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Research Article

Some New Oscillation Criteria for Fourth-Order Nonlinear Delay Difference Equations

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In this paper, the authors studied oscillatory behavior of solutions of fourth-order delay difference equation $\Delta(c_3(n)\Delta(c_2(n)\Delta(c_1(n)\Delta u(n)))) + p(n)f(u(n-k)) = 0$ under the conditions $\sum_{n=n_0}^{\infty} c_i(n) < \infty$, $i = 1, 2, 3$. New oscillation criteria have been obtained which greatly reduce the number of conditions required for the studied equation. Some examples are presented to show the strength and applicability of the main results.

1. Introduction

In this paper, we are concerned with the fourth-order delay difference equation of the form

$$D_4 u(n) + p(n) f(u(n-k)) = 0, \quad n \ge n_0 \ge 0, \tag{1}$$

where

$$\begin{aligned} D_0 u &= u, \\ D_i u &= c_i(n)\Delta(D_{i-1}u), \quad i = 1, 2, 3, \\ D_4 u &= \Delta(D_3 u). \end{aligned} \tag{2}$$

In the sequel, we will assume the following:

$(H1)$ $\{c_1(n)\}$, $\{c_2(n)\}$ and $\{c_3(n)\}$ are the positive real sequence for all $n \ge n_0$ and satisfy

$$w_i(n_0) = \sum_{n=n_0}^{\infty} \frac{1}{c_i(n)} < \infty, \quad i = 1, 2, 3. \tag{3}$$

$(H2)$ $\{p(n)\}$ is a nonnegative real sequence and does not vanish eventually.

$(H3)$ $f \in C(\mathbb{R}, \mathbb{R})$ is nondecreasing, and $(f(x)/x) \ge M_1 > 0$ for all $x \ne 0$.

$(H4)$ k is a nonnegative integer.

Under a solution of (1), we mean a real-valued sequence $\{u(n)\}$ defined for all $n \ge n_0 - k$ and satisfies equation (1) for all $n \ge n_0$. We restrict our attention to only those solutions of (1) which exist for all $n \ge N \ge n_0$ and satisfy the following condition:

$$\sup\{|u(n)|: n \ge N_1\} > 0 \text{ for any } N_1 \ge N. \tag{4}$$

A solution $\{u(n)\}$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

Fourth-order difference equations naturally appear in discrete-type models concerning physical, biological, and chemical phenomena (see for example [1]). In mechanical and engineering problems, questions concerning the existence of oscillatory solutions play an important role. During the last several years, there has been a constant interest in getting sufficient conditions for oscillatory behavior of different classes of fourth-order difference equations with or without deviating arguments, (see [2–19], and the references cited therein). In particular, the authors in [5, 8, 13, 15, 18, 19] established oscillation results for (1) under the following assumptions:

$$c_1(n) = c_2(n) = 1 \text{ and } w_3(n_0) < \infty,$$
$$c_1(n) = c_3(n) = 1 \text{ and } w_2(n_0) < \infty, \tag{5}$$

respectively.

In [3, 4, 6, 7, 9–12, 14, 16, 17], the authors studied the oscillation properties of solutions of equation (1) under the following condition:

$$w_i(n_0) = \infty, \quad i = 1, 2, 3. \tag{6}$$

From the review of the literature, it seems that there is nothing known about the oscillation of equation (1) when condition (3) holds. Inspired by the ideas adopted in [20], our main aim is to fill this gap by presenting easily verifiable criteria for the oscillation of all solutions of (1). Examples illustrating the importance of the results obtained are presented.

2. Main Results

For the sake of convenience, we use the following notations throughout the paper. We denote

$$w_{12}(n) = \sum_{s=n}^{\infty} \frac{w_2(s)}{c_1(s)},$$

$$w_{23}(n) = \sum_{s=n}^{\infty} \frac{w_3(s)}{c_2(s)},$$

$$w_{123}(n) = \sum_{s=n}^{\infty} \frac{w_{23}(s)}{c_1(s)}, \tag{7}$$

$$B(n, N) = \sum_{s=N}^{n-1} \frac{1}{c_2(s)} \sum_{i=N}^{s-1} \frac{1}{c_3(i)} \sum_{t=N}^{i-1} p(t),$$

$$\overline{B}(n, N) = \sum_{s=N}^{n-1} \frac{p(s) w_{123}(s-k)}{c_3(s-k)},$$

where $n \geq N \geq n_0$. In what follows, we only need to consider eventually positive solutions of (1), since if $\{u(n)\}$ satisfies (1), then so does $\{-u(n)\}$.

