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Research Article

Some Fixed Point Theorems in Modular Function Spaces Endowed with a Graph

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The aim of this paper is to give fixed point theorems for G-monotone ρ-nonexpansive mappings over ρ-compact or ρ-a.e. compact sets in modular function spaces endowed with a reflexive digraph not necessarily transitive. Examples are given to support our work.

1. Introduction

Let X be a nonempty set. We denote by 2^X the set of subsets of X. An element x of X is said to be a fixed point of a self-mapping T on X, if $Tx = x$. For a set-valued mapping $T : X \to 2^X$, we call a fixed point of the set-valued mapping T every element x of X that verify $x \in Tx$.

The two most important results in fixed point theory, are without contest, the Banach contraction principle (BCP for short) and Tarski's fixed point theorem. Since their appearances, they were subject of many generalizations, either by extending the contractive condition for the B.C.P., or changing the structure of the space itself. For example, in the case of B.C.P., Ran and Reurings in [1] have obtained a fixed point result for a relaxed contraction condition in metric spaces endowed with a partially order relation, i.e., a contraction only for comparable elements. Jachymski in [2] got a further generalization of Ran and Reurings result by replacing the partial order by a relaxed type of graph in metric spaces, where Nadler [3] managed to give an equivalent form of the B.C.P. for multi-valued mappings.

In the beginning of 1930's, Orlicz and Birnbaum considered the space

$$L_\varphi = \left\{ f : \mathbb{R} \to \mathbb{R} : \quad \exists \lambda < 0 \int_{\mathbb{R}} \varphi(\lambda |f(x)|) dx < \infty \right\}, \quad (1)$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex increasing function, such that $\lim_{x\to\infty} \varphi(x) = \infty$. So, L_φ become a generalization of L^p spaces, which corresponds to the particular case $\varphi(t) = t^p$ where $t \in [0, \infty)$ and $1 \le p \le \infty$. Nevertheless, the formula that gives the norm over L^p, does not establish a norm over L_φ. Thereby, Nakano in [4, 5], captured the essence of the good behavior of the quantity $\int_{\mathbb{R}} \varphi(\lambda |f(x)|) dx$, what he called modular function, and gave some characterization of the geometry of these spaces, see [6].

Fixed point theory in modular function spaces was first studied by Khamsi et al. in [7], we find there an outline of a fixed point theory for ρ-nonexpansive mappings defined on some subsets of modular function spaces. Recently, Alfuraidan in [8], gave some extensions of fixed point theorems in modular function spaces endowed with partial order relation, namely Ran and Reurings result in this context.

On the other hand, the combination of metric fixed point theory and graph theory allows Jachymski in [2] to give an extension of Banach Contraction Principle in a metric space endowed with a graph. Souayah et al. in [9], after introducing the notion of G_m-contraction they were able to investigate the existence and uniqueness of the fixed point for such contractions in M-metric space endowed with a graph, in particular they generalize the results of Jachymski. Recently, in [10] the authors proved the existence of a unique best proximity point for some contractive type mappings in rectangular metric spaces endowed with a graph structure.

In this paper, we generalize all the results obtained by Alfuraidan in [8]. We consider modular function spaces endowed with graph which satisfy a path connectivity instead of adjacency, and we have got new fixed point theorems for G-monotone set-valued mappings. Examples are given to support our work.

2. Preliminaries

Let (L_ρ, ρ) be a modular function space where ρ is a nonzero regular modular function see [6, 11, 12] for more details. Recall that for all $f \in L_\rho$, ρ satisfies the following properties:

(i) $\rho(f) = 0 \iff f = 0$;
(ii) $\rho(\alpha f) = \rho(f)$ if $|\alpha| = 1$;
(iii) $\rho(\alpha f + (1-\alpha)g) \le \alpha\rho(f) + (1-\alpha)\rho(g)$.

If ρ satisfies the above conditions, we note then $\rho \in \mathscr{R}$. Furthermore, the associated norm $\|.\|_\rho$ defined using modular function ρ is called Luxembourg norm and we have:

$$\|f\|_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{f}{\alpha}\right) \le 1\right\}. \quad (2)$$

The following definitions will be needed in the sequel.

Definition 1. (see [11]). Let $\rho \in \mathscr{R}$.

(1) We say that $(f_n)_n \in L_\rho$ ρ-converges to f, and write: $f_n \to f(\rho)$, if $\rho(f_n - f) \to 0$, and a sequence $(f_n)_n \in L_\rho$ is called ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as $(n,m) \to \infty$.
(2) A set $B \subset L_\rho$ is called ρ-closed, if for any sequence $(f_n)_n \in B$, $f_n \to f(\rho)$ implies $f \in B$.
(3) A set $B \subset L_\rho$ is called ρ-bounded, if his diameter $\delta_\rho(B) = \sup\{\rho(f-g)/f, g \in B\}$ is finite.
(4) A set $B \subset L_\rho$ is called ρ-compact, if for any sequence $(f_n)_n \in L_\rho$ there exists a subsequence $(f_{k_n})_n$ and $f \in B$ such that f_{k_n} ρ-converges to f.

If $C \subset L_\rho$, $\mathscr{C}(C)$ will denote the set of ρ-closed subsets of C, and $\mathscr{K}(C)$, the set of ρ-compact subsets of C.

Definition 2. (see [11]). Let $\rho \in \mathscr{R}$, We say that ρ has the Δ_2-type condition, if there exists $k \in [0, +\infty)$ such that: $\rho(2f) \le k\rho(f)$, for any $f \in L_\rho$.

It is known that the ρ-convergence and $\|.\|_\rho$-convergence are different in general, but if we assume that ρ has the Δ_2-type condition we have the equivalence. That is, the Luxembourg norm convergence and ρ-convergence are equivalent (see [11, Lemma 3.2]).

The following lemma will be very useful along this work.

Lemma 1 (see [13]). *Let ρ be convex and satisfy the Δ_2-type condition. Let $(f_n)_n \in L_\rho$, such that*

$$\rho(f_{n+1} - f_n) \le K\alpha^n, \quad \forall n, \quad (3)$$

where K is arbitrary nonzero constant and $\alpha \in (0,1)$, then $(f_n)_n$ is Cauchy for $\|.\|_\rho$ and ρ-Cauchy.

Theorem 1. (see [11]). *Let $\rho \in \mathscr{R}$.*

(i) *$(L_\rho, \|.\|_\rho)$ is a complete normed space, and $L\rho$ is ρ-complete.*
(ii) *$\|f_n\|_\rho \to 0$ iff $\rho(\alpha f_n) \to 0$ for every $\alpha > 0$.*
(iii) *If ρ has the Δ_2-property and $\rho(\alpha f_n) \to 0$ for $\alpha > 0$, then $\|f_n\|_\rho \to 0$.*

We recall the definition of digraph, the interested reader can consult the book [14] for more details.

Definition 3. A directed graph or digraph G is determined by a nonempty set $V(G)$ of its vertices and the set $E(G) \subset V(G) \times V(G)$ of its directed edges. A digraph is reflexive if each vertex has a loop. Given a digraph $G = (V, E)$.

(i) If whenever $(x, y) \in E(G) \Rightarrow (y, x) \notin E(G)$, then the digraph G is called an oriented graph.
(ii) A digraph G is transitive whenever $[(x,y) \in E(G)$ and $(y,z) \in E(G)] \Rightarrow (x,z) \in E(G)$, for any $x, y, z \in V(G)$.
(iii) A dipath of G is a sequence $a_0, a_1, \ldots, a_n, \ldots$ with $(a_i, a_{i+1}) \in E(G)$ for each $i \in \mathbb{N}$.
(iv) A finite dipath of length n from x to y is a sequence of $n+1$ vertices $(a_0, a_1, \ldots, a_n)$ with $(a_i, a_{i+1}) \in E(G)$ and $x = a_0$, $y = a_n$.
(v) A closed directed path of length $n > 1$ from x to y, i.e., $x = y$, is called a directed cycle.
(vi) A digraph is connected if there is a finite (di)path joining any two of its vertices and it is weakly connected if $\widetilde{G}$ is connected.
(vii) $[x]_G$ is the set of all vertices which are contained in some path beginning at x. (i.e. $y \in [x]_G \iff$ there exist $(a_0, a_1, \ldots, a_n)$ with $(a_i, a_{i+1}) \in E(G)$ and $x = a_0$, $y = a_n$).

It seems that the graph theory when coupled with the classical metric fixed point theory leads to a new interesting theory, following the works of [2, 8] we introduce the following definitions.

Definition 4. Let $\rho \in \mathscr{R}$, $C \subset L_\rho$ a nonempty subset and $T : C \to 2^C$ a set-valued mapping.

(i) We say that T is a G-monotone ρ-contraction, if there exists $\alpha \in [0,1)$ such that for any $f, h \in C$ with $h \in [f]_G$, and any $F \in T(f)$; there exists $H \in T(h)$ verifying $H \in [F]_G$ and $\rho(F - H) \le \alpha\rho(f - h)$.
(ii) We say that T is a G-monotone ρ-nonexpansive mapping, if for any $f, h \in C$ with $h \in [f]_G$, and any $F \in T(f)$; there exists $H \in T(h)$ verifying $H \in [F]_G$ and $\rho(F - H) \le \rho(f - h)$.

Note that if T is only a single valued and the graph is transitive we get the notion of edge preserving ρ-contraction (resp. ρ-nonexpansive) mapping.

Definition 5. Let C be a nonempty subset of L_ρ and G be a digraph such that $V(G) = C$. If for any sequence $(f_n)_n$ in C that ρ-converges to f, we have $f \in [f_n]_G$ for every n, provided $f_{n+1} \in [f_n]_G$ for every n; then we say that G has the (P)-property.

3. Fixed Point Results for ρ-Contractions Mappings

We start this section by the following result which will be useful in the sequel.

Proposition 1. *Let $\rho \in \mathscr{R}$. Let $C \subset L_\rho$ be a nonempty ρ-closed subset, G a reflexive digraph such that $V(G) = C$. Let $T : C \to 2^C$ be a G-monotone ρ-contraction mapping and suppose that T has a fixed point, then if $f \in [\overline{f}]_G$ for some $f \in C$ where $\overline{f}$ is a fixed point for T such that $\rho(f - \overline{f}) < \infty$; there exists a sequence $(f_n)_n \in C$ such that $f_{n+1} \in T(f_n), \forall n$; and $(f_n)_n$ ρ-converges to $\overline{f}$.*

Proof. Since $f \in [\overline{f}]_G$ and $\overline{f} \in T(\overline{f})$ where T is G-monotone ρ-contraction, there exists $f_1 \in T(f)$ such that $f_1 \in [\overline{f}]_G$ and $\rho(\overline{f} - f_1) \le \alpha\rho(\overline{f} - f)$.

By induction we construct a sequence $(f_n)_n$ such that $f_{n+1} \in T(f_n)$ and $\rho(\overline{f} - f_{n+1}) \le \alpha\rho(\overline{f} - f_n)$ for each $n \in \mathbb{N}$ i.e.

$$\rho(\overline{f} - f_n) \le \alpha^n \rho(\overline{f} - f), \tag{4}$$

for all $n \in \mathbb{N}$. And since $\alpha \in [0, 1)$, we get $(f_n)_n$, ρ-converges to $\overline{f}$. □

If ρ has the Δ_2-type condition then $\rho(f) < \infty$ for every $f \in L_\rho$. Thus, we get the following result which is a generalization of [8, Theorem 3.1].

Theorem 2. *Let $\rho \in \mathscr{R}$ be convex and satisfies the Δ_2-type condition. Let $C \subset L_\rho$ be a nonempty ρ-closed subset, G a reflexive digraph such that $V(G) = C$, with the (P)-property. Let $T : C \to \mathscr{C}(C)$ be a G-monotone ρ-contraction mapping and $C_T := \{f \in C : g \in [f]_G \text{ for some } g \in T(f)\}$.*

If $C_T \ne \emptyset$, then T has a fixed point, moreover, if $f \in C_T$ then $T|_{[f]_G}$ has a fixed point.

Proof. Let $f_0 \in C_T$, there exists then $f_1 \in T(f_0)$ such that $f_1 \in [f_0]_G$ and as T is G-monotone ρ-contraction, there exists $f_2 \in T(f_1)$ such that $f_2 \in [f_1]_G$ and

$$\rho(f_2 - f_1) \le \alpha\rho(f_1 - f_0), \quad \alpha \in (0, 1). \tag{5}$$

In the same way, there exists $f_3 \in T(f_2)$ such that $f_3 \in [f_2]_G$ and $\rho(f_3 - f_2) \le \alpha\rho(f_2 - f_1)$ and by induction we construct a sequence $(f_n) \in C$ with $f_{n+1} \in T(f_n)$ and $f_{n+1} \in [f_n]_G$ such that $\rho(f_{n+1} - f_n) \le \alpha\rho(f_n - f_{n-1}), \forall n \ge 1$, which implies $\rho(f_{n+1} - f_n) \le \alpha^n\rho(f_1 - f_0)$. Hence by the Lemma 1, $(f_n)_n$ is ρ-Cauchy, and as L_ρ is ρ-complete then $(f_n)_n$ ρ-converges to some $f \in C$ as C is ρ-closed. And since G has the (P)-property then for each $n, f \in [f_n]_G$ and as T is G-monotone ρ-contraction, there exists $g_n \in T(f)$ such that for every n:

$$g_n \in [f_{n+1}]_G \text{ and } \rho(f_{n+1} - g_n) \le \alpha\rho(f_n - f). \tag{6}$$

And then clearly, $(f_{n+1} - g_n)_n$ ρ-converges to zero in L_ρ thus $(f_n - f)_n$ and $(f_{n+1} - g_n)_n$ both converge to zero in $(L_\rho, \|.\|_\rho)$ which mean that $(g_n)_n$ converges to f in $(L_\rho, \|.\|_\rho)$ as $\|.\|_\rho$ is a norm on L_ρ, therefore $(g_n)_n$ ρ-converges to f and then $f \in T(f)$ as $(g_n)_n \in T(f)$ and $T(f)$ is ρ-closed i.e., f is a fixed point of T.

Note that as $f \in [f_n]_G$ for every n, in particular $f \in [f_0]_G$ thus $T|_{[f_0]_G}$ has a fixed point. □

Example 1. Let $L_\rho = L^\infty([0, 1])$ the set of measurable functions $f : [0, 1] \to \mathbb{R}$ such that $\exists M \ge 0$ with $|f(x)| \le M$ a.e. in $[0, 1]$. It is known that $L^\infty([0, 1])$ is a normed linear space with the norm

$$\rho(f) = \|f\|_\infty = \inf\{M < 0 : |f(x)| \le M \text{ a.e. in } [0, 1]\}. \tag{7}$$

Then, (L_ρ, ρ) is a modular function space with the Δ_2-condition. Let $C = B_\rho(0, 1)$ be the ρ-closed ball centered at 0 with radius 1, that is the set

$$\{f \in L^\infty : \|f\|_\infty \le 1\}. \tag{8}$$

Consider the digraph $G = (C, E)$ defined by

$$\begin{aligned}&(f_1, f_2) \in E \iff \exists c \in [0, 1] \text{ such that}\\ &|f_2(x)| \le c|f_1(x)|, \text{ a.e. in } [0, 1],\end{aligned} \tag{9}$$

for every $f_1, f_2 \in C$. Thus we get,

$$\begin{aligned}&h \in [f]_G \iff \exists c \in [0, 1] \text{ such that}\\ &|f(x)| \le c|h(x)| \text{ a.e. in } [0, 1].\end{aligned} \tag{10}$$

It is clear that this binary relation defines a reflexive digraph without being a partially order (i.e., without antisymmetric condition).

Let $f \in C$ and define $T : C \to \mathscr{C}(C)$ by

$$Tf = \left\{\frac{1}{2^n} f : n \ge 1\right\} \cup \{0\}. \tag{11}$$

(i) For all $F \in Tf$, we have $\|F\|_\infty = \|1/2^n f\| \le \|f\| \le 1$, which implies that $F \in C$. Moreover, Tf is ρ-closed since $Tf = \overline{\{(1/2^n) f : n \ge 1\}}$.

(ii) $\forall f \in C$, if $h \in [f]_G$, let for any $F \in Tf$ we have two cases: $F = 1/2^n f$ or $F = 0$. If $F = 1/2^n f$ choose $H = 1/2^n h$, hence we have the G-monotonocity:

$$|F(x)| = \frac{1}{2^n}|f(x)| \le c.\frac{1}{2^n}|h(x)| = c|H(x)|, \tag{12}$$

and the ρ-contraction condition for $\alpha = 1/2$

$$\rho(F - H) = \frac{1}{2^n}\rho(f - h) \le \frac{1}{2}\rho(f - h). \tag{13}$$

Now, if $F = 0$ let $H = 0$ the G-monotonocity and ρ-contraction conditions are obvious.

For the (P)-property. Let (f_n) be a sequence in C such that

$$f_{n+1} \in [f_n]_G \ \forall n \text{ and } \|f_n - f\|_\infty \to 0 \text{ as } n \to \infty. \quad (14)$$

$f_{n+1} \in [f_n]_G \Rightarrow \exists c_n \in [0,1]$ such that $|f_n(x)| \le c_n |f_{n+1}(x)|$, and since $0 \le c_n \le 1$ we get for all $n \in \mathbb{N}$,

$$|f_n(x) \le |f_{n+1}(x)|, \forall n. \quad (15)$$

Furthermore,

$$\left||f_n(x)| - |f(x)|\right| \le |f_n(x) - f(x)| \le \|f_n - f\|_\infty, \quad (16)$$

implies that $|f_n(x)| \to |f(x)|$, thus $|f_n(x)| \le |f(x)|$ for all $n \in \mathbb{N}$, hence $f \in [f_n]_G$. That is G has the (P)-property. Then the set-valued mapping T satisfies all the conditions of Theorem 2 hence it has a fixed point, namely $0 \in T0$.

Corollary 1. *Let $\rho \in \mathscr{R}$ be convex and satisfies the Δ_2-type condition. Let $C \subset L_\rho$ be a nonempty ρ-closed subset, G a reflexive digraph such that $V(G) = C$, with the (P)-property. Let $T : C \to \mathscr{C}(C)$ be a G-monotone ρ-contraction mapping. If G has a mother vertex, (a vertex f such that all other vertices in G can be reached by a path from f), then T has a fixed point.*

Proof. Indeed for such $f, g \in [f]_G$, for all $g \in T(f)$ and then $f \in C_T \neq \emptyset$, we then get the result using Theorem 2. □

4. Fixed Point Results for ρ-Nonexpansive Mappings

Recall the notion of approximated fixed point sequence.

Definition 6. We say that $T : C \to 2^C$ has an approximated fixed point sequence, if there exists $(f_n)_n \in C$ such that for every n, there exists $F_n \in T(f_n)$ satisfying $\lim_n \rho(f_n - F_n) = 0$.

Definition 7. A digraph G is said to be G-convex if for all $f \in [F]_G$ and $h \in [H]_G$ we have $\lambda f + (1-\lambda)h \in [\lambda F + (1-\lambda)H]_G$.

If the reflexive digraph G is transitive and antisymmetric, i.e., being a partially order, then the most ordered Banach spaces enjoy this property. The following result gives a sufficient condition to obtain an approximated fixed point sequence for G-convex digraph.

Proposition 2. *Let $\rho \in \mathscr{R}$ be convex and satisfies the Δ_2-type condition. Let $C \subset L_\rho$ be a nonempty convex, ρ-closed, and ρ-bounded subset, G a G-convex reflexive digraph such that $V(G) = C$, with the (P)-property.*

Let $T : C \to \mathscr{C}(C)$ be a G-monotone ρ-nonexpansive mapping, if $C_T := \{f \in C : f \in [g]_G \text{ for some } g \in T(f)\}$ is nonempty; then T has an approximated fixed point sequence.

Proof. Let $f_0 \in C_T$ and for each $\lambda \in (0,1)$ define:

$$T_\lambda(f) = \lambda f_0 + (1-\lambda)T(f), \quad (17)$$

for any $f \in C$. Then $T_\lambda(f)$ is ρ-closed since if

$$g \in T_\lambda(f) \iff \frac{1}{1-\lambda}(g - \lambda f_0) \in T(f). \quad (18)$$

Moreover, C_{T_λ} is nonempty for any $\lambda \in (0,1)$ as $f_0 \in C_{T_\lambda}$.

Let $f, g \in C$ such that $g \in [f]_H$, since T is H-monotone ρ-nonexpansive, we get for all $F \in T(f)$ there exists $H \in T(g)$ such that $H \in [F]_H$ and

$$\rho(F - H) \le \rho(f - g), \quad (19)$$

thus, $\lambda f_0 + (1-\lambda)H \in [\lambda f_0 + (1-\lambda)F]_H$ and as ρ is convex:

$$\begin{aligned}\rho((\lambda f_0 + (1-\lambda)F) - (\lambda f_0 + (1-\lambda)H)) \\ = \rho((1-\lambda)(F - H)) \le (1-\lambda)\rho(f - g),\end{aligned} \quad (20)$$

i.e., T_λ is H-monotone ρ-contraction. Using Theorem 2, T_λ has a fixed point $f_\lambda \in C$. Thus $\exists F_\lambda \in T(f_\lambda)$ such that $f_\lambda = \lambda f_0 + (1-\lambda)F_\lambda$ and then, as ρ is convex:

$$\rho(f_\lambda - F_\lambda) = \rho(\lambda(f_0 - F_\lambda)) \le \lambda \delta_\rho(C). \quad (21)$$

Choosing $\lambda = 1/n$ for $n \ge 1$ gives the result. □

Now with more restrictions on the graph H, we obtain a stronger version of the above result (Proposition 2).

Definition 8. Let $\rho \in \mathscr{R}$, a digraph G is said to be compatible with the vector structure of L_ρ, if for every $f, g, \tilde{f}, \in L_\rho$ such that $\tilde{f} \in [f]_G$ then $\tilde{f} + g \in [f + g]_G$ and $\alpha \cdot \tilde{f} \in [\alpha \cdot f]_G, \forall \alpha \in \mathbb{R}_+$.

Notice that if G is compatible with the vector structure of L_ρ then it is G-convex. Indeed, for every $f, g, \tilde{f}, \tilde{g} \in L_\rho$ such that $\tilde{f} \in [f]_G$ and $\tilde{g} \in [g]_G$ then $\tilde{f} + \tilde{g} \in [f + g]_G$, and thus for every $\alpha \in [0,1], \alpha\tilde{f} + (1-\alpha)\tilde{g} \in [\alpha f + (1 - \alpha g)]_G$.

Lemma 2. *Let $\rho \in \mathscr{R}$ be convex and satisfies the Δ_2-type condition. Let $C \subset L_\rho$ be a nonempty convex, ρ-closed, and ρ-bounded subset, G a reflexive digraph compatible with the vector structure of L_ρ such that $V(G) = C$, with the (P)-property. Let $T : C \to \mathscr{C}(C)$ be a G-monotone ρ-nonexpansive mapping, if $C_T := \{f \in C : f \in [g]_G \text{ for some } g \in T(f)\}$ is nonempty; then there exist two sequences $(f_n)_n$ and $(F_n)_n \in L_\rho$ such that*

$$F_n \in T(f_n), f_{n+1} \in [f_n]_{G,} F_{n+1} \in [f_{n+1}]_{G,} \text{ for every } n, \quad (22)$$

and $\lim_n \rho(F_n - f_n) