach algebra $M_\varphi(A)$ under some suitable conditions. We introduce the notion of Jordan φ-multiplier and prove that every Jordan φ-multiplier is a φ-multiplier whenever the range of φ is dense in the algebra.

In Section 3 we extend the notion of φ-multipliers on Banach algebras to topological modules and investigate some of their properties. We discuss the φ-strict and φ-uniform topologies on $M_\varphi(A)$ and apply our results to $L_1(G)$-module $L_p(G)$.

Let X be a topological vector space and let A be a topological algebra, both over the same field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). Then X is called a topological left A-module if it is a left A-module and the module multiplication $(a, x) \to a \cdot x$ from $A \times X$ into X is separately continuous. If $b(A)$ denotes the collection of all bounded sets in A, then module multiplication $(a, x) \to a \cdot x$ is called $b(A)$-hypocontinuous [5] if, given any neighborhood G of 0 in X and any $D \in b(A)$, there exists a neighborhood H of 0 in X such that $D \cdot H \subseteq G$. Clearly, joint continuity $\Rightarrow$ hypocontinuity $\Rightarrow$ separate continuity. A mapping ψ from a left A-module X into another left A-module Y is called an A-module homomorphism if $\psi(a \cdot x) = a \cdot \psi(x)$ for all $a \in A$ and $x \in X$.

2. Some Properties of φ-Multipliers on Banach Algebras

Let us start with the following result proved in [4].

Theorem 1 (see [4, Theorem 2.2]). *Let A be a faithful commutative Banach algebra and let φ be an idempotent homomorphism on A. Then $M_\varphi(A)$ is a Banach algebra. Moreover, if $A^2 = A$ and $\varphi \circ T = T \circ \varphi$ for all $T \in M_\varphi(A)$, then $M_\varphi(A)$ is a faithful commutative Banach algebra.*

Definition 2. Let A be a Banach algebra and let φ be a homomorphism from A to A. The mapping $T : A \to A$ is called a left (resp., right) Jordan φ-multiplier on A if for all $x \in A$

$$T\left(x^2\right) = T(x)\varphi(x) \quad \left(\text{resp., } T\left(x^2\right) = \varphi(x)T(x)\right). \tag{3}$$

T is called a Jordan φ-multiplier on A if it is both a left Jordan φ-multiplier and a right Jordan φ-multiplier on A.

Theorem 3. *Let A be a faithful commutative Banach algebra and let φ be a homomorphism from A to A with dense range. Then T is a φ-multiplier if and only if T is a Jordan φ-multiplier.*

Proof. It is clear that every φ-multiplier is a Jordan φ-multiplier. Conversely, suppose T is a Jordan φ-multiplier. Then

$$\begin{aligned} T\left((x+y)^2\right) &= \varphi(x+y)T(x+y) \\ &= \varphi(x)T(x) + \varphi(x)T(y) \\ &\quad + \varphi(y)T(x) + \varphi(y)T(y) \end{aligned} \tag{4}$$

for all $x, y \in A$.

On the other hand, we have

$$\begin{aligned} T\left((x+y)^2\right) &= T\left(x^2 + 2xy + y^2\right) \\ &= \varphi(x)T(x) + 2T(xy) + \varphi(y)T(y). \end{aligned} \tag{5}$$

Comparing (4), (5) we obtain

$$2T(xy) = \varphi(x)T(y) + \varphi(y)T(x). \tag{6}$$

From (6) and using commutativity of A, for each sequence $\{z_n\}_{n=1}^{\infty} \subset A$ we have

$$\begin{aligned} 2T(xyz_n) &= \varphi(y)T(xz_n) + \varphi(xz_n)T(y) \\ &= \frac{\varphi(y)\left[\varphi(x)T(z_n) + \varphi(z_n)T(x)\right]}{2} \\ &\quad + \varphi(xz_n)T(y), \end{aligned} \tag{7}$$

so we have

$$\begin{aligned} 2T(xyz_n) &= \left[\varphi(y)\varphi(x)T(z_n) + \varphi(y)\varphi(z_n)T(x)\right. \\ &\quad \left. + 2\varphi(x)\varphi(z_n)T(y)\right] \cdot 2^{-1}; \end{aligned} \tag{8}$$

similarly by using (6) we have

$$\begin{aligned} 2T(xyz_n) &= \left[\varphi(x)\varphi(y)T(z_n) + \varphi(x)\varphi(z_n)T(y)\right. \\ &\quad \left. + 2\varphi(y)\varphi(z_n)T(x)\right] \cdot 2^{-1}; \end{aligned} \tag{9}$$

comparing (8), (9) we obtain

$$\lim_{n\to\infty} \varphi(x)\varphi(z_n)T(y) = \lim_{n\to\infty} \varphi(y)\varphi(z_n)T(x) \tag{10}$$

for all $x, y, z_n \in A$. Since φ has dense range and A is a faithful commutative Banach algebra, we have

$$\varphi(x)T(y) = T(x)\varphi(y); \tag{11}$$

hence T is a φ-multiplier. □

We mention that Theorem 3 holds for certain noncommutative cases, but not in general. For instance, Zalar has proved in [6] that any left (right) Jordan multiplier on a 2-torsion free semiprime ring is a left (right) multiplier. Vukman [7] has shown that an additive map $\varphi : R \to R$, where R is a 2-torsion free semiprime ring, with the property that $2\varphi(a^2) = a\varphi(a) + \varphi(a)a$ for all $a \in A$, is a multiplier.

The following example shows that, in general, the above theorem need not hold for noncommutative Banach algebras.

Example 4. Consider the subalgebra

$$\mathscr{A} = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\} \tag{12}$$

of the algebra of all 3×3 matrices. It is obvious that $\mathscr{A}$ is a Banach algebra with respect to the norm given by

$$\left\| \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \right\| = |a| + |b| + |c| + |d|. \tag{13}$$

Let

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

and define a continuous linear map $\varphi : \mathscr{A} \to \mathscr{A}$ by $\varphi(A) = AX + XA$. By a straightforward calculation one can prove that

$$BAX + XAB = BXA + AXB, \quad A, B \in \mathscr{A}. \tag{15}$$

If $'\circ'$ denotes the Jordan product $A \circ B = AB + BA$, then we have $\varphi(A \circ B) = A \circ \varphi(B)$ for each $A, B \in \mathscr{A}$ and hence φ is a right Jordan multiplier. Also $A \circ \varphi(B) = \varphi(1)$ for all $A, B \in \mathscr{A}$ with $A \circ B = 1$. If we consider

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

then $\varphi(A) \neq 0$ and $A\varphi(1) = 0$, where $0, 1$ are the zero matrix and the identity matrix, respectively. Thus φ is not a right multiplier.

Lemma 5 (see [8]). *Let A be an amenable Banach algebra and let ψ be a continuous homomorphism of A onto a dense subalgebra of a Banach algebra B. Then B is amenable.*

Theorem 6. *(a) Let A be a unital commutative Banach algebra and let φ be an idempotent homomorphism on A such that φ commutes with each $S \in M_\varphi(A)$. If A is Arens regular then $M_\varphi(A)$ is Arens regular.*

(b) Let A be a commutative Banach algebra and let φ be as in part (a). If A is amenable then $M_\varphi(A)$ is amenable.

Proof. (a) Define $\mu : A \to M_\varphi(A)$ by $\mu(a) = {}_a\varphi$, where ${}_a\varphi(b) = \varphi(ab)$. The homomorphism μ is onto. Namely, if $S \in M_\varphi(A)$, then $S = {}_{S(e)}\varphi$. Of course, $\mu^{**} : A^{**} \to (M_\varphi(A))^{**}$ has the same property, as well. Let $F', G' \in (M_\varphi(A))^{**}$. Then there exist $F, G \in A^{**}$ such that $\mu^{**}(F) = F'$, $\mu^{**}(G) = G'$. Let $\circ$ and $\circ'$ be two Arens multiplications on the second dual A^{**}. Thus,

$$\begin{aligned} F' \circ G' &= \mu^{**}(F) \circ \mu^{**}(G) = \mu^{**}(F \circ G) = \mu^{**}\left(F \circ' G\right) \\ &= \mu^{**}(F) \circ' \mu^{**}(G) = F' \circ' G'. \end{aligned} \tag{17}$$