Lemma 1. *Let $\{u(n)\}$ be an eventually positive solution of (1). Then, there is an integer $n_1 \geq n_0$ such that $\{u(n)\}$ satisfies one of the following cases:*

(i) $u(n) > 0$, $D_1u(n) > 0$, $D_2u(n) > 0$, $D_3u(n) > 0$, and $D_4u(n) \leq 0$

(ii) $u(n) > 0$, $D_1u(n) > 0$, $D_2u(n) > 0$, $D_3u(n) < 0$, and $D_4u(n) \leq 0$

(iii) $u(n) > 0$, $D_1u(n) > 0$, $D_2u(n) < 0$, $D_3u(n) > 0$, and $D_4u(n) \leq 0$

(iv) $u(n) > 0$, $D_1u(n) > 0$, $D_2u(n) < 0$, $D_3u(n) < 0$, and $D_4u(n) \leq 0$

(v) $u(n) > 0$, $D_1u(n) < 0$, $D_2u(n) > 0$, $D_3u(n) > 0$, and $D_4u(n) \leq 0$

(vi) $u(n) > 0$, $D_1u(n) < 0$, $D_2u(n) > 0$, $D_3u(n) < 0$, and $D_4u(n) \leq 0$

(vii) $u(n) > 0$, $D_1u(n) < 0$, $D_2u(n) < 0$, $D_3u(n) > 0$, $D_4u(n) \leq 0$

(viii) $u(n) > 0$, $D_1u(n) < 0$, $D_2u(n) < 0$, $D_3u(n) < 0$, and $D_4u(n) \leq 0$

for all $n \geq n_1$.

Proof. The proof is quite obvious, and hence, we omit it.

First, we start with a lemma which ensures the nonexistence of solutions of types (i) – (iv). □

Lemma 2. *Let $\{u(n)\}$ be a positive solution of (1). If*

$$B(\infty, n_0) = \infty, \tag{8}$$

then cases (i) – (iv) of Lemma 1 are not possible.

Proof. From (H_1) and (8), one can see that

$$\sum_{n=n_0}^{\infty} \frac{1}{c_3(n)} \sum_{s=n_0}^{n-1} p(s) = \sum_{n=n_0}^{\infty} p(n) = \infty. \tag{9}$$

Now, let us assume that $\{u(n)\}$ is an eventually positive solution of (1) satisfying one of the cases (i) – (iv) from Lemma 1 and there is an integer $n_1 \geq n_0$ such that $u(n-k) > 0$ for $n \geq n_1$. Since $\{u(n)\}$ is increasing, there is a constant $M > 0$ and an integer $n_2 \geq n_1$ such that $u(n-k) \geq M$ for all $n \geq n_2$. Substituting this in (1), one gets

$$-D_4u(n) \geq f(M) p(n), \quad n \geq n_2. \tag{10}$$

Summing (10) from n_2 to $n-1$, one can find

$$-D_3u(n) + D_3u(n_2) \geq f(M) \sum_{s=n_2}^{n-1} p(s). \tag{11}$$

If we assume that $\{u(n)\}$ belongs to either case (i) or case (iii), then from (9) and (11), we obtain

$$D_3u(n_2) \geq f(M) \sum_{s=n_2}^{n-1} p(s) \longrightarrow \infty \text{ as } n \longrightarrow \infty, \tag{12}$$

which contradicts the fact that $D_3u(n)$ is nonincreasing.

