Research Article

Variable Exponent Function Spaces related to a Sublinear Expectation

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In this paper, variable exponent function spaces $\mathbb{L}^{p(\cdot)}$, $\mathbb{L}_b^{p(\cdot)}$, and $\mathbb{L}_c^{p(\cdot)}$ are introduced in the framework of sublinear expectation, and some basic and important properties of these spaces are given. A version of Kolmogorov's criterion on variable exponent function spaces is proved for continuous modification of stochastic processes.

1. Introduction

Variable exponent spaces are extensively applied in the study of some nonlinear problems in natural science and engineering. Basic properties of the spaces are first given by Kováčik and Rákosník in [1]. Some theories of variable exponent spaces can also be found in [2, 3]. Harjulehto et al. present an overview of applications to differential equations with non-standard growth in [4]. Diening et al. [5] summarize most of the existing literature of theory of variable exponent function spaces and applications to partial differential equations. In [6], Aoyama proves some important probability inequalities in variable exponent Lebesgue spaces.

Nonlinear expectations play an important role in the research of financial markets. One of the most important application is that a coherent risk measure (the basic theory about coherent risk measure can be found in [7]) is a sublinear expectation $\mathbb{E} : \mathcal{H} \rightarrow \mathbb{R}$ defined on $\mathcal{H}$, which is a linear space of financial losses. In this paper, we are interested in behavior of sublinear expectation spaces with variable exponents. By the following representation theorem which can be found in Peng ([8], p. 4), we know that a sublinear expectation can be expressed as a supremum of linear expectations, i.e., there exists a family of linear expectations $\{E_\theta\}_{\theta \in \Theta}$ such that

$$\mathbb{E}[X] = \sup_{\theta \in \Theta} E_\theta[X], \quad X \in \mathcal{H}. \quad (1)$$

Thus, we consider the upper expectation only in this paper. Some other important theories about nonlinear expectations can be found in Peng's [9, 10].

The remainder of the paper is divided as follows: in Section 2, motivated by Fu [11] and Denis et al. [12], we first introduce $\mathbb{L}^{p(\cdot)}$, $\mathbb{L}_b^{p(\cdot)}$ and $\mathbb{L}_c^{p(\cdot)}$ and give some properties of these spaces. And Each element of $\|\cdot\|_{p(\cdot)}$ -completion of $C_b(\Omega)$ has a quasi-continuous version is proved. In Section 3, applying the results of Section 2, we discuss a version of Kolmogorov's criterion for continuous modification of stochastic processes, which are in the variable exponent function space related to a sublinear expectation, after proving the situation under a linear expectation.

2. Variable Exponent Function Spaces

Let Ω be a complete metric space equipped with the distance d , $\mathcal{B}(\Omega)$ the Borel σ -algebra of Ω and $\mathcal{M}$ the collection of all probability measures on $(\Omega, \mathcal{B}(\Omega))$.

$L^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$ -measurable real functions;

$B_b(\Omega)$: all bounded functions in $L^0(\Omega)$;

$C_b(\Omega)$: all continuous functions in $B_b(\Omega)$.

For a given subset $\mathcal{P} \subseteq \mathcal{M}$, we denote

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega). \quad (2)$$

Definition 1. A set A is a polar if $c(A) = 0$ and a property holds “quasi-surely” (q.s.) if it holds outside a polar set.

The upper expectation of $\mathcal{P}$ is defined as follows: for each $X \in L^0(\Omega)$ such that $E_p[X]$ exists for each $P \in \mathcal{P}$,

$$\mathbb{E}[X] := \sup_{P \in \mathcal{P}} E_P[X]. \quad (3)$$

About $\mathbb{E}$, we know the following properties.

Theorem 2 (see Theorem 9 in [12]). *The upper expectation $\mathbb{E}[\cdot]$ of family $\mathcal{P}$ is a sublinear expectation on $B_b(\Omega)$ as well as on $C_b(\Omega)$, i.e.,*

- (1) For all X, Y in $B_b(\Omega)$, $X \geq Y$, $\mathbb{E}[X] \geq \mathbb{E}[Y]$.
- (2) For all X, Y in $B_b(\Omega)$, $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.
- (3) For all $\lambda \geq 0$, $X \in B_b(\Omega)$, $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$.
- (4) For all $c \in \mathbb{R}$, $X \in B_b(\Omega)$, $\mathbb{E}[X + c] = \mathbb{E}[X] + c$.

Let a $\mathcal{B}(\Omega)$ -measurable real function $p : \Omega \rightarrow [1, \infty)$, be a variable exponent. In the space $L^0(\Omega)$, the moduli ρ are defined by

$$\begin{aligned} \rho_{p(\cdot)}(X) &:= E[|X|^p] = \sup_{P \in \mathcal{P}} E_P[|X|^p] \\ &= \sup_{P \in \mathcal{P}} \int_{\Omega} |X|^p(\omega) P(d\omega). \end{aligned} \quad (4)$$

Definition 3. The space $\mathcal{L}^{p(\cdot)}$ is the set of $X \in L^0(\Omega)$ satisfying $\rho_{p(\cdot)}(X) < \infty$, and it is endowed with the Luxemburg norm:

$$\|X\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}\left(\frac{X}{\lambda}\right) \leq 1 \right\}. \quad (5)$$

We set $\mathcal{N}^{p(\cdot)} := \{X \in L^0(\Omega) : \rho(X) = 0\}$, and denote $\mathbb{L}^{p(\cdot)} := \mathcal{L}^{p(\cdot)} / \mathcal{N}^{p(\cdot)}$. As usual, we do not take care about the distinction between classes and their representatives.

Proposition 4. *If the variable exponent p satisfies $1 < p^- \leq p(\omega) \leq p^+ < \infty$, then the inequality*

$$\mathbb{E}[|XY|] \leq \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right) \|X\|_{p(\cdot)} \|Y\|_{p'(\cdot)} \quad (6)$$

holds for every $X \in \mathbb{L}^{p(\cdot)}$ and $Y \in \mathbb{L}^{p'(\cdot)}$, where $1/p(\omega) + 1/p'(\omega) = 1$.

Proof. By Young inequality, we have

$$\frac{|X|}{\|X\|_{p(\cdot)}} \frac{|Y|}{\|Y\|_{p'(\cdot)}} \leq \frac{1}{p(\omega)} \left(\frac{|X|}{\|X\|_{p(\cdot)}}\right)^p + \frac{1}{p'(\omega)} \left(\frac{|Y|}{\|Y\|_{p'(\cdot)}}\right)^{p'}. \quad (7)$$

By the monotonicity, sub-additivity and positive homogeneity of $\mathbb{E}$, and the property of the norm, we have

$$\begin{aligned} & \mathbb{E} \left[\frac{|X|}{\|X\|_{p(\cdot)}} \frac{|Y|}{\|Y\|_{p'(\cdot)}} \right] \\ & \leq \mathbb{E} \left[\frac{1}{p(\omega)} \left(\frac{|X|}{\|X\|_{p(\cdot)}}\right)^p + \frac{1}{p'(\omega)} \left(\frac{|Y|}{\|Y\|_{p'(\cdot)}}\right)^{p'} \right] \\ & \leq \mathbb{E} \left[\frac{1}{p(\omega)} \left(\frac{|X|}{\|X\|_{p(\cdot)}}\right)^p \right] + \mathbb{E} \left[\frac{1}{p'(\omega)} \left(\frac{|Y|}{\|Y\|_{p'(\cdot)}}\right)^{p'} \right] \\ & \leq \frac{1}{p^-} \mathbb{E} \left[\left(\frac{|X|}{\|X\|_{p(\cdot)}}\right)^p \right] + \left(1 - \frac{1}{p^+}\right) \mathbb{E} \left[\left(\frac{|Y|}{\|Y\|_{p'(\cdot)}}\right)^{p'} \right] \\ & \leq 1 + \frac{1}{p^-} - \frac{1}{p^+}. \end{aligned} \quad (8)$$

Thus, the inequality follows. $\square$

Proposition 5. *Suppose that the variable exponent p satisfies $1 < p^- \leq p(\omega) \leq p^+ < \infty$. If $X, X_k \in \mathbb{L}^{p(\cdot)}$, then*

- (1) If $\|X\|_{p(\cdot)} \geq 1$, then $\|X\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(X) \leq \|X\|_{p(\cdot)}^{p^+}$.
- (2) If $\|X\|_{p(\cdot)} \leq 1$, then $\|X\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(X) \leq \|X\|_{p(\cdot)}^{p^-}$.
- (3) $\lim_{k \rightarrow \infty} \|X_k\|_{p(\cdot)} = 0$, if and only if $\lim_{k \rightarrow \infty} \rho_{p(\cdot)}(X_k) = 0$.
- (4) $\lim_{k \rightarrow \infty} \|X_k\|_{p(\cdot)} = \infty$, if and only if $\lim_{k \rightarrow \infty} \rho_{p(\cdot)}(X_k) = \infty$.