(b) Let $T \in M_\varphi(A)$ and $\{e_\alpha : \alpha \in I\}$ be a bounded approximate identity in A (see [9]). A simple computation shows that $T = \lim_\alpha {}_{T(e_\alpha)}\varphi$, which means $\overline{\mu(A)} = M_\varphi(A)$. So by Lemma 5 we conclude that $M_\varphi(A)$ is amenable. □

Theorem 7. *Let A be a unital Banach algebra and let $\varphi : A \to A$ be a spectrum preserving homomorphism with dense range. Then each $T \in M_\varphi(A)$ is spectrum preserving.*

Proof. Let $a \in A$ and $\lambda \notin \sigma(T(a))$. Since φ has dense range, there exists a sequence $\{c_n\}_n \in A$ such that $(T(a) - \lambda)\lim_n \varphi(c_n) = 1$. Thus

$$\begin{aligned} &(\varphi(a) - \lambda)\lim_n T(c_n) \\ &\quad = \lim_n (\varphi(a) - \lambda) T(c_n) = \lim_n \varphi(a - \lambda) T(c_n) \\ &\quad = \lim_n T(a - \lambda)\varphi(c_n) = (T(a) - \lambda)\lim_n \varphi(c_n) \\ &\quad = 1. \end{aligned} \tag{18}$$

Similarly $\lim_n T(c_n)(\varphi(a) - \lambda) = 1$. Thus $\lambda \notin \sigma(\varphi(a))$. Since $\sigma(\varphi(a)) = \sigma(a)$, we have $\lambda \notin \sigma(a)$.

Now, let $\lambda \notin \sigma(a)$. Then there exists $b \in A$ such that $(a - \lambda)b = 1$. Thus

$$(T(a) - \lambda)\varphi(b) = T(a - \lambda)\varphi(b) = T((a - \lambda)b) = 1. \tag{19}$$

Similarly $\varphi(b)(T(a) - \lambda) = 1$. Hence $\lambda \notin \sigma(T(a))$, which means $\sigma(T(a)) = \sigma(a)$. □

3. φ-Multipliers on Topological Modules and Their Properties

Now, we consider φ-multipliers in the general topological module setting and investigate some of their properties.

Definition 8. Let A be a topological algebra and let X, Y be two topological A-bimodules and let φ be a nonzero and continuous idempotent A-module homomorphism on X. A linear and bounded mapping $T : X \to Y$ is called a left (resp., right) φ-multiplier if $T(a \cdot x) = T(\varphi(x)) \cdot a$ (resp., $T(x \cdot a) = a \cdot T(\varphi(x))$) for all $a \in A, x \in X$. We say T is a φ-multiplier if it is both a left φ-multiplier and a right φ-multiplier.

We denote by $M_\varphi(X, Y)$ (resp., $M_\varphi^l(X, Y), M_\varphi^r(X, Y)$) the collection of all φ-multipliers (resp., left φ-multipliers, right φ-multipliers).

It is easy to check that $\varphi \in M_\varphi(X, X)$. So $M_\varphi(X, X) \neq \{0\}$.

Example 9. Let A be a topological algebra, X an A-bimodule, and φ an idempotent A-module homomorphism on X. For each $a \in A$ the mapping ${}_a\varphi : X \to X$ defined by ${}_a\varphi(x) = a \cdot \varphi(x)$ is a left φ-multiplier on X.