Next, assume that case (ii) holds. From (11), we have

$$-D_3u(n) \geq f(M) \sum_{s=n_2}^{n-1} p(s), \tag{13}$$

or

$$-\Delta(D_2u(n)) \geq \frac{f(M)}{c_3(n)} \sum_{s=n_2}^{n-1} p(s). \tag{14}$$

Summing (14) from n_2 to $n-1$, we have

$$D_2u(n_2) - D_2u(n) \geq f(M) \sum_{s=n_2}^{n-1} \frac{1}{c_3(s)} \sum_{t=n_2}^{s-1} p(t), \tag{15}$$

which, in view of (9), yields

$$D_2u(n_2) \geq f(M) \sum_{s=n_2}^{n-1} \frac{1}{c_3(s)} \sum_{t=n_2}^{s-1} p(t) \longrightarrow \infty \text{ as } t \longrightarrow \infty, \tag{16}$$

which obviously contradicts the fact that $D_2u(n)$ is decreasing.

Finally, we assume that case (iv) holds. Proceeding the same way as in the above case, we get (15), that is,.

$$-\Delta(D_1u(n)) \geq \frac{f(M)}{c_2(n)} \sum_{s=n_2}^{n-1} \frac{1}{c_3(s)} \sum_{t=n_2}^{s-1} p(t). \tag{17}$$

Summing this inequality from n_2 to $n-1$, one obtains

$$\begin{aligned} D_1u(n_2) - D_1u(n) &\geq f(M) \sum_{s=n_2}^{n-1} \frac{1}{c_2(s)} \sum_{t=n_2}^{s-1} \frac{1}{c_3(t)} \sum_{j=n_2}^{t-1} p(j) \\ &= f(M)B(n, n_2), \end{aligned} \tag{18}$$

which in view of (8) implies that $D_1u(n_2) \geq f(M)B(n, n_2) \longrightarrow \infty$ as $n \longrightarrow \infty$, which contradicts the fact that $D_1u(n)$ is decreasing. This completes the proof.

In our first main result with a simple condition, we show that any nonoscillatory solution of (1) converges to zero as $n \longrightarrow \infty$. □

Theorem 1. *If*

$$\sum_{n=n_0}^{\infty} \frac{B(n, n_0)}{c_1(n)} = \infty, \tag{19}$$

then any solution of (1) is oscillatory or converges to zero as $n \longrightarrow \infty$.

Proof. Let $\{u(n)\}$ be a nonoscillatory solution of (1). With no loss of generality, we may take $n_1 \geq n_0$ such that $u(n) > 0$ and $u(n-k) > 0$ for all $n \geq n_1$. From Lemma 1, one may have eight possible cases for $n \geq n_1$.

From (19) and (H_1), we see that $\sum_{n=n_0}^{\infty} B(n, n_0)$ cannot be bounded, and by Lemma 2, case (i) – (iv) are impossible.

Now, let us assume that one of the cases (v) – (viii) holds. Since $\{u(n)\}$ is decreasing, there is a finite nonnegative limit $u(\infty) = \lim_{n \longrightarrow \infty} u(n) = M$. Assume that $M > 0$. Then, there is a $n_2 \geq n_1$ such that $u(n-k) \geq M$ for $n \geq n_2$ and inequality (10) holds. Then, one can arrive at contradiction to (12) in cases (v) and (vii) and a contradiction to (16) in case (vi). Thus, we conclude that $M = 0$.

If we assume that case (viii) holds, then we get (18), that is,

$$\begin{gathered} -D_1u(n) \geq f(M)B(n, n_2) \\ \text{or } -\Delta u(n) \geq \frac{f(M)}{c_1(n)} B(n, n_2). \end{gathered} \tag{20}$$

Summing the last inequality from n_2 to n, we obtain

$$u(n_2) \geq f(M) \sum_{s=n_2}^{n} \frac{B(s, n_2)}{c_1(s)}. \tag{21}$$

But the term on the right side of the above inequality tends to ∞ as $n \longrightarrow \infty$ due to (19), which contradicts the fact that $\{u(n)\}$ is decreasing. This completes the proof.

In the following, we present oscillation criteria for (1). □

Theorem 2. *If*

$$\lim_{n \longrightarrow \infty} \sup A(n, N) > \frac{1}{M_1}, \tag{22}$$

for any $N \geq n_0$, *where*

$$A(n, N) = \min\{w_1(n)B(n, N), w_3(n)\overline{B}(n, N)\}, \tag{23}$$

then (1) oscillates.

Proof. Let $\{u(n)\}$ be a nonoscillatory solution of (1). With no loss of generality, we may take $n_1 \geq n_0$ such that $u(n) > 0$ and $u(n-k) > 0$ for all $n \geq n_1$. By Lemma 1, eight possible cases may occur for all $n \geq n_1$.