In particular, the linear expectation E_p also follows this proposition.

Proof. (1) By $\|X\|_{p(\cdot)} \geq 1$ and the definition of the norm,

$$\mathbb{E} \left[\frac{|X|^p}{\|X\|_{p(\cdot)}^{p^+}} \right] \leq \mathbb{E} \left[\left(\frac{|X|}{\|X\|_{p(\cdot)}}\right)^p \right] \leq 1. \quad (9)$$

Thus, $\rho_{p(\cdot)}(X) \leq \|X\|_{p(\cdot)}^{p^+}$. As

$$\|X\|_{p(\cdot)}^{p^-/p} \leq \|X\|_{p(\cdot)}, \quad (10)$$

we also have

$$\mathbb{E} \left[\left(\frac{|X|}{\|X\|_{p(\cdot)}^{p^-/p}}\right)^p \right] \geq 1. \quad (11)$$

That is to say $\|X\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(X)$. The proof is completed. (2) can be easily proved in the similar method. By (1) and (2), we can easily get (3) and (4).

Proposition 6. *Suppose that the variable exponent p satisfies $1 < p^- \leq p(\omega) \leq p^+ < \infty$. Let $X, X_k \in \mathbb{L}^{p(\cdot)}$. Then for each $\alpha > 0$*

$$c(\{|X| > \alpha\}) \leq (\alpha^{-p^-} \vee \alpha^{-p^+}) \mathbb{E}[|Xp|]. \quad (12)$$

Proof. For each $P \in \mathcal{P}$, by Markov inequality, we have

$$P\left(\left|\frac{X}{\alpha}\right|^p > 1\right) \leq E_p\left[\left|\frac{X}{\alpha}\right|^p\right]. \quad (13)$$

Take the supremum in $\mathcal{P}$ both sides,

$$c\left(\left\{\left|\frac{X}{\alpha}\right|^p > 1\right\}\right) \leq \mathbb{E}\left[\left|\frac{X}{\alpha}\right|^p\right] \quad (14)$$

Thus,

$$c(\{|X| > \alpha\}) \leq \mathbb{E}\left[\left|\frac{X}{\alpha}\right|^p\right] \leq (\alpha^{-p^-} \vee \alpha^{-p^+})\mathbb{E}[|X|^p]. \quad (15)$$

□

Lemma 7 (Proposition 17 in [12]). *Let $p \in (0, \infty]$ and $\{X_n\}$ be a sequence in $\mathbb{L}^p$ which converges to X in $\mathbb{L}^p$. Then there exists a subsequence $\{X_{n_k}\}$ which converges to X quasi-surely in the sense that it converges to X outside a polar set.*

Proposition 8. *If the variable exponent p satisfies $1 < p^- \leq p(\omega) \leq p^+ < \infty$, $\mathbb{L}^{p(\cdot)}$ is a Banach space.*

Proof. Let $\{X_n\}$ be a Cauchy sequence in $\mathbb{L}^{p(\cdot)}$. Then, by Proposition 4,

$$\mathbb{E}[|X_m - X_n|] \leq C\|X_m - X_n\|_{p(\cdot)}, \quad (16)$$

where C is a constant. That is to say $\{X_n\}$ is a Cauchy sequence in $\mathbb{L}^1$. By Proposition 14 in [12], $\mathbb{L}^1$ is a Banach Space. Thus, $\{X_n\}$ converges in $\mathbb{L}^1$. Suppose that $X_n \rightarrow X$, $X \in \mathbb{L}^1$ and further by Lemma 7, we suppose $X_n \rightarrow X$ quasi-surely (subtracting a subsequence if necessary). For each $0 < \varepsilon < 1$, there exists n_0 such that $\|X_m - X_n\|_{p(\cdot)} < \varepsilon$ for $m, n \geq n_0$. Fix n , by Fatou's lemma