Proof. Let $a, b \in A$ and $x \in X$,

$$\begin{aligned} {}_a\varphi(b \cdot x) &= a \cdot \varphi(b \cdot x) = a \cdot b \cdot \varphi(x) = b \cdot a \cdot \varphi(\varphi(x)) \\ &= b \cdot {}_a\varphi(\varphi(x)), \\ {}_a\varphi(x \cdot b) &= a \cdot \varphi(x \cdot b) = a \cdot \varphi(x) \cdot b = a \cdot \varphi(\varphi(x)) \cdot b \\ &= {}_a\varphi(\varphi(x)) \cdot b. \end{aligned} \tag{20}$$

Hence ${}_a\varphi \in M_\varphi(X, X)$. □

In the sequel, A denotes a topological algebra and X, Y are two topological A-bimodules. In general, φ is an A-module homomorphism on X such that it is also idempotent, linear, and continuous. Sometimes φ is on A; it will be mentioned when this happens.

Lemma 10. *$M_\varphi^l(X, Y)$ is a left A-module.*

Proof. $M_\varphi^l(X, Y)$ denotes the vector space of all left φ-multipliers from X to Y. Let $T \in M_\varphi^l(X, Y)$ and $a \in A$ be arbitrary. Define $a * T$ as $(a * T)(x) := T(x \cdot a)$ where $x \in X$ is arbitrary. Since the equalities

$$\begin{aligned} (a * T)(b \cdot x) &= T(b \cdot x \cdot a) = b \cdot T(\varphi(x \cdot a)) \\ &= b \cdot T(\varphi(x) \cdot a) = b \cdot (a * T)(\varphi(x)) \end{aligned} \tag{21}$$

hold for all $b \in A$ and $x \in X$, we conclude that $a * T$ is a left φ-multiplier. Then, since X is an A-bimodule and T is linear, $M^l_\varphi(X, Y)$ is a left A-module. □

Definition 11. An A-bimodule X is said to be commutative if $a \cdot x = x \cdot a$ holds for all $a \in A$ and $x \in X$.

Definition 12 (see [10, 11]). Let X be a left (resp., right) A-module. A is said to be left (resp., right) faithful in X if, for any $x \in A$, $a \cdot x = 0$ (resp., $x \cdot a = 0$) for all $a \in A$ implies that $x = 0$. If X is an A-bimodule then A is said to be faithful in X if it is both left and right faithful in X.

The following definition generalizes Definition 12.

Definition 13. Let X be a left (resp., right) A-module. A is said to be left (resp., right) φ-faithful in X if, for any $x \in A$, $\varphi(a) \cdot x = 0$ (resp., $x \cdot \varphi(a) = 0$) for all $a \in A$ implies that $x = 0$. If X is an A-bimodule then A is said to be φ-faithful in X if it is both left and right φ-faithful in X.

Definition 14. Let A be a topological algebra, X a commutative A-bimodule, and φ an idempotent A-module homomorphism on A. For any $x \in X$, define

$$ {}_x\varphi : A \longrightarrow X \quad \text{by } {}_x\varphi(a) = x \cdot \varphi(a), \ a \in A. \tag{22}$$

It is easy to see that ${}_x\varphi \in M^l_\varphi(A, X)$. Now, we define $\psi : X \to M^l_\varphi(A, X)$, by $\psi(x) = {}_x\varphi$.

Lemma 15. *Let A be a topological algebra with an approximate identity $\{e_\lambda : \lambda \in I\}$ and let X be a topological A-bimodule. If A is φ-faithful in X and $\psi : X \to M^l_\varphi(A, X)$ is onto, then A is left faithful in $M^l_\varphi(A, X)$.*

Proof. Let $T \in M^l_\varphi(A, X)$. In view of Lemma 10, it is enough to show that $T = 0$ if $a * T = 0$ for all $a \in A$. Since ψ is onto, there exist $x \in X$ such that $T = {}_x\varphi$. Therefore for any $b \in A$ and $\lambda \in I$,

$$ x \cdot \varphi(b \cdot e_\lambda) = {}_x\varphi(b \cdot e_\lambda) = (e_\lambda * {}_x\varphi)(b) = 0. \tag{23}$$

