Hindawi
Abstract and Applied Analysis
Volume 2020, Article ID 2127071, 11 pages
https://doi.org/10.1155/2020/2127071

Research Article

Positive Solution for the Integral and Infinite Point Boundary Value Problem for Fractional-Order Differential Equation Involving a Generalized ϕ-Laplacian Operator

Nadir Benkaci-Ali

Faculty of Sciences, University M'Hmed Bouguerra, Boumerdes, Algeria

Correspondence should be addressed to Nadir Benkaci-Ali; radians_2005@yahoo.fr

Received 9 June 2020; Accepted 31 July 2020; Published 28 August 2020

Academic Editor: Changbum Chun

In this paper, we establish the existence of nontrivial positive solution to the following integral-infinite point boundary-value problem involving ϕ-Laplacian operator $D_{0^+}^{\alpha}\phi(x, D_{0^+}^{\beta}u(x)) + f(x, u(x)) = 0, x \in (0,1), D_{0^+}^{\sigma}u(0) = D_{0^+}^{\beta}u(0) = 0, u(1) = \int_0^1 g(t)u(t)dt + \sum_{n=1}^{n=+\infty}\alpha_n u(\eta_n)$, where $\phi : [0,1] \times R \to R$ is a continuous function and $D_{0^+}^{p}$ is the Riemann-Liouville derivative for $p \in \{\alpha, \beta, \sigma\}$. By using some properties of fixed point index, we obtain the existence results and give an example at last.

1. Introduction

Our aim in this article is to study the existence of a nontrivial positive solution to the following integral and infinite point boundary-value problem involving a two-dimensional ϕ-Laplacian operator

$$\begin{cases} D_{0^+}^{\alpha}\phi\left(x, D_{0^+}^{\beta}u(x)\right) + f(x, u(x)) = 0, & x \in (0,1), \\ D_{0^+}^{\sigma}u(0) = D_{0^+}^{\beta}u(0) = 0, & \\ u(1) = \int_0^1 g(t)u(t)dt + \sum_{n=1}^{n=+\infty}\alpha_n u(\eta_n), & \end{cases} \quad (1)$$

where $D_{0^+}^{p}$ is the Riemann-Liouville derivative for $p \in \{\alpha, \beta, \sigma\}$, $0 < \alpha \leq 1 \leq \beta \leq 2$ and

$$\sigma = \begin{cases} 0 & \text{if } 1 < \beta \leq 2, \\ 1 & \text{if } \beta = 1, \end{cases} \quad (2)$$

and $\alpha_n, \eta_n \in (0,1)$ for $n \geq 1$ such that

$$\sum_{n=1}^{n=+\infty}\alpha_n < \infty. \quad (3)$$

Throughout this paper, we assume that the following conditions are satisfied;

(A1) $\eta_0 = 0 < \eta_n < \eta_{n+1} < 1$ for $n \in N$ with

$$\lim_{n\to+\infty}\eta_n = \eta \leq 1. \quad (4)$$

(A2) $f : [0,1] \times R^+ \to R^+$ is continuous and $g : [0,1] \to R^+$ is an integrable function.

(A3) $\int_0^1 t^{\beta-1}g(t)dt + \sum_{n=1}^{n=+\infty}\alpha_n\eta_n^{\beta-1} < 1$.

(A4) $\phi : [0,1] \times R \to R$ is continuous and for $t \in 0, 1]$, the function $\phi(t,.)$ is odd and increasing, $\phi^{-1}(t,.)$ is the inverse function of $\phi(t,.)$ denoted by $\psi(t,.)$ where $\psi : [0,1] \times R \to R$ is continuous.

(A5) There exist $p^+, p^- \in \mathbb{R}$ with $p^+ \geq p^- > 1$ such that

$$\phi^-(x) \leq \phi(.,x) \leq \phi^+(x) \text{for} (t,x) \in 0,1] \times \mathbb{R}, \quad (5)$$

with

$$\phi^-(x) = \begin{cases} \phi_{p^+}(x) \text{if} x \in 0,1] \cup (-\infty, -1], \\ \phi_{p^-}(x) \text{if} x \in -1, 0] \cup 1, +\infty), \end{cases} \quad (6)$$

and

$$\phi^+(x) = \begin{cases} \phi_{p^-}(x) \text{if} x \in 0,1\Big] \cup (-\infty,-1], \\ \phi_{p^+}(x) \text{if} x \in -1, 0\Big] \cup 1,+\infty). \end{cases} \quad (7)$$

Boundary value problems involving a $p(t)$-Laplacian operator have attracted a great deal of attention in the last ten years (see [1] [2–9]). At the same time, boundary value problems with fractional-order differential equations involving $p(t)$-Laplacian are of great importance and are an interesting class of problems. Such kind of BVPs in Banach space has been studied by many authors, see, for example [10–13] and the references therein. Noting that the generalized ϕ-Laplacian operator can turn into the well-known $p(t)$-Laplacian operator when we replace ϕ by $\phi_{p(t)}(x) = |x|^{p(t)-2}x$, so our results extend and enrich some existing papers.

By using the homotopy deformation property of the fixed point index, our paper aims at investigating the existence of at least one positive solution for bvp 1.

The paper is organized as follows. In the first section, we recall some lemmas giving fixed point index calculations. In the second section, we present a fixed point formulation for bvp (1), and we close this section by some lemmas making use of homotopical arguments. After that, we give our main results and their proofs and we end by giving as an example, a problem involving a sum of many $p(t)$-Laplacian operators.

2. Preliminaries

For the sake of completeness, let us recall some basic facts needed in this paper. Let E be a real Banach space equipped with its norm noted $\|.\|$, $L(E)$ is the set of all linear continuous mapping from E into E . For $L \in L(E)$, $r(L) = \lim_{n\to\infty}\| L^n\|^{1/n}$ denotes the spectral radius of L. A nonempty closed convex subset K of E is said to be a cone if $K \cap (-K) = 0$ and $(tK) \subset K$ for all $t \geq 0$.

Let K be a cone in E. A cone K induces a partial ordering "$\leq$", defined so that $x \leq y$ if and only if $y - x \in K$.

K is said to be normal if there exists a positive constant N such that for all $u, v \in K$,

$$u \leq v \quad \text{implies} \|u\| \leq N\|v\|. \quad (8)$$

$L \in L(E)$ is said to be positive in K if $L(K) \subset K$, it is said to be strongly positive in K if int $(K) \neq \varnothing$ and $L(K\{0\}) \subset$ int (K), and it is said to be K-normal if for all $u, v \in K$,

$$u \leq v \quad \text{implies} \|Lu\| \leq \|Lv\|. \quad (9)$$

Let E be a real Banach space and let K be a cone.

Let $R > 0$, $B(0, R)$ be the ball of radius R in E and $A : K_R \to K$ a completely continuous mapping, where $K_R = B(0, R) \cap K$. We will use the following lemmas concerning computations of the fixed point index, i, for a compact map A (see [14]).

Lemma 1. *If $\|Ax\| < \|x\|$ for all $x \in \partial B(0, R) \cap K$, then*

$$i(A, K_R, K) = 1. \quad (10)$$

Lemma 2. *If $\|Ax\| > \|x\|$ for all $x \in \partial B(0, R) \cap K$, then*

$$i(A, K_R, K) = 0. \quad (11)$$

Lemma 3. *If $Ax \geq x$ for all $x \in \partial B(0, R) \cap K$, then*

$$i(A, K_R, K) = 1. \quad (12)$$

Lemma 4. *If $Ax \leq x$ for all $x \in \partial B(0, R) \cap K$, then*

$$i(A, K_R, K) = 0. \quad (13)$$

Lemma 5. *If $Ax \neq \lambda x$ for all $x \in \partial B(0, R) \cap K$ and $\lambda > 1$, then*

$$i(A, K_R, K) = 1. \quad (14)$$

3. Related Lemmas

Let $N : E \to E$ be an operator and K be a cone of a real Banach space E, and consider the partial ordering "$\leq$" in E, defined so that $x \leq y$ if and only if $y - x \in K$.

Let $\rho \in K^*$, and consider the following cone.

$P = K(\rho) = \{u \in K : u \geq \|u\|\rho\}$ and the positive value

$$\lambda_0(K) = \inf \Lambda^-(K), \quad (15)$$

where

$$\Lambda^-(K) = \{\lambda \geq 0 : \text{there exists } u \in K \cap \partial B(0, 1) \text{ such that } Nu \leq \lambda u\}. \quad (16)$$

Remark 6. It is clear that If N is completely continuous, then from Lemmas 4 and 5, there exist $\lambda \geq 1$ such that $\lambda \in \Lambda^-(K)$.

Lemma 7. *Assume that $N : E \to E$ is increasing, positively 1-homogeneous, and completely continuous, such that $N(K\{0\}) \subset K \setminus \{0\}$.*

If there exist $\rho \in K^*$ such that $NK \subset P = K(\rho)$, then

$$\lambda_0(K) = \lambda_0(P) > 0. \quad (17)$$

Proof. In first, we claim $\lambda_0(K) = \lambda_0(P)$.

Let $\lambda \geq 0$, $u \in K \cap \partial B(0, 1)$ such that $Nu \leq \lambda u$. Since $NK \subset P$, N is strictly increasing and positively 1-homogenuous, we have

$$N\left(\frac{Nu}{\|Nu\|}\right) \leq \lambda \frac{Nu}{\|Nu\|}, \quad (18)$$

then

$$\Lambda^-(K) \subset \Lambda^-(P), \quad (19)$$

with $P \subset K$ we deduce

$$\Lambda^{-}(P) = \Lambda^{-}(K), \tag{20}$$

and so

$$\lambda_0(K) = \lambda_0(P). \tag{21}$$

Now, we show that

$$\lambda_0(P) > 0. \tag{22}$$

In the contrary, we assume that there exist $(\lambda_n) \in R^+$ and $u_n \in P \cap \partial B(0,1)_n$ with $\lim_{n\to\infty}\lambda_n = 0$. Such that

$$Nu_n \leq \lambda_n u_n. \tag{23}$$

For $n \in N$,

$$\lambda_n u_n \geq Nu_n \geq N(\|u_n\|\rho) = N(\rho), \tag{24}$$

and so

$$\lambda_n u_n - N(\rho) \in K. \tag{25}$$

Then,
$\lim_{n\to\infty}\lambda_n u_n - N(\rho) = -N(\rho) \in K$, and we obtain

$$N(\rho) = 0, \tag{26}$$

which is a contradiction.

Remark 8. If K is a normal cone in a Banach space E, with the constant of normality $n = 1$ (i.e, $\|u\| \geq \|v\|$ if $u \geq v \geq 0$), then

$$\lambda_0(K) \geq \|N(\rho)\|. \tag{27}$$

Since for $\lambda \in \Lambda(P)$, $u \in P \cap B(0,1)$

$$\lambda u \geq Nu \geq N(\|u\|\rho) = N(\rho). \tag{28}$$

In the following lemma, we assume that $N_0 : E \to E$ is a positively 1-homogeneous and completely continuous operator, and $N : E \to E$ is a completely continuous, increasing and positively 1-homogeneous operator, such that

$$N(K \setminus \{0\}) \subset P \setminus \{0\}, \tag{29}$$

where $P = K(\rho)$, $\rho \in K^*$, and K is a normal cone in a Banach space E, with the constant of normality $n = 1$.

Lemma 9. *Let $Q, Q_0, G_2 : K \to K$ be continuous mappings with*

$$\lim_{\|u\|\to+\infty} \frac{\|Qu\|}{\|u\|} < +\infty \quad \textit{and} \lim_{\|u\|\to+\infty} \frac{\|G_2 u\|}{\|u\|} = 0 \leq \lim_{\|u\|\to+\infty} \frac{\|Q_0 u\|}{\|u\|} < +\infty, \tag{30}$$

such that

$$NQu - G_2 u \leq N_0 Q_0 u, \textit{for} u \in K. \tag{31}$$

Suppose that there exist $\lambda_1 \in \mathbb{R}^+$ and $G_1 : K \to K$ with

$$\lim_{\|u\|\to+\infty} \frac{G_1 u}{\|u\|} = 0, \tag{32}$$

such that

$$Qu \geq \lambda_1 u - G_1(u), \quad \text{for} \quad u \in K. \tag{33}$$

If

$$\lambda_1 > \lambda_0^{-1}(K), \tag{34}$$

then there exist $R_1 > 0$ such that for all $R \geq R_1$

$$i(N_0 Q_0, K_R, K) = i(NQ, P_R, P). \tag{35}$$

Moreover, if

$$\lambda_1 > \|N(\rho)\|^{-1}, \tag{36}$$

then there exist $R_2 > 0$ such that for all $R \geq R_2$

$$i(N_0 Q_0, K_R, K) = 0. \tag{37}$$

Proof. In first, we show that there exist $R_1 > 0$ such that for all $R \geq R_1$

$$i(NQ, K_R, K) = i(N_0 Q_0, K_R, K). \tag{38}$$

We consider the homotopy

$$H(t,u) = tN_0 Q_0 u + (1-t) NQu. \tag{39}$$

We show that there exist $R_1 > 0$ such that for all $R \geq R_1$ the equation $H(t,u) = u$ has not solutions in $[0,1] \times (K \cap \partial B(0,R))$. In the contrary, we assume that for all $n \in N$, there exist $R_n \geq n$ and $(t_n, u_n) \in 0,1] \times (K \cap \partial B(0,R_n))$ such that

$$u_n = H(t_n, u_n) = t_n N_0 Q_0 u_n + (1-t_n) NQu_n. \tag{40}$$

By dividing the above equation by $\|u_n\|$, we obtain

$$v_n = \frac{u_n}{\|u_n\|} = t_n N_0 \left(\frac{Q_0 u_n}{\|u_n\|}\right) + (1-t_n) N \left(\frac{Qu_n}{\|u_n\|}\right). \tag{41}$$

From (5),

$$\lim_{n\to\infty} \frac{\|Q_0 u_n\|}{\|u_n\|} < \infty \text{and} \lim_{n\to\infty} \frac{\|Qu_n\|}{\|u_n\|} < \infty, \tag{42}$$

then the sequences $(Q_0 u_n/\|u_n\|)_n$, $(Qu_n/\|u_n\|)_n$ are bounded, and we deduce from the compactness of N and N_0, that $(v_n)_n$ admits a convergent subsequence also denoted by $(v_n)_n$.

Let $v = \lim_{n\to\infty} v_n \in K \cap \partial B(0,1)$ and $t = \lim_{n\to\infty} t_n$.

By using the conditions (31) and (33), it follows from 12 that for all $n \in N$

$$v_n \geq N\left(\frac{Qu_n}{\|u_n\|}\right) - t_n \frac{G_2 u_n}{\|u_n\|} \geq N\left(\lambda_1 v_n - \frac{G_1 u_n}{\|u_n\|}\right) - t_n \frac{G_2 u_n}{\|u_n\|}. \quad (43)$$

With the fact that

$$\lim_{n\to+\infty} \frac{G_1 u_n}{\|u_n\|} = \lim_{n\to+\infty} \frac{G_2 u_n}{\|u_n\|} = 0, \quad (44)$$

we have

$$v \geq \lambda_1 N v, \quad (45)$$

and so

$$\lambda_1^{-1} \in \Lambda^-(K), \quad (46)$$

where

$$\Lambda^-(K) = \{\lambda \geq 0\,;\ \text{there exists}\ u \in K \cap \partial B(0,1)\ \text{such that}\ Nu \leq \lambda u\}. \quad (47)$$

Then

$$\lambda_1^{-1} \geq \lambda_0(K), \quad (48)$$

which contradicts (10).

Then, there exist $R_1 > 0$ such that for all $R \geq R_1$ the equation $H(t,u) = u$ has not the solutions in

$$0, 1] \times (K \cap \partial B(0,R)), \quad (49)$$

and by invariance property of fixed point index, we deduce that for all $R \geq R_{1.}$

$$i(NQ, K_R, K) = i(N_0 Q_0, K_R, K). \quad (50)$$

By the fact that $NQ(K) \subset P$, we have from the excision property of the fixed point index that

$$i(NQ, K_R, K) = i(NQ, P_R, P). \quad (51)$$

Then

$$i(N_0 Q_0, K_R, K) = i(NQ, P_R, P). \quad (52)$$

Now, we assume that the condition (36) holds.

By using Lemma 4, we prove that there exists $R_0 > 0$ such that, for all $R \geq R_0$

$$i(NQ, P_R, P) = 0. \quad (53)$$

In the contrary, we assume that for all $n \in N$, there exist $R_n \geq n$ and $u_n \in P \cap \partial B(0, R_n)$ such that

$$u_n \geq NQu_n. \quad (54)$$

By the condition (33), we have

$$v_n = \frac{u_n}{\|u_n\|} \geq N\left(\frac{Qu_n}{\|u_n\|}\right) - t_n \frac{G_2 u_n}{\|u_n\|} \geq N\left(\lambda_1 v_n - \frac{G_1(u_n)}{\|u_n\|}\right) - t_n \frac{G_2 u_n}{\|u_n\|}, \quad (55)$$

with

$$\lim_{n\to\infty} \frac{G_1(u_n)}{\|u_n\|} = \lim_{n\to\infty} \frac{G_2(u_n)}{\|u_n\|} = 0. \quad (56)$$

As

$$u_n \geq \rho \|u_n\|, \quad (57)$$

we have

$$v_n \geq N\left(\lambda_1 \rho - \frac{G_1(u_n)}{\|u_n\|}\right) - t_n \frac{G_2 u_n}{\|u_n\|}. \quad (58)$$

Set

$$A_n = N\left(\lambda_1 \rho - \frac{G_1(u_n)}{\|u_n\|}\right) - t_n \frac{G_2 u_n}{\|u_n\|} - N(\lambda_1 \rho). \quad (59)$$

We have

$$\lim_{n\to\infty} A_n = 0, \qquad v_n - A_n \geq N(\lambda_1 \rho) \geq 0. \quad (60)$$

Since K is normal with the constant of normality $N = 1$, then for $n \in N$,

$$\|v_n - A_n\| \geq \|N(\lambda_1 \rho))\|, \quad (61)$$

and so

$$1 = \lim_{n\to\infty} \|v_n\| = \lim_{n\to\infty} \|v_n - A_n\| \geq \lambda_1 \|N(\rho))\|, \quad (62)$$

then

$$1 \geq \lambda_1 \|N(\rho)\|, \quad (63)$$

which contradicts (10).

Consequently, for $R \geq R_2 = \max\{R_1, R_0\}$,

$$i(N_0 Q_0, K_R, K) = i(NQ, P_R, P) = 0. \quad (64)$$

Definition 10. [10, 11] The Riemann-Liouville fractional integral of order $p > 0$ of $f \in L^1([a,b], \mathbb{R}^+)$ is defined by

$$I_{a^+}^p f(x) = \frac{1}{\Gamma(p)} \int_a^x (x-t)^{p-1} f(t)\,dt, \quad (65)$$

where Γ is the gamma function.

Definition 11. [10, 11] The Riemann-Liouville fractional derivative of order $p \geq 0$ of a function f is defined by

$$D_{a^+}^{p} f(x) = \frac{d^n}{dx^n} I_{a^+}^{n-p} f(x), n = [\alpha] + 1, \tag{66}$$

where $[n]$ is the integer part of α.

Remark 12. If $p \in \mathbb{N}$, then

$$D_{a^+}^{p} f = \frac{\delta^p}{\delta x^p} f, \tag{67}$$

and for $p = 1$,

$$I_{a^+}^{1} f(x) = \int_a^x f(t)dt. \tag{68}$$

Lemma 13. *[15] Let $p > 0$, and let $u(t)$ be an integrable function in $[a, b]$.*

$$I_{a^+}^{p} D_{a^+}^{p} u(x) = u(x) + c_1(x-a)^{p-1} + c_2(x-a)^{p-2} \cdots + c_n(x-a)^{p-n}, \tag{69}$$

where $c_k \in \mathbb{R}$, $k \in \{1, 2, \cdots, n\}$, $n = [\alpha] + 1$ and $[n]$ is the integer part of α.

Lemma 14. *Let $h \in L(0, 1)$, $0 < \alpha \leq 1 \leq \beta \leq 2$ and*

$$\sigma = \begin{cases} 0 & \text{if } 1 < \beta \leq 2, \\ 1 & \text{if } \beta = 1. \end{cases} \tag{70}$$

Then the unique solution of

$$\begin{cases} D_{0^+}^{\alpha} \phi\left(x, D_{0^+}^{\beta} u(x)\right) + h(x) = 0, \quad x \in (0, 1), \\ D_{0^+}^{\sigma} u(0) = D_{0^+}^{\beta} u(0) = 0, \quad u(1) = \int_0^1 g(t)u(t)dt + \sum_{n=1}^{n=+\infty} \alpha_n u(\eta_n), \end{cases} \tag{71}$$

is given by

$$u(x) = \int_0^1 G(x,t)H(t)dt, \tag{72}$$

with

$$H(t) = \psi\left(t, \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds\right), \tag{73}$$

and

$$G(x,t) = \frac{1}{\Gamma(\beta)} \begin{cases} x^{\beta-1} G_m(t) - (x-t)^{\beta-1} & \text{if} 0 \leq t < \min\{x, \eta\}, \\ x^{\beta-1} G_\eta(t) - (x-t)^{\beta-1} & \text{if} \eta \leq t \leq x, \\ x^{\beta-1} G_m(t) & \text{if} x \leq t < \eta, \\ x^{\beta-1} G_\eta(t) & \text{if} t \geq \max\{x, \eta\} \end{cases} \tag{74}$$

with

$$\eta = \lim_{n\to\infty} \eta_n, , \tag{75}$$

and $m \in \mathbb{N}^*$ such that

$$\eta_{m-1} \leq t \leq \eta_m, \tag{76}$$

where

$$\begin{gathered} G_m(t) = \frac{\mu(t) - \sum_{n\geq m} \alpha_n (\eta_n - t)^{\beta-1}}{1-L}, \\ G_\eta(t) = \frac{\mu(t)}{1-L}, \\ \mu(t) = (1-t)^{\beta-1} - \int_t^1 (s-t)^{\beta-1} g(s)ds, \end{gathered} \tag{77}$$

with

$$L = \sum_{n\geq 1} \alpha_n \eta_n^{\beta-1} + \int_0^1 s^{\beta-1} g(s)ds < 1. \tag{78}$$

Proof. By Lemma 13, equation

$$D_{0^+}^{\alpha} \phi\left(x, D_{0^+}^{\beta} u(x)\right) + h(x) = 0, \tag{79}$$

gives

$$\phi\left(x, D_{0^+}^{\beta} u(x)\right) = \begin{cases} -I_{0^+}^{\alpha}(h)(x) + c_1 x^{\alpha-1} & \text{if} 0 < \alpha < 1, \\ -I_{0^+}^{1}(h)(x) + c_1 + c_2 x^{-1} & \text{if} \alpha = 1. \end{cases} \tag{80}$$

Since $D_{0^+}^{\beta} u(0) = 0$, we have that $c_1 = c_2 = 0$ and

$$D_{0^+}^{\beta} u(t) = -H(t), \tag{81}$$

with

$$H(t) = \psi\left(t, \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds\right). \tag{82}$$

And also from Lemma 13, we have

$$u(x) = \begin{cases} -I_{0^+}^{\beta} H(x) + d_1 x^{\beta-1} + d_2 x^{\beta-2} & \text{if } 1 \le \beta < 2, \\ -I_{0^+}^{\beta} H(x) + d_1 x + d_2 + d_3 x^{-1} & \text{if } \beta = 2. \end{cases} \quad (83)$$

If $\beta \neq 1$, then the condition $u(0) = 0$ leads $d_2 = d_3 = 0$, and if $\beta = 1$, the equation $D_{0^+}^{1} u = (\delta u/\delta x) = -H$ leads

$$u(x) = -I_{0^+}^{1} H(x) + d_1, \quad (84)$$

with

$$d_1 = u(0). \quad (85)$$

Then

$$u(x) = -I_{0^+}^{\beta} H(x) + d_1 x^{\beta-1}, \text{for} \beta \in 1, 2]. \quad (86)$$

In addition, from equation

$$u(1) = \sum_{n\ge1} \alpha_n u(\eta_n) + \int_0^1 g(s) u(s) ds, \quad (87)$$

we deduce that

$$\Gamma(\beta)(1-L) d_1 = -\sum_{n\ge1} \alpha_n \int_0^{\eta_n} (\eta_n - t)^{\beta-1} H(t) dt - \int_0^1 g(t) \int_0^t (t-s)^{\beta-1} H(s) ds dt + \int_0^1 (1-t)^{\beta-1} H(t) dt, \quad (88)$$

with $L = \sum_{n\ge1} \alpha_n \eta_n^{\beta-1} + \int_0^1 s^{\beta-1} g(s) ds$. The Fubini's theorem gives

$$\int_0^1 H(t) \int_t^1 (s-t)^{\beta-1} g(s) ds dt = \int_0^1 H(t) \int_t^1 (s-t)^{\beta-1} g(s) ds dt. \quad (89)$$

Then

$$u(x) = \frac{1}{\Gamma(\beta)} \left[\frac{C.x^{\beta-1}}{1-L} - \int_0^x (x-t)^{\beta-1} H(t) dt \right], \quad (90)$$

where

$$C = \int_0^1 (1-t)^{\beta-1} H(t) dt - \int_0^1 H(t) \int_t^1 (t-s)^{\beta-1} g(s) ds dt - \sum_{n\ge1} \alpha_n \int_0^{\eta_n} (\eta_n - t)^{\beta-1} H(t) dt. \quad (91)$$

Consequently, the solution of (19) is

$$u(x) = \int_0^1 G(x,t) H(t) dt, \quad (92)$$

with

$$G(x,t) = \frac{1}{\Gamma(\beta)} \begin{cases} x^{\beta-1} G_m(t) - (x-t)^{\beta-1} & \text{if} 0 \le t < \min\{x, \eta\}, \\ x^{\beta-1} G_\eta(t) - (x-t)^{\beta-1} & \text{if} \eta \le t \le x, \\ x^{\beta-1} G_m(t) & \text{if} x \le t < \eta, \\ x^{\beta-1} G_\eta(t) & \text{if} t \ge \max\{x, \eta\}, \end{cases} \quad (93)$$

with

$$\eta = \lim_{n\to\infty} \eta_n, , \quad (94)$$

and $m \in N^*$ such that

$$\eta_{m-1} \le t \le \eta_m, \quad (95)$$

where

$$G_m(t) = \frac{\mu(t) - \sum_{n\ge m} \alpha_n (\eta_n - t)^{\beta-1}}{1-L}, \quad G_\eta(t) = \frac{\mu(t)}{1-L}, \quad \mu(t) = (1-t)^{\beta-1} - \int_t^1 (s-t)^{\beta-1} g(s) ds. \quad (96)$$

This finishes the proof.

Lemma 15. *For $x, t \in [0, 1]$, we have*

$$h_1(t) x^{\beta} \le G(x,t) \le h_2(t) x^{\beta-1}, \quad (97)$$

where

$$h_1(t) = \frac{(1-t)^{\beta-1} \int_0^t s^{\beta-1} g(s) ds}{\Gamma(\beta)(1-L)}, \quad h_2(t) = \frac{\mu(t)}{\Gamma(\beta)(1-L)}, \quad (98)$$

with

$$L = \int_0^1 s^{\beta-1} g(s) ds + \sum_{n\ge1} \alpha_n \eta_n^{\beta-1}. \quad (99)$$

Proof. It is clear that the right hand inequality

$$G(x,t) \le h_2(t) x^{\beta-1}, \quad (100)$$

is obvious.

Now, we show that

$$G(x,t) \geq h_1(t)x^{\beta}, \tag{101}$$

where

$$h_1(t) = \frac{(1-t)^{\beta-1}\int_0^t s^{\beta-1}g(s)ds}{\Gamma(\beta)(1-L)}, \tag{102}$$

with

$$L = \int_0^1 s^{\beta-1}g(s)ds + \sum_{n\geq 1}\alpha_n\eta_n^{\beta-1}. \tag{103}$$

Let $x, t \in 0, 1]$.

For $n \in N^*$, as $t \geq t\eta_n$ and $t \geq tx$, we have

$$\begin{aligned}(\eta_n - t)^{\beta-1} &\leq \eta_n^{\beta-1}(1-t)^{\beta-1},\\ (x-t)^{\beta-1} &\leq x^{\beta-1}(1-t)^{\beta-1},\end{aligned} \tag{104}$$

and $t \geq ts$ gives

$$\int_t^1 (s-t)^{\beta-1}g(s)ds \leq (1-t)^{\beta-1}\int_t^1 s^{\beta-1}g(s)ds. \tag{105}$$

For $t, x \in 0, 1]$, we have

$$G(x,t) \geq \frac{x^{\beta-1}(1-t)^{\beta-1}}{\Gamma(\beta)}\left[\frac{1-\int_t^1 s^{\beta-1}g(s)ds - \sum_{n\geq 1}\alpha_n\eta_n^{\beta-1}}{1-L} - 1\right], \tag{106}$$

and with $x^{\beta-1} \geq x^{\beta}$ leads

$$G(x,t) \geq \frac{x^{\beta}(1-t)^{\beta-1}}{\Gamma(\beta)}\left[\frac{\int_0^1 s^{\beta-1}g(s)ds - \int_t^1 s^{\beta-1}g(s)ds}{1-L}\right], \tag{107}$$

then

$$G(x,t) \geq x^{\beta}h_1(t). \tag{108}$$

Remark 16. The function $G : [0,1]\times[0,1] \to \mathbb{R}$ defined by (20) is continuous, and from Lemma 15, we have $G(x,t) \geq 0$.

According to Lemma 14, u is solution of 1 if and only if u is a fixed point of the operator

$$Tu(x) = \int_0^1 G(x,t)\psi\left(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))ds\right)dt. \tag{109}$$

T can be written as

$$T = N_0 Q_0, \tag{110}$$

where

$$\begin{aligned}N_0 u(x) &= \int_0^1 G(x,\mathrm{t})u(t)dt\\ Q_0 u(t) &= \psi\left(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))ds\right).\end{aligned} \tag{111}$$

Remark 17. Let $E = C([0,1])$ be the Banach space equipped with the sup-norm $\|u\| = sup_{x\in 0,1]}|u(x)|$, and the cone

$$K = E^+ = \{u \in E\,; u \geq 0\}. \tag{112}$$

We have from 2 of the condition (A5) that

$$\psi^+(x) \leq \psi(.,x) \leq \psi^-(x)\mathrm{for}t \in [0,1], \tag{113}$$

where ψ^-, ψ^+ are the inverse functions of ϕ^-, ϕ^+, respectively, defined by

$$\psi^-(x) = \begin{pmatrix}\psi_{p^+}(x)\mathrm{if}x \in [0,1]\cup(-\infty,-1],\\ \psi_{p^-}(x)\mathrm{if}x \in [-1,0]\cup[1,+\infty),\end{pmatrix} \tag{114}$$

and

$$\psi^+(x) = \begin{pmatrix}\psi_{p^-}(x)\mathrm{if}x \in [0,1]\cup(-\infty,-1],\\ \psi_{p^+}(x)\mathrm{if}x \in [-1,0]\cup[1,+\infty).\end{pmatrix} \tag{115}$$

where ψ^+ is the inverse function of ϕ^+ defined in the condition (A5).

Remark 18. There exist $c, e > 0$ such that for all $(t,x) \in [0,1] \times \mathbb{R}^+$,

$$\psi_{p^-}(x) + e \geq \psi(t,x) \geq \psi_{p^+}(x) - c. \tag{116}$$

Remark 19. By Lemma 15 and Remark 18, we have for $u \in K$

$$N_0 Q_0 u \geq NQu - G_2 u, \tag{117}$$

where

$$Nu(x) = x^{\beta}\int_0^1 h_1(t)\psi_{p^+}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_{p^+}(u(s))ds\right)dt, \tag{118}$$

and

$$Qu(x) = \psi_{p^+}(f(x,u(x))), \tag{119}$$

with

$$G_2u(x) = c.x^{\beta}\int_0^1 h_1(t)dt, \qquad (120)$$

Moreover, the linear operator $N_0 : E \to E$ is compact, and $N : E \to E$ is completely continuous, increasing, positively 1- homogeneous and verifying

$$N(K \setminus \{0\}) \subset P \setminus \{0\}, \qquad (121)$$

where

$$P = K(\rho) = \{u \in K\,;\, u \geq \rho\|u\|\}, \qquad (122)$$

with

$$\rho(x) = x^{\beta} \in K^{*}. \qquad (123)$$

4. Main Results

Set

$$\lambda = \phi_{p^+}\left[\left[\int_0^1 h_2(t)\psi_{p^+}\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)dt\right]^{-1}\right]. \qquad (124)$$

Theorem 20. *Assume that there exist $r_0 > 0, r_1 > 0$ and*

$$\gamma > \phi_{p^+}\left(\|N(\rho)\|^{-1}\right), \qquad (125)$$

such that

$$f(t,x) < \lambda\phi_{p^+}(x), for (t,x) \in [0,1] \times [0,r_0], \qquad (126)$$

and

$$f(t,x) \geq \gamma\phi_{p^+}(x), for (t,x) \in [0,1] \times [r_1,+\infty), \qquad (127)$$

with

$$\lim_{x\to+\infty}\frac{\sup_{t\in[0,1]}\{f(t,x)\}}{\phi_{p^-}(x)} < \infty, \qquad (128)$$

then problem 1 has at least one nontrivial positive solution.

Proof. In first, we show that $i(T, K_r, K) = 1$ where

$$r = \min\left\{r_0, 1, \psi_{p^+}\left(\frac{\Gamma(\alpha+1)}{\lambda}\right\}. \qquad (129)$$

From (25), we have

$$f(t,x) < \lambda\phi_{p^+}(x), (t,x) \in 0,1] \times 0,r]. \qquad (130)$$

For $u \in \partial B(0,r) \cap K$,

$$\begin{aligned} Tu(x) &= \int_0^1 G(t,x)\psi\left(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))ds\right)dt \\ &\leq \int_0^1 x^{\beta-1}h_2(t)\psi\left(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))ds\right)dt \\ &\leq \int_0^1 h_2(t)\psi^-\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))ds\right)dt \\ &< \int_0^1 h_2(t)\psi^-\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda\phi_{p^+}(u(s))ds\right)dt \\ &< \int_0^1 h_2(t)\psi^-\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\lambda\phi^-(r)\right)\right)dt, \end{aligned} \qquad (131)$$

and from the definition of the constant r, we have that

$$\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\lambda\phi_{p^+}(r)\right)\right) \leq 1, \qquad (132)$$

then

$$Tu(x) < \int_0^1 h_2(t)\psi_{p^+}\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\lambda\phi_{p^+}(r)\right)\right)dt. \qquad (133)$$

Then,

$$\|Tu\| < \|u\|. \qquad (134)$$

By Lemma 1,

$$i(T, K_r, K) = 1. \qquad (135)$$

Now, by using Lemma 9, we show that there exists $R > 0$ such that

$$i(T, K_R, K) = 0. \qquad (136)$$

In first, we have from Remark 19 that

$$T = N_0Q_0 \geq NQ - G_2, \qquad (137)$$

where

$$G_2u = c. \qquad (138)$$

As $\lim_{\|u\|\to+\infty}(G_2(u)/\|u\|) = 0$, then the condition (31) of Lemma 9 is satisfied.

Now, we have from (26), for $x \geq r_1$

$$\psi_{p^+}(f(t,x)) \geq \lambda_1 x, \qquad (139)$$

with

$$\lambda_1 = \psi_{p^+}(\gamma) > \|N(\rho)\|^{-1}. \qquad (140)$$

Then, there exists $d \in R$ such that

$$\psi_{p^+}(f(t,x)) \geq \lambda_1 x - d, \text{for} x \geq 0. \tag{141}$$

and set

$$G_1(u) = d. \tag{142}$$

We have for $u \in K$

$$Q(u)(t) \geq \lambda_1 u(t) - G_1(u)(t), \tag{143}$$

with

$$\lim_{\|u\|\to\infty} \frac{G_1(u)}{\|u\|} = 0. \tag{144}$$

Moreover, from Remark 18, for $u \in K$,

$$Qu(t) = \psi_{p^+}(f(t,u(t))) \leq \psi_{p^-}(f(t,u(t))) + e + c, \text{for} t \in 0, 1], \tag{145}$$

and

$$\begin{aligned} Q_0 u(t) &= \psi\left(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s))ds\right) \\ &\leq \psi_{p^-}\left(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s))ds\right) + e. \end{aligned} \tag{146}$$

Then, from (27), we have

$$\lim_{\|u\|\to+\infty} \frac{\|Q(u)\|}{\|u\|} < \infty \text{and} \lim_{\|u\|\to+\infty} \frac{\|Q_0(u)\|}{\|u\|} < \infty. \tag{147}$$

By Lemma 9, there exist $R > r_0$ such that

$$i(N_0 Q_0, K_R, K) = 0. \tag{148}$$

Consequently, $T = N_0 Q_0$ has at least one fixed point u in $K \cap (\bar{B}(0,R) \setminus B(0,r))$, which is a nontrivial positive solution for problem 1.

Set

$$\lambda_2 = \phi_{p^-}\left[\int_0^1 h_2(t)\psi_{p^-}\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)dt\right]^{-1}, \tag{149}$$

and

$$N_2(u)(x) = x^\beta \int_0^1 h_1(t)\psi_{p^-}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_{p^-}(u(s))ds\right)dt. \tag{150}$$

Theorem 21. *Assume that there exist $r_2 > 0, r_3 > 0$ and*

$$\gamma > \phi_{p^-}\left(\|N_2(\rho)\|^{-1}\right), \tag{151}$$

such that

$$f(t,x) < \lambda_2 \phi_{p^-}(x), for (t,x) \in [0,1] \times [r_2, +\infty), \tag{152}$$

and

$$f(t,x) \geq \gamma \phi_{p^-}(x), for (t,x) \in [0,1] \times [0, r_3], \tag{153}$$

then problem 1 has at least one nontrivial positive solution.

Proof. In first, by using Lemma 3, we show that there exists $R \geq r_2$ such that $i(T, P_R, P) = 1$. In the contrary, we assume that there exists a sequence $(u_n)_n$ in P with

$$\lim_{n\to\infty} \|u_n\| = \infty, \tag{154}$$

such that

$$Tu_n \geq u_n. \tag{155}$$

From (28), there exist $\varepsilon > 0$ and $b \in R$ such that

$$f(t,x) \leq (\lambda_2 - \varepsilon)\phi_{p^-}(x) + b, (t,x) \in 0, 1] \times 0, +\infty). \tag{156}$$

Then, for $n \in N$

$$\begin{aligned} u_n \leq Tu_n(x) &= \int_0^1 G(t,x)\psi\left(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u_n(s))ds\right)dt \\ &\leq \int_0^1 x^{\beta-1} h_2(t)\psi\left(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u_n(s))ds\right)dt \\ &\leq \int_0^1 h_2(t)\psi_{p^-}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u_n(s))ds\right)dt + e\int_0^1 h_2(t)dt \\ &\leq \int_0^1 h_2(t)\psi_{p^-}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[(\lambda-\varepsilon)\phi_{p^-}(u_n(s)) + b\right]ds\right)dt \\ &\quad + e\int_0^1 h_2(t)\mathrm{d}t \leq \|u_n\|\psi_{p^-}(\lambda-\varepsilon)\int_0^1 h_2(t)\psi_{p^-} \\ &\quad \cdot \left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(1+r_n)ds\right)dt + e\int_0^1 h_2(t)dt, \end{aligned} \tag{157}$$

where

$$r_n = \frac{b}{\phi_{p^-}(\|u_n\|)(\lambda-\varepsilon)}. \tag{158}$$

Then,

$$1 \leq \psi_{p^-}(\lambda-\varepsilon)\int_0^1 h_2(t)\psi_{p^-}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(1+r_n)ds\right)dt, \tag{159}$$

with

$$\lim_{n\to\infty} r_n = 0 = \lim_{n\to\infty} \frac{e\int_0^1 h_2(t)dt}{\|u_n\|}, \tag{160}$$

it follows the following contradiction

$$1 \leq \psi_{p^-}(\lambda - \varepsilon)\psi_{p^-}\left(\lambda^{-1}\right) < 1. \tag{161}$$

Then, there exists $R \geq r_2$ such that

$$i(T, P_R, P) = 1. \tag{162}$$

Now, we prove that $i(T, P_{r_3}, P) = 0$.
Let $u \in P \cap \partial B(0, r_0)$, with

$$r_0 = \min\left\{1, r_3, \psi_{p^-}\left(\frac{\Gamma(\alpha+1)}{\gamma}\right)\right\}. \tag{163}$$

We have $u \geq \rho\|u\|$, and from (29)

$$\begin{aligned} Tu(1) &\geq \int_0^1 h_1(t)\psi\left(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\gamma\phi_{p^-}(u(s))\right)ds\right)dt \\ &\geq \int_0^1 h_1(t)\psi_{p^-}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\gamma\phi_{p^-}(\rho(s)\|u\|)ds\right)\right)dt. \end{aligned} \tag{164}$$

Then,

$$\left\|Tu\right| \geq \psi_{p^-}(\gamma)\|N_2(\rho)\|\left|\|u\| > \|u\|. \tag{165}$$

From Lemma 1, we have

$$i\left(T, P_{r_0}, P\right) = 0. \tag{166}$$

Consequently, $T = N_0 Q_0$ has at least one fixed point u in $K \cap (\bar{B}(0, R) \setminus B(0, r_0))$, which is a nontrivial positive solution for problem 1.

Example 22. We consider the following $(p_1(x), p_2(x), \cdots, p_n(x))$-Laplacian boundary value problem

$$\begin{cases} \sum_{k=1}^{k=N} D_{0^+}^{\alpha}\phi_{p_k(x)}\left(x, \left(D_{0^+}^{\beta}u(x)\right) + h(x, u(x)) = 0, x \in (0,1),\right. \\ D_{0^+}^{\sigma}u(0) = D_{0^+}^{\beta}u(0) = 0, \\ u(1) = \int_0^1 g(t)u(t)dt + \sum_{n=1}^{n=+\infty} \alpha_n u(\eta_n), \end{cases} \tag{167}$$

where $\phi_{p_k(t)}$ is the $p_k(t)$-Laplacian operator defined in $[0,1] \times \mathbb{R}$ as

$$\phi_{p_k(t)}(t, x) = |x|^{p_k(t)-2}.x, \text{for} k \in \{1, 2, \cdots, N\}, N \in \mathbb{N}^*, \tag{168}$$

with

$$p_{k(t)} \in C^1([0,1], (1, +\infty)). \tag{169}$$

We consider the problem 1 with $f(t,x) = h(t,x)/N$ and $\phi(t,x) = 1/N\sum_{k=1}^{k=N}\phi_{p_k(t)}(t,x)$. We assume that the conditions (A1), (A2), and (A3) are satisfied, and ϕ verifies (A4) and (A5) with

$$p^+ = \max\{p_k(t), t \in [0,1], k \in 1, 2, .., N\}, \tag{170}$$

and

$$p^- = \min\{p_k(t), t \in [0,1], k \in 1, 2, .., N\}. \tag{171}$$

Set

$$\begin{aligned} \lambda &= \phi_{p^+}\left[\int_0^1 h_2(t)\psi_{p^+}\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)dt\right]^{-1}, \\ \lambda_2 &= \phi_{p^-}\left[\int_0^1 h_2(t)\psi_{p^-}\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)dt\right]^{-1}, \end{aligned} \tag{172}$$

and

$$\gamma_0 = \max\left\{\phi_{p^+}\left(\|N(\rho)\|^{-1}\right), \phi_{p^-}\left(\|N_2(\rho)\|^{-1}\right)\right\}. \tag{173}$$

We deduce from Theorems 20 and 21 that, if there exist $R_0 > 0, R_1 > 0$ and $\gamma > \gamma_0$ such that h verifies one of the following conditions;

(H1)

$$\begin{aligned} &h(t,x) < N\lambda\phi_{p^+}(x), \text{for}(t,x) \in [0,1] \times [0, R_0]. \\ &h(t,x) \geq N\gamma\phi_{p^+}(x), \text{for}(t,x) \in [0,1] \times [R_1, +\infty), \end{aligned} \tag{174}$$

and

$$\lim_{x \to +\infty} \frac{\sup_{t \in [0,1]}\{h(t,x)\}}{\phi_{p^-}(x)} < \infty, \tag{175}$$

or

(H2)

$$h(t,x) < N\lambda_2\phi_{p^-}(x), \text{for}(t,x) \in [0,1] \times [R_1, +\infty), \tag{176}$$

and

$$h(t,x) \geq N\gamma\phi_{p^-}(x), \text{for}(t,x) \in [0,1] \times [0, R_0], \tag{177}$$

then problem (30) has at least one nontrivial positive solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

References

[1] A. Benmezai and S. Mechrouk, "Positive solution for Dirichlet p(t)-Laplacian Bvps," *Bulletin of the Iranian Mathematical Society*, vol. 42, no. 1, pp. 155–173, 2016.

[2] X. Fan, "On the sub-supersolution method for p(x)-Laplacian equations," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 665–682, 2007.

[3] X. Fan, "Remarks on eigenvalue problems involving the p(x)-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 352, no. 1, pp. 85–98, 2009.

[4] X. Fan, H. Wu, and F. Wang, "Hartman-type results for $p\ (t)$-Laplacian systems," *Nonlinear Analysis*, vol. 52, no. 2, pp. 585–594, 2003.

[5] X. Fan and Q. Zhang, "Existence of solutions for $p\ (x)$-Laplacian Dirichlet problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 52, no. 8, pp. 1843–1852, 2003.

[6] Q. Zhang, X. Liu, and Z. Qiu, "Existence of solutions and multiple solutions for a class of weighted p(r)-Laplacian system," *Journal of Mathematical Analysis and Applications*, vol. 355, no. 2, pp. 620–633, 2009.

[7] Q. Zhang, X. Liu, and Z. Qiu, "Existence of solutions for weighted p(t)-Laplacian system multi-point boundary value problems," *Nonlinear Anal*, vol. 71, no. 9, pp. 3715–3727, 2009.

[8] Q. Zhang, Y. Wang, and Z. Qiu, "Existence of solutions and boundary asymptotic behavior of p(r)-Laplacian equation multi-point boundary value problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 6, pp. 2950–2973, 2010.

[9] Q. Zhang, "Slowly oscillating solutions for differential equations with strictly monotone operator," *Journal of Inequalities and Applications*, vol. 2007, 10 pages, 2007.

[10] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)*, Academic Press, San Diego, 1988.

[11] I. Podlubny, *Fractional Differential Equations. Mathematics in Science and Engineering*, vol. 198, Academic Press, San Diego, 1999.

[12] T. Shen, W. Liu, and R. Zhao, "Fractional boundary value problems with p(t)-Laplacian operator," *Advances in Difference Equations*, vol. 118, 2016.

[13] T. Shen and W. Liu, "Solvability of fractional p-Laplacian boundary value problems with controlled parameters," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 5, pp. 2366–2383, 2017.

[14] D. Guo and V. Lakshmikantaham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[15] D. Jiang and C. Yuan, "The positive properties of the green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 2, pp. 710–719, 2010.