$$\begin{aligned} \mathbb{E}\left[\left(\frac{|X_n - X|}{\varepsilon}\right)^p\right] &= \sup_{p \in \mathcal{P}} E_p\left[\left(\frac{|X_n - X|}{\varepsilon}\right)^p\right] \\ &\leq \sup_{p \in \mathcal{P}} \liminf_{m \rightarrow \infty} E_p\left[\left(\frac{|X_n - X_m|}{\varepsilon}\right)^p\right] \\ &\leq \liminf_{m \rightarrow \infty} \mathbb{E}\left[\left(\frac{|X_n - X_m|}{\varepsilon}\right)^p\right] \leq 1. \end{aligned} \quad (17)$$

Thus, $\|X_n - X\|_{p(\cdot)} \leq \varepsilon$, and further $\rho_{p(\cdot)}(X_n - X) \leq \|X_n - X\|_{p(\cdot)}^{p^-} \leq \varepsilon^{p^-}$, that is to say, $X_n - X \in \mathbb{L}^{p(\cdot)}$, and $X \in \mathbb{L}^{p(\cdot)}$. The proof is completed. □

We denote by $\mathbb{L}_b^{p(\cdot)}$ the completion of $B_b(\Omega)$ and by $\mathbb{L}_c^{p(\cdot)}$ the completion of $C_b(\Omega)$. By Proposition 8, we have $\mathbb{L}_c^{p(\cdot)} \subset \mathbb{L}_b^{p(\cdot)} \subset \mathbb{L}^{p(\cdot)}$ for $1 < p^- \leq p(\omega) \leq p^+ < \infty$.

Proposition 9. *Suppose that the variable exponent p satisfies $1 < p^- \leq p(\omega) \leq p^+ < \infty$, then*

$$\mathbb{L}_b^{p(\cdot)} = \left\{X \in \mathbb{L}^{p(\cdot)} : \lim_{n \rightarrow \infty} \mathbb{E}[|X|^p 1_{\{|X|>n\}}] = 0\right\}. \quad (18)$$

Proof. We denote $J_{p(\cdot)} = \{X \in \mathbb{L}^{p(\cdot)} : \lim_{n \rightarrow \infty} \mathbb{E}[|X|^p 1_{\{|X|>n\}}] = 0\}$.

For each $X \in J_{p(\cdot)}$ let $X_n = (X \wedge n) \vee (-n) \in B_b(\Omega)$. We have

$$\rho_{p(\cdot)}(X - X_n) \leq \mathbb{E}[|X|^p 1_{\{|X|>n\}}], \quad (19)$$

so $\lim_{n \rightarrow \infty} \rho_{p(\cdot)}(X - X_n) = 0$. Thus, $X \in \mathbb{L}_b^{p(\cdot)}$. On the other hand, for any $X \in \mathbb{L}_b^{p(\cdot)}$, we can find a sequence $\{Y_n\} \subset B_b(\Omega)$ such that $\lim_{n \rightarrow \infty} \rho_{p(\cdot)}(X - Y_n) = 0$. Let $y_n = \sup_{\omega \in \Omega} |Y_n(\omega)|$ and $X_n = (X \wedge y_n) \vee (-y_n)$. Since $|X - X_n| \leq |X - Y_n|$, we have $\lim_{n \rightarrow \infty} \rho_{p(\cdot)}(X - X_n) = 0$. This implies that for any sequence $\{\alpha_n\}$ such that $\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} \rho_{p(\cdot)}(X - (X \wedge \alpha_n) \vee (-\alpha_n)) = 0$. And for all $n \in \mathbb{N}$,

$$\mathbb{E}[|X|^p 1_{\{|X|>n\}}] < 2^{p^+ - 1} \mathbb{E}[(|X| - n)^p 1_{\{|X|>n\}}] + \mathbb{E}[2^{p-1} n^p 1_{\{|X|>n\}}]. \quad (20)$$

For the first term of right hand side, we have

$$\lim_{n \rightarrow \infty} \mathbb{E}[(|X| - n)^p 1_{\{|X|>n\}}] = \lim_{n \rightarrow \infty} \rho_{p(\cdot)}(X - (X \wedge n) \vee (-n)) = 0. \quad (21)$$

For the second term,

$$\frac{n^p}{2^p} 1_{\{|X|>n\}} \leq \left(|X| - \frac{n}{2}\right)^p 1_{\{|X|>n\}} \leq \left(|X| - \frac{n}{2}\right)^p 1_{\{|X|>n/2\}}, \quad (22)$$

so

$$\begin{aligned} \mathbb{E}[2^{p-1} n^p 1_{\{|X|>n\}}] &= \mathbb{E}\left[2^{2p-1} \frac{n^p}{2^p} 1_{\{|X|>n\}}\right] \\ &\leq \mathbb{E}\left[2^{2p-1} \left(|X| - \frac{n}{2}\right)^p 1_{\{|X|>n/2\}}\right] \\ &\leq 2^{2p^+ - 1} \mathbb{E}[(|X| - (n/2))^p 1_{\{|X|>n/2\}}], \end{aligned} \quad (23)$$

and $\lim_{n \rightarrow \infty} \mathbb{E}[2^{p-1} n^p 1_{\{|X|>n\}}] = 0$. Thus, $X \in J_{p(\cdot)}$. □

Proposition 10. *If the variable exponent p satisfies $1 < p^- \leq p(\omega) \leq p^+ < \infty$, let $X \in \mathbb{L}_b^{p(\cdot)}$, then for each $\varepsilon > 0$, there exists a $\delta > 0$, such that for all $A \in \mathcal{B}(\Omega)$ with $c(A) \leq \delta$, we have $\mathbb{E}[|X|^p 1_A] \leq \varepsilon$.*

Proof. For each $\varepsilon > 0$, by Proposition 9, there exists $N > 0$ such that $\mathbb{E}[|X|^p 1_{\{|X|>N\}}] \leq \varepsilon/2$. Set $\delta = \varepsilon/2N^{p^+}$, then for $A \in \mathcal{B}(\Omega)$ with $c(A) \leq \delta$, we have.

$$\begin{aligned} \mathbb{E}[|X|^p 1_A] &\leq \mathbb{E}[|X|^p 1_A 1_{\{|X|>N\}}] + \mathbb{E}[|X|^p 1_A 1_{\{|X| \leq N\}}] \\ &\leq \mathbb{E}[|X|^p 1_{\{|X|>N\}}] + N^{p^+} c(A) \leq \varepsilon. \end{aligned} \quad (24)$$

□

Definition 11. A mapping X on Ω with values in topological space is said to be quasi-continuous (q.c.) if $\forall \varepsilon > 0$, there exists an open set O with $c(O) < \varepsilon$ such that $X|_O$ is continuous.

Definition 12. We say that $X : \Omega \rightarrow \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous $Y : \Omega \rightarrow \mathbb{R}$ with $X = Y$ q.s.

Proposition 13. *If the variable exponent p satisfies $1 < p^- \leq p(\omega) \leq p^+ < \infty$. Then each element in $\mathbb{L}_c^{p(\cdot)}$ has a quasi-continuous version.*

Proof. For each $X \in \mathbb{L}_c^{p(\cdot)}$, there exists $\{X_n\} \subset C_b(\Omega)$ such that $X_n \rightarrow X$ in $\mathbb{L}_c^{p(\cdot)}$. Let us choose a subsequence $\{X_{n_k}\}$ such that

$$\rho_{p(\cdot)}(X_{n_{k+1}} - X_{n_k}) \leq 2^{-2k}, \quad \forall k \geq 1, \quad (25)$$

and set for all k ,

$$A_k = \bigcup_{i=k}^{\infty} \{|X_{n_{i+1}} - X_{n_i}| > 2^{-i/p^+}\}. \quad (26)$$

Because of the subadditivity property and Proposition 6,

$$\begin{aligned} c(A_k) &\leq \sum_{i=k}^{\infty} c(|X_{n_{i+1}} - X_{n_i}| < 2^{-i/p^+}) \\ &\leq \sum_{i=k}^{\infty} 2^i \mathbb{E}[|X_{n_{i+1}} - X_{n_i}|^p] \leq \sum_{i=k}^{\infty} 2^{-i} = 2^{-k+1}. \end{aligned} \quad (27)$$

Thus, $\lim_{k \rightarrow \infty} c(A_k) = 0$, so the Borel set $A = \bigcap_{k=1}^{\infty} A_k$ is polar. As each X_{n_k} is continuous, for all $k \geq 1$, A_k is an open set. And for all k , $\{X_{n_k}\}$ converges uniformly on A_k^c so that the limit is continuous on each A_k^c . $\square$

Proposition 14. Suppose that the variable exponent p satisfies $1 < p^- \leq p(\omega) \leq p^+ < \infty$, then

$$\begin{aligned} \mathbb{L}_c^{p(\cdot)} &= \{X \in \mathbb{L}^{p(\cdot)} : X \text{ has a quasicontinuous version,} \\ &\text{and } \lim_{n \rightarrow \infty} \mathbb{E}[|X|^p 1_{\{|X| < n\}}] = 0\}. \end{aligned} \quad (28)$$

Proof. Let

$$\begin{aligned} J_{p(\cdot)} &= \{X \in \mathbb{L}^{p(\cdot)} : X \text{ has a quasicontinuous version,} \\ &\text{and } \lim_{n \rightarrow \infty} \mathbb{E}[|X|^p 1_{\{|X| < n\}}] = 0\}. \end{aligned} \quad (29)$$

For each $X \in \mathbb{L}_c^{p(\cdot)}$, X has a quasi-continuous version by Proposition 13. Since $X \in \mathbb{L}_c^{p(\cdot)} \subset \mathbb{L}_b^{p(\cdot)}$, by Proposition 9 we have $\lim_{n \rightarrow \infty} \mathbb{E}[|X|^p 1_{\{|X| > n\}}] = 0$. Thus, $X \in J_{p(\cdot)}$.

On the other hand, Let $X \in J_{p(\cdot)}$ be quasi-continuous. For all $n \in \mathbb{N}$, define

$$Y_n = (X \wedge n) \vee (-n). \quad (30)$$

Since $\lim_{n \rightarrow \infty} \mathbb{E}[|X|^p 1_{\{|X| > n\}}] = 0$ and

$$\begin{aligned} E[|X - Y_n|^p] &\leq E[|X - Y_n|^p 1_{\{|X| < n\}}] + E[|X - Y_n|^p 1_{\{|X| \geq n\}}] \\ &\leq E[|X|^p 1_{\{|X| < n\}}], \end{aligned} \quad (31)$$

we have $\lim_{n \rightarrow \infty} \mathbb{E}[|X - Y_n|^p] = 0$.

For all $n \in \mathbb{N}$, Y_n is quasi-continuous, so there exists a closed set F_n^c such that $c(F_n^c) < 1/n^{p^+ + 1}$ and Y_n is continuous on F_n^c . By Tietze's extension theorem there exists $Z_n \in C_b(\Omega)$ such that $|Z_n| \leq n$ and $Z_n = Y_n$ on F_n^c . Then, we have

$$\begin{aligned} E[|Y_n - Z_n|^p] &\leq E[|Y_n - Z_n|^p 1_{F_n^c}] + E[|Y_n - Z_n|^p 1_{F_n^c}] \\ &\leq E[(2n)^p 1_{F_n^c}] \leq (2n)^{p^+} c(F_n^c) \leq \frac{2^{p^+}}{n}. \end{aligned} \quad (32)$$

Thus,

$$\begin{aligned} E[|X - Z_n|^p] &\leq 2^{p^+ - 1} (E[|X - Y_n|^p] + E[|Y_n - Z_n|^p]) \\ &\leq 2^{p^+ - 1} E[|X - Y_n|^p] + \frac{2^{2p^+ - 1}}{n}. \end{aligned} \quad (33)$$

As a consequence, $X \in \mathbb{L}_c^{p(\cdot)}$. $\square$

Proposition 15. Suppose that $X : \Omega \rightarrow \mathbb{R}$ has a quasi-continuous version and that there exists a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying $\lim_{n \rightarrow \infty} f(t)/t^p = \infty$ uniformly on Ω and $\mathbb{E}[f(|X|)] < \infty$. Then $X \in \mathbb{L}_c^{p(\cdot)}$, where $1 < p^- \leq p(\omega) \leq p^+ < \infty$.

Proof. For each $\varepsilon > 0$, there exists $N > 0$ which is independent of ω such that for all $t \geq N$,

$$\frac{f(t)}{t^p} \geq \frac{1}{\varepsilon}. \quad (34)$$

So

$$\mathbb{E}[|X|^p 1_{\{|X| > N\}}] \leq \varepsilon \mathbb{E}[f(|X|) 1_{\{|X| > N\}}] \leq \varepsilon \mathbb{E}[f(|X|)]. \quad (35)$$

Thus, $\lim_{N \rightarrow \infty} \mathbb{E}[|X|^p 1_{\{|X| > N\}}] = 0$.

3. Kolmogorov's Criterion on Variable Exponent Function Spaces

Definition 16. Let I be a set of indices. $\{X_t\}_{t \in I}$ and $\{Y_t\}_{t \in I}$ be two processes indexed by I . We say that Y is a quasi-modification of X if for all $t \in I$, $X_t = Y_t$ q.s.

To prove a Kolmogorov's criterion for a process indexed by $\mathbb{R}^d$ with $d \in \mathbb{N}$ on variable exponent function spaces, we give the following lemma first.

Lemma 17. Let $1 < p^- \leq p(\omega) \leq p^+ < \infty$ and $\{X_t\}_{t \in [0,1]^d}$ be a process such that for all $t \in [0,1]^d$, $X_t \in \mathbb{L}^{p(\cdot)}$. Assume that for a fixed $P \in \mathcal{P}$ there exist positive constants c and ε such that

$$E_P[|X_t - X_s|^p] \leq c|t - s|^d + \varepsilon. \quad (36)$$

Then X admit a modification $\tilde{X}$ such that

$$E_P \left[\left(\sup_{s \neq t} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t - s|^\alpha} \right)^p \right] < \infty, \quad (37)$$

for every $\alpha \in [0, \varepsilon/p^+]$

Proof. Fix $P \in \mathcal{P}$. For $m \in \mathbb{N}$, define the set of dyadic points in $[0,1]^d$:

$$D_m := \left\{ \left(\frac{i_1}{2^m}, \dots, \frac{i_d}{2^m} \right) : i_1, \dots, i_d \in \{0, 1, \dots, 2^m\} \right\}, \quad (38)$$

and $D = \bigcup_m D_m$. Set $\Delta_m = \{(s, t) : |s - t| = 2^{-m}, s, t \in D_m\}$, and there are fewer than $2^{(m+1)d}$ such pairs in Δ_m . For $s, t \in D$, we say that $s \leq t$ if each component of s is less than or equal to the corresponding component of t .

Set $K_i = \sup_{(s,t) \in \Delta_i} |X_s - X_t|$. And by the hypothesis in the lemma,

$$E_P[K_i^p] \leq \sum_{(s,t) \in \Delta_i} E_P[|X_s - X_t|^p] \leq 2^{(i+1)d} c 2^{-i(d+\varepsilon)} = c 2^d 2^{-i\varepsilon}. \quad (39)$$

It is easy to see $c 2^d 2^{-i\varepsilon} < 1$ from some i_0 on.

For $s, t \in D$, there exists increasing sequences $\{s_n\} \subset D$ and $\{t_n\} \subset D$ such that $s_n, t_n \in D_n$, $s_n \leq s$, $t_n \leq t$ and $s_n = s$, $t_n = t$ from some n on.

Let now $s, t \in D$ and $|s - t| \leq 2^{-m}$, and either $s_m = t_m$ or $(s_m, t_m) \in \Delta_m$ and in any case,

$$X_s - X_t = \sum_{i=m}^{\infty} (X_{s_{i+1}} - X_{s_i}) + X_{s_m} - X_{t_m} + \sum_{i=m}^{\infty} (X_{t_i} - X_{t_{i+1}}), \quad (40)$$

where the series are actually finite sums. It follows that

$$|X_s - X_t| \leq K_m + 2 \sum_{i=m+1}^{\infty} K_i \leq 2 \sum_{i=m}^{\infty} K_i. \quad (41)$$

Thus, setting $M_\alpha = \sup_{s,t \in D, s \neq t} |X_s - X_t|/|s - t|^\alpha$, we have

$$\begin{aligned} M_\alpha &\leq \sup_{m \in \mathbb{N}} \left\{ 2^{(m+1)\alpha} \sup_{s,t \in D, s \neq t, |s-t| \leq 2^{-m}} |X_s - X_t| \right\} \\ &\leq \sup_{m \in \mathbb{N}} \left\{ 2^{(m+1)\alpha+1} \sum_{i=m}^{\infty} K_i \right\} \leq 2^{\alpha+1} \sum_{i=0}^{\infty} 2^{i\alpha} K_i. \end{aligned} \quad (42)$$

As

$$\begin{aligned} K_{ip(\cdot)} &\leq \max \left\{ (E_P[|K_i|p])^{1/p^-}, (E_P[|K_i|p])^{1/p^+} \right\} \\ &\leq \max \left\{ (c 2^d 2^{-i\varepsilon})^{1/p^-}, (c 2^d 2^{-i\varepsilon})^{1/p^+} \right\}, \end{aligned} \quad (43)$$

for every $\alpha \in [0, \varepsilon/p^+]$, we have

$$\begin{aligned} \|M_\alpha\|_{p(\cdot)} &\leq 2^{\alpha+1} \sum_{i=0}^{\infty} 2^{i\alpha} \|K_i\|_{p(\cdot)} \\ &\leq 2^{\alpha+1} \left(\sum_{i=0}^{i_0-1} 2^{i\alpha} (c 2^d 2^{-i\varepsilon})^{1/p^-} + \sum_{i=i_0}^{\infty} 2^{i\alpha} (c 2^d 2^{-i\varepsilon})^{1/p^+} \right) \\ &\leq C_{\alpha, p^-, \varepsilon} + C_{d, \alpha, p^+, \varepsilon} \sum_{i=i_0}^{\infty} 2^{i(\alpha - \varepsilon/p^+)} < \infty, \end{aligned} \quad (44)$$

where $\alpha \in [0, \varepsilon/p^+]$. It follows in particular that for almost every ω , X is uniformly continuous on D and it makes sense to set

$$\tilde{X}_t(\omega) = \lim_{s \rightarrow t, s \in D} X_s(\omega). \quad (45)$$

By Fatou's lemma and the hypothesis,

$$E_P[|\tilde{X}_t - X_t|^p] \leq \liminf_{s \rightarrow t} E_P[|X_s - X_t|^p] \leq \liminf_{s \rightarrow t} c|t - s|^{d+\varepsilon}, \quad (46)$$

so, $\tilde{X}_t = X_t$ a.s. and $\tilde{X}_t$ is a modification. $\square$

Theorem 18. Let $1 < p^- \leq p(\omega) \leq p^+ < \infty$ and $\{X_t\}_{t \in [0,1]^d}$ be a process such that for all $t \in [0,1]^d$, $X_t \in \mathbb{L}^{p(\cdot)}$. Assume that there exists a positive constants c and ε such that

$$E[|X_t - X_s|^p] \leq c|t - s|^{d+\varepsilon}. \quad (47)$$

Then X admit a modification $\tilde{X}$ such that

$$E \left[\left(\sup_{s \neq t} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t - s|^\alpha} p \right) \right] < \infty, \quad (48)$$

for every $\alpha \in [0, \varepsilon/p^+]$.

Proof. Let set D the same as the proof of Lemma 17. We set.

$$M_\alpha = \sup_{s,t \in D, s \neq t} \frac{|X_s - X_t|}{|s - t|^\alpha}, \quad (49)$$

where $\alpha \in [0, \varepsilon/p^+]$. By Lemma 17, we know that for any $P \in \mathcal{P}$, $E_P[M^p]$ is finite and uniformly bounded with respect to P so that

$$\mathbb{E}[M^p] = \sup_{P \in \mathcal{P}} E_P[M^p] < \infty. \quad (50)$$

Thus, X is uniformly continuous on D q.s. and set

$$\tilde{X}_t(\omega) = \lim_{s \rightarrow t, s \in D} X_s(\omega), \quad t \in [0,1]^d. \quad (51)$$

In the similar way in Lemma 17, $\tilde{X}_t = X_t$ q.s. and $\tilde{X}_t$ is a modification. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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