The continuity of φ implies that $x \cdot \varphi(b) = 0$. Now, since A is φ-faithful in X, we conclude that $x = 0$. Hence $T = {}_x\varphi = 0$. □

Definition 16. Let A be a Hausdorff topological algebra and (X, τ) a Hausdorff topological A-bimodule. Let φ be an A-module homomorphism on A. The φ-uniform operator topology u_φ (resp., φ-strong operator topology s_φ) on $M^l_\varphi(A, X)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

$$ N(\varphi(B), V) = \{T \in M^l_\varphi(A, X) : T(\varphi(B)) \subseteq V\}, \tag{24}$$

where B is a bounded (resp., finite) subset of A and V is a neighborhood of 0 in X. Clearly $s_\varphi \le u_\varphi$.

Theorem 17. *Let φ be an A-module homomorphism on A and let (X, τ) be a topological A-bimodule with $b(A)$-hypocontinuous module multiplication.*

Then $(M^l_\varphi(A, X), s_\varphi)$ and $(M^l_\varphi(A, X), u_\varphi)$ are topological left A-modules.

Proof. By Lemma 10, $M^l_\varphi(A, X)$ is a left A-module. Now, let us prove that the module multiplication $(a, T) \to a * T$ from $A \times M^l_\varphi(A, X)$ into $M^l_\varphi(A, X)$ is separately continuous in u_φ-topology. Let $T \in M^l_\varphi(A, X)$ and $\{a_\alpha : \alpha \in I\}$ be a net in A with $a_\alpha \to a \in A$ and let D be a bounded subset of A and let V be a neighborhood of 0 in X. By $b(A)$-hypocontinuity, there exists a balanced neighborhood H of 0 in X such that $\varphi(D) \cdot H \subset V$. Since T and φ are continuous, there exist $\alpha_0 \in I$ such that

$$\begin{aligned} (a_\alpha * T)(\varphi(b)) - (a * T)(\varphi(b)) &= T(\varphi(b) \cdot a_\alpha) - T(\varphi(b) \cdot a) \\ &= \varphi(b) \cdot [T(\varphi(a_\alpha)) - T(\varphi(a))] \\ &\in \varphi(D) \cdot H \subset V \end{aligned} \tag{25}$$

for all $b \in D$ and $\alpha \ge \alpha_0$. Hence $a_\alpha * T \to_{u_\varphi} a * T$.

Next, let $a \in A$ and $\{T_\alpha : \alpha \in I\}$ be a net in $M^l_\varphi(A, X)$ such that $T_\alpha \to_{u_\varphi} T \in M^l_\varphi(A, X)$ and let D be a bounded subset of A and let V be a neighborhood of 0 in X. Since the mappings φ and $R_a(x) = xa$ are continuous, it follows that $\varphi(D) \cdot a$ is a bounded subset in A. So there exist $\alpha_0 \in I$ such that

$$\begin{aligned} (a * T_\alpha - a * T)(\varphi(b)) &= T_\alpha(\varphi(b) \cdot a) - T(\varphi(b) \cdot a) \\ &= (T_\alpha - T)(\varphi(D) \cdot a) \subseteq V \end{aligned} \tag{26}$$

for all $\alpha \ge \alpha_0$ and $b \in D$. Hence $a * T_\alpha \to_{u_\varphi} a * T$. That means $(M^l_\varphi(A, X), u_\varphi)$ is a left topological module. A similar computation shows that $(M^l_\varphi(A, X), s_\varphi)$ is a left topological module. □

Lemma 18. *Let A be a topological algebra with an approximate identity $\{e_\lambda : \lambda \in I\}$ and let (X, τ) be a commutative A-bimodule and φ an idempotent A-module homomorphism on A. Then $\overline{\psi(X)}^{s_\varphi} = M^l_\varphi(A, X)$.*