First, we note that, in view of (H_1), it is necessary for the validity of (22) that

$$B(\infty, n_0) = \overline{B}(\infty, n_0) = \infty. \tag{24}$$

In view of Lemma 2, the above condition implies that cases (i) – (iv) of Lemma 1 are not possible. Next, we shall consider the remaining possible cases (v) – (viii) separately.

Assume that case (v) holds. From the monotonicity of $D_2u(n)$, we have

$$-D_1u(n) \geq D_1u(\infty) - D_1u(n) = \sum_{s=n}^{\infty} \frac{1}{c_2(s)} D_2u(s) \geq w_2(n) D_2u(n), \tag{25}$$

that is,

$$-\Delta u(n) \geq D_2u(n) \frac{w_2(n)}{c_1(n)}. \tag{26}$$

Summing the above inequality from n to ∞, one gets

$$u(n) \geq D_2u(n) \sum_{s=n}^{\infty} \frac{w_2(s)}{c_1(s)} = D_2u(n) w_{12}(n). \tag{27}$$

Using (27) and the increasing property of $D_2u(n)$ in (1), there is a constant $M > 0$ and an integer $n_2 \geq n_1$ such that

$$\begin{aligned} -D_4u(n) = p(n) f(u(n-k)) &\geq M_1 p(n) D_2u(n-k) w_{12}(n-k) \\ &\geq M_2 p(n) w_{12}(n-k), \quad n \geq n_2, \end{aligned} \tag{28}$$

where $M_2 = MM_1$. Summing the above inequality from n_2 to $n-1$, we have

$$D_3u(n_2) \geq D_3u(n) + M_2 \sum_{s=n_2}^{n-1} p(s) w_{12}(s-k). \tag{29}$$

From (H_1) and (24), one can easily see that

$$\infty = \overline{B}(\infty, n_0) = \sum_{n=n_0}^{\infty} \frac{p(n)w_{123}(n-k)}{w_3(n-k)} \le \sum_{n=n_0}^{\infty} p(n)w_{12}(n-k). \tag{30}$$

Using (30) in (29), we arrive at a contradiction with the fact that $\{D_3u(n)\}$ is nonincreasing.

Assume that case (vi) holds. From the monotonicity of $D_3u(n)$, we have

$$D_2u(n) - D_2u(\infty) = -\sum_{s=n}^{\infty} \frac{1}{c_3(s)} D_3u(s) \ge -D_3u(n)w_3(n). \tag{31}$$

Therefore,

$$\Delta\left(\frac{D_2u(n)}{w_3(n)}\right) = \frac{D_3u(n)w_3(n) + D_2u(n)}{w_3(n)w_3(n+1)c_3(n)} \ge 0, \tag{32}$$

which implies that $\{D_2u(n)/w_3(n)\}$ is nondecreasing. Using this property, one obtains

$$-D_1u(n) \ge \sum_{s=n}^{\infty} \frac{1}{c_2(s)} D_2u(s) \ge \frac{D_2u(n)}{w_3(n)} \sum_{s=n}^{\infty} \frac{w_3(s)}{c_2(s)} = \frac{D_2u(n)}{w_3(n)} w_{23}(n). \tag{33}$$

Hence,

$$\Delta\left(-\frac{D_1u(n)}{w_{23}(n)}\right) = \frac{-D_2u(n)w_{23}(n) - D_1u(n)w_3(n)}{w_{23}(n+1)w_{23}(n)c_2(n)} \ge 0, \tag{34}$$

and so $-(D_1u(n)/w_{23}(n))$ is nondecreasing. Finally, we get

$$u(n) \ge -\sum_{s=n}^{\infty} \frac{1}{c_1(s)} D_1u(s) \ge -\frac{D_1u(n)}{w_{23}(n)} \sum_{s=n}^{\infty} \frac{w_{23}(s)}{c_1(s)} = -\frac{D_1u(n)}{w_{23}(n)} w_{123}(n). \tag{35}$$

Using (33) in the above inequality, we get

$$u(n) \ge \frac{D_2u(n)}{w_3(n)} w_{123}(n). \tag{36}$$

Therefore,

$$-D_4u(n) \ge M_1 p(n)u(n-k) \ge M_1 \frac{p(n)w_{123}(n-k)}{w_3(n-k)} D_2u(n-k). \tag{37}$$

Summing this inequality from n_1 to $n-1$ and using the monotonicity of $D_2u(n)$, we obtain

$$\begin{aligned} -D_3u(n) &\ge \sum_{s=n_1}^{n-1} M_1 \frac{p(s)w_{123}(s-k)}{w_3(s-k)} D_2u(s-k) \\ &\ge M_1 D_2u(n-k) \sum_{s=n_1}^{n-1} \frac{p(s)w_{123}(s-k)}{w_3(s-k)} \\ &\ge M_1 D_2u(n)\overline{B}(n, n_1). \end{aligned} \tag{38}$$

From (31) and (38), one obtains

$$-D_3u(n) \ge -M_1 D_3u(n)\overline{B}(n, n_1)w_3(n). \tag{39}$$

Dividing the above inequality by $-D_3u(n)$ and then taking the lim sup on both sides, one arrives at a contradiction with (22).

Assume that case (vii) holds. From the decreasing property of $D_1u(n)$, one obtains

$$u(n) \ge u(n) - u(\infty) = -\sum_{s=n}^{\infty} \frac{1}{c_1(s)} D_1u(s) \ge -w_1(n)D_1u(n). \tag{40}$$

Thus,

$$\Delta\left(\frac{u(n)}{w_1(n)}\right) = \frac{D_1u(n)w_1(n) + u(n)}{w_1(n)w_1(n+1)c_1(n)} \ge 0, \tag{41}$$

which means that $\{u(n)/w_1(n)\}$ is nondecreasing. Summing (1) from n_1 to $n-1$ and using the above property of $\{u(n)/w_1(n)\}$, we obtain

$$D_3u(n_1) \ge D_3u(n) + M_1 \sum_{s=n_1}^{n-1} p(s)u(s-k) \ge \frac{u(n_1)}{w_1(n_1)} M_1 \cdot \sum_{s=n_1}^{n-1} p(s)w_1(s). \tag{42}$$

Furthermore, on using (H_1) and (30), it is easy to see that for any constant $b > 0$,

$$\infty = \sum_{n=n_1}^{\infty} p(s)w_{12}(s) \le b \sum_{n=n_1}^{\infty} p(s)w_1(s). \tag{43}$$

This, by virtue of (42), contradicts the fact that $\{D_3u(n)\}$ is nonincreasing.

Finally, assume that case (viii) holds. Summing (1) from n_1 to $n-1$, we have

$$-D_3u(n) \ge M_1 \sum_{s=n_1}^{n-1} p(s)u(s-k) \ge M_1 u(n-k) \sum_{s=n_1}^{n-1} p(s). \tag{44}$$

Dividing both sides of the above inequality by $c_3(n)$ and summing the resulting inequality again from n_1 to $n-1$, one obtains

$$-D_2u(n) \ge M_1 u(n-k) \sum_{s=n_1}^{n-1} \frac{1}{c_3(s)} \sum_{t=n_1}^{s-1} p(t). \tag{45}$$

Similarly, we obtain

$$\begin{aligned}-D_1u(n) &\geq M_1u(n-k)\sum_{s=n_1}^{n-1}\frac{1}{c_2(s)}\sum_{t=n_1}^{s-1}\frac{1}{c_3(t)}\sum_{j=n_1}^{t-1}p(j)\\ &= M_1u(n-k)B(n,n_1)\\ &\geq M_1u(n)B(n,n_1)\\ &\geq -M_1D_1u(n)w_1(n)B(n,n_1)\\ \text{or }\frac{1}{M_1}&\geq w_1(n)B(n,n_1),\end{aligned} \tag{46}$$

which contradicts (22). This completes the proof.

Finally, we obtain an oscillation criterion using classical Riccati-type transformation technique. □

Theorem 3. *If for all sufficiently large $N \geq n_0$,*

$$\lim_{n\longrightarrow\infty}\sum_{s=N}^{n}\left(M_1p(s)w_{123}(s+1)-\frac{w_{23}(s)}{4c_1(s)w_{123}(s+1)}\right)=\infty, \tag{47}$$

$$\lim_{n\longrightarrow\infty}\sup\sum_{s=N}^{n}\left(\frac{M_1w_1(s+1)}{c_2(s)}\sum_{t=N}^{s-1}\frac{1}{c_3(t)}\sum_{j=N}^{t-1}p(j)-\frac{1}{4w_1(s+1)c_1(s)}\right)=\infty, \tag{48}$$

then equation (1) oscillates.

Proof. Assume the contrary that $\{u(n)\}$ is a nonoscillatory solution of (1) for all $n \geq n_0$. With no loss of generality, we may take $n_1 \geq n_0$ such that $u(n) > 0$ and $u(n-k) > 0$ for all $n \geq n_1$. By Lemma 1, eight possible cases may occur for all $n \geq n_1$. From (48), one may see that

$$\sum_{n=n_0}^{\infty}\frac{w_1(n)}{c_2(n)}\sum_{s=n_1}^{n-1}\frac{1}{c_3(s)}\sum_{t=n_1}^{s-1}p(t)=\infty, \tag{49}$$

which in view of (H_1) implies that $B(\infty, n_0) = \infty$. So by Lemma 2, cases (i) – (iv) from Lemma 1 are not possible. Therefore, it is enough to consider cases (v) – (viii).

First, assume that case (v) holds. From (47), we have

$$\infty=\sum_{n=n_0}^{\infty}p(n)w_{123}(n+1)\leq\sum_{n=n_0}p(n)w_{123}(n). \tag{50}$$

Then, arguing as in the proof of Theorem 2 case (v), we obtain a contradiction.

Next, assume case (vi) holds. Define the function

$$y(n)=\frac{D_3u(n)}{u(n)}<0. \tag{51}$$

In view of (31) and (36), one can obtain

$$u(n)\geq D_3u(n)w_{123}(n), \tag{52}$$

and hence,

$$-1\leq y(n)w_{123}(n)<0. \tag{53}$$

Also, arguing as in the proof of Theorem 2 case (vi), we obtain from (31) and (33) that

$$\Delta u(n)\geq -D_3u(n)\frac{w_{23}(n)}{c_1(n)}. \tag{54}$$

By (1), (54), and the monotonicity of $u(n)$, we conclude that

$$\begin{aligned}\Delta y(n)&=\frac{D_4u(n)}{u(n+1)}-\frac{D_3u(n)\Delta u(n)}{u(n)u(n+1)}\\ &\leq -M_1p(n)\frac{u(n-k)}{u(n+1)}-\frac{(D_3u(n))^2w_{23}(n)}{c_1(n)u(n)u(n+1)}\\ &\leq -M_1p(n)-y^2(n)\frac{w_{23}(n)}{c_1(n)},\\ &\text{or }\Delta y(n)+M_1p(n)+y^2(n)\frac{w_{23}(n)}{c_1(n)}\leq 0.\end{aligned} \tag{55}$$

Multiplying the above inequality by w_{123} $(n+1)$ and summing the resulting inequality from n_1 to $n-1$, we obtain

$$\begin{aligned}&\sum_{s=n_1}^{n-1}w_{123}(s+1)\Delta y(s)+\sum_{s=n_1}^{n-1}M_1w_{123}(s+1)p(s)\\ &+\sum_{s=n_1}^{n-1}w_{123}(s+1)y^2(s)\frac{w_{23}(s)}{c_1(s)}\leq 0.\end{aligned} \tag{56}$$

Now, applying the summation by parts formula and then rearranging, we obtain

$$\begin{aligned}&y(n)w_{123}(n)-y(n_1)w_{123}(n_1)+\sum_{s=n_1}^{n-1}M_1w_{123}(s+1)p(s)\\ &+\sum_{s=n_1}^{n-1}y(s)\frac{w_{23}(s)}{c_1(s)}\\ &+\sum_{s=n_1}^{n-1}y^2(s)\frac{w_{23}(s)w_{123}(s+1)}{c_1(s)}\leq 0.\end{aligned} \tag{57}$$

Therefore, in view of (53) that

$$\sum_{s=n_1}^{n-1}\left(M_1 w_{123}(s+1)p(s)-\frac{w_{23}(s)}{4c_1(s)w_{123}(s)}\right)\le y(n_1)w_{123}(n_1)+1, \tag{58}$$

which contradicts (47).

Assume now that case (vii) holds. Note that

$$\sum_{n=n_0}^{\infty} p(n)w_{123}(n)=\infty \tag{59}$$

is necessary for (47). Then, for any constant $M>0$, we have

$$\infty=\sum_{n=n_0}^{\infty} p(n)w_{123}(n)\le M\sum_{n=n_0}^{\infty} p(n)w_{12}(n). \tag{60}$$

Proceeding as in the proof of Theorem 2 case (vii), we obtain a contradiction.

Finally, assume that case (viii) holds. Define

$$v(n)=\frac{D_1u(n)}{u(n)}<0. \tag{61}$$

From (45), we have

$$-D_2u(n)\ge M_1u(n+1)\sum_{s=n_1}^{n-1}\frac{1}{c_3(s)}\sum_{t=n_1}^{s-1}p(t). \tag{62}$$

On the other hand, from the monotonicity of $D_1u(n)$, we have

$$u(\infty)-u(n)=\sum_{s=n}^{\infty}\frac{D_1u(s)}{c_1(s)}\le D_1u(n)w_1(n), \tag{63}$$

or

$$-1\le v(n)w_1(n)<0. \tag{64}$$

Then, using (62), we have

$$\begin{aligned}\Delta v(n)&=\frac{D_2u(n)}{c_2(n)u(n+1)}-\frac{(D_1u(n))^2}{c_1(n)u(n)u(n+1)}\\&\le-\frac{M_1}{c_2(n)}\sum_{s=n_1}^{n-1}\frac{1}{c_3(s)}\sum_{t=n_1}^{s-1}p(t)-\frac{v^2(n)}{c_1(n)}.\end{aligned} \tag{65}$$

Now, multiplying the last inequality by $w_1\ (n+1)$ and then summing from n_1 to $n-1$, one obtains

$$\begin{aligned}&v(n)w_1(n)-v(n_1)w_1(n_1)+\sum_{s=n_1}^{n-1}\frac{v(s)}{c_1(s)}+\sum_{s=n_1}^{n-1}\frac{M_1w_1(s+1)}{c_2(s)}\\&\cdot\sum_{t=n_1}^{s-1}\frac{1}{c_3(t)}\sum_{j=n_1}^{t-1}p(j)+\sum_{s=n_1}^{n-1}w_1(s+1)\frac{v^2(s)}{c_1(s)}\le 0.\end{aligned} \tag{66}$$

Hence, in view of (64),

$$\begin{aligned}&\sum_{s=n_1}^{n-1}\left(\frac{M_1w_1(s+1)}{c_2(s)}\sum_{t=n_1}^{s-1}\frac{1}{c_3(t)}\sum_{j=n_1}^{t-1}p(j)-\frac{1}{4w_1(s+1)c_1(s)}\right)\\&\quad\le v(n_1)w_1(n_1)+1,\end{aligned} \tag{67}$$

which contradicts (48). The proof is now complete. □

3. Examples

In this section, we provide some examples to illustrate the applicability and strength of the results obtained in the previous section.

Example 1. Let us consider the following fourth-order delay difference equation:

$$\Delta\left(n^2\Delta\left(n^2\Delta\left(n^2\Delta u(n)\right)\right)\right)+q_0n^2u(n-k)=0,\quad n\ge 1, \tag{68}$$

where $q_0>0$ and k is a positive integer. It is easy to verify that condition (19) is satisfied and by Theorem 1, one can conclude that any nonoscillatory solution of (68) converges to zero as $n\longrightarrow\infty$.

Example 2. Consider the following fourth-order delay difference equation:

$$\Delta(n(n+1)\Delta(n(n+1)\Delta(n(n+1)\Delta u(n))))+q_0n^2u(n-k)=0,\quad n\ge 1, \tag{69}$$

where $q_0>0$ and k is a positive integer. By simple calculation, we see that $w_1(n)=w_2(n)=w_3(n)=(1/n)$ and $B(n,1)\approx q_0(n/6)$. Hence, by Theorem 2, equation (69) is oscillatory if $q_0>6$. The same conclusion follows from Theorem 3 if $q_0>(3/2)$.

Data Availability

No data were used to support this study

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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