Proof. Let $T \in M^l_\varphi(A, X)$, and let B be a finite subset of A and let V be a neighborhood of 0 in X. For each $a \in A$ we have $e_\lambda\varphi(a) \to \varphi(a)$. Then, since T is continuous and B is finite, there exist $\lambda_0 \in I$ such that $T(e_\lambda\varphi(a)) - T(\varphi(a)) \in V$, for all $a \in B$ and $\lambda \ge \lambda_0$. Then, for any $a \in B$ and $\lambda \ge \lambda_0$

$$\begin{aligned} {}_{\{T(\varphi(e_\lambda))\}_\lambda}\varphi(\varphi(a)) - T(\varphi(a)) &= T(\varphi(e_\lambda)) \cdot \varphi(\varphi(a)) - T(\varphi(a)) \\ &= T(e_\lambda\varphi(a)) - T(\varphi(a)) \in V. \end{aligned} \tag{27}$$

Therefore ${}_{\{T(\varphi(e_\lambda))\}_\lambda}\varphi \to_{s_\varphi} T$. □

Theorem 19. *Let A be a topological algebra with an approximate identity $\{e_\lambda : \lambda \in I\}$ and let X be a commutative* *topological A-bimodule such that (X,τ) is complete. Suppose φ is an idempotent A-module homomorphism on A and A is φ-faithful in X. Then (X,τ) is isomorphic to $(M^l_\varphi(A,X), s_\varphi)$.*

Proof. Let ψ be as in Definition 14. It is obvious that ψ is a continuous module homomorphism. We first show that ψ is onto. In view of Lemma 18, it is enough to prove that $\psi(X)$ is s_φ-closed. Let $T \in \overline{\psi(X)}^{s_\varphi}$. Then there exists a net $\{x_\alpha\}_\alpha \subseteq X$ such that ${}_{x_\alpha}\varphi \rightarrow_{s_\varphi} T$. It follows that the net $\{x_\alpha \cdot \varphi(a)\}_\alpha = \{{}_{x_\alpha}\varphi(\varphi(a))\}_\alpha$ is τ-Cauchy in X for every $a \in A$. Now, since A is φ-faithful in X, the net $\{x_\alpha\}$ is τ-Cauchy in X. By completeness of (X,τ), there exist $x \in X$ such that $x_\alpha \rightarrow x$. Hence ${}_{x_\alpha}\varphi \rightarrow_{s_\varphi} {}_x\varphi$. By uniqueness of limit in Hausdorff space $T = {}_x\varphi$. Therefore $\psi(X)$ is s_φ-closed.

To show that ψ is one-to-one, let $x, y \in X$ such that ${}_x\varphi = {}_y\varphi$. Then for any $a \in A$, $(x-y)\cdot\varphi(a) = 0$. Since A is φ-faithful in X, this implies that $x = y$. Thus ψ is one-to-one. □

Definition 20. Let A be a topological algebra and let X be a topological A-bimodule. The uniform topology γ_φ (strict topology β_φ) on $M^l_\varphi(A,X)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all sets

$$N'(\varphi(D), G) = \left\{T \in M^l_\varphi(A,X) : \varphi(D) * T \subset G\right\}, \quad (28)$$

where D is a bounded (finite) subset of A and G is a neighborhood of 0 in $(M^l_\varphi(A,X), u_\varphi)[(M^l_\varphi(A,X), s_\varphi)]$.

Lemma 21. *Let A be a topological algebra with a bounded approximate identity $\{e_\lambda : \lambda \in I\}$ and let X be a topological A-bimodule. Then $u_\varphi = \gamma_\varphi$ and $s_\varphi = \beta_\varphi$.*

Proof. Let $\{T_\alpha\}_\alpha$ be a net in $M^l_\varphi(A,X)$ with $T_\alpha \rightarrow_{u_\varphi} T$. Let $G = N(\varphi(C), V)$ be a neighborhood of 0 in u_φ-topology. Since φ is continuous, $\varphi(C)\varphi(D)$ is a bounded subset of A for each bounded subset D of A. Then there exist α_0 such that

$$(\varphi(D) * (T_\alpha - T))(\varphi(C)) = (T_\alpha - T)(\varphi(C)\varphi(D)) \in V \quad (29)$$

for all $\alpha \geq \alpha_0$. That means $\varphi(D) * (T_\alpha - T) \in G$. Hence $T_\alpha \rightarrow_{\gamma_\varphi} T$.

Conversely, let $\{T_\alpha\}_\alpha$ be a net in $M^l_\varphi(A,X)$ with $T_\alpha \rightarrow_{\gamma_\varphi} T$. Let D be a bounded subset of A and let V be a closed neighborhood of 0 in X. Choose $C = \{e_\lambda\}_\lambda$. Then there exist α_0 such that

$$\begin{aligned}(T_\alpha - T)(\varphi(D)) &= \lim_\lambda (T_\alpha - T)(\varphi(C)\varphi(D)) \\ &= \lim_\lambda (\varphi(D) * (T_\alpha - T))(\varphi(C)) \in V\end{aligned} \quad (30)$$

for all $\alpha \geq \alpha_0$. That means $T_\alpha - T \in N(\varphi(D), V)$. Hence $T_\alpha \rightarrow_{u_\varphi} T$. □

At the end we characterize the φ-multipliers on $L_p(G)$, where G is a compact Abelian group. Of course, $L_p(G)$ is a Banach algebra and several authors studied its multipliers. For instance, Larsen [5] showed that a linear transformation $T : L_p(G) \rightarrow L_p(G)$, where G is a locally compact Abelian group, is a multiplier if and only if there exists a unique $\varphi \in L_\infty(\widehat{G})$ such that $\widehat{Tf} = \varphi\widehat{f}$ for each $f \in L_p(G)$.

However, we now consider $L_p(G)$ as a left Banach module over the group algebra $L_1(G)$. Namely, the algebra $L_1(G)$ acts on $L_p(G)$ through the convolution $L_1(G) * L_p(G) = L_p(G)$.

Example 22. Let G be a compact Abelian group and let $\varphi : L_1(G) \rightarrow L_p(G)$ be an idempotent $L_1(G)$-module homomorphism with dense range. If $T : L_p(G) \rightarrow L_p(G)$ is a φ-multiplier then there exists a unique function $H_T \in L_p(G)$ such that

$$T(\varphi(f)) = \varphi(f) * H_T, \quad (f \in L_1(G)). \quad (31)$$

Proof. Let $\{e_\beta\}_\beta$ be a bounded approximate identity in $L_1(G)$. Then $\{T(\varphi(e_\beta))\}_\beta \subseteq \mathrm{Ball}(L_q(G))^*$. By the Alaoglu theorem, there exists a function $H_T \in L_p(G)$ such that $T(\varphi(e_\beta)) \rightarrow_{\text{weak}*} H_T$. Then for each $f \in L_1(G)$

$$\varphi(f) * T(\varphi(e_\beta)) \longrightarrow_{\text{weak}*} \varphi(f) * H_T. \quad (32)$$

On the other hand, since T is a φ-multiplier,

$$\varphi(f) * T(\varphi(e_\beta)) = T(\varphi(f * e_\beta)) \longrightarrow T(\varphi(f)) \quad (33)$$

for each $f \in L_1(G)$. By uniqueness of limit, $T(\varphi(f)) = \varphi(f) * H_T$.

To show that H_T is unique, let ψ be a second function in $L_p(G)$ such that $T(\varphi(f)) = \varphi(f) * \psi$ for each $f \in L_1(G)$. Since φ has dense range, $T(\varphi(f)) = f * \psi$ for each $f \in L_1(G)$. Therefore

$$\widehat{f}(\gamma)\,(\widehat{\psi - H_T})(\gamma) = 0 \quad (34)$$

for each $f \in L_1(G)$ and $\gamma \in \widehat{G}$. By compactness of G, for each $\gamma \in \widehat{G}$ there exist $f \in L_1(G)$ such that $\widehat{f}(\gamma) \neq 0$. Hence the semisimplicity of $L_p(G)$ implies that $H_T = \psi$. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The author wishes to thank the referee for several useful suggestions which significantly improved the quality of the paper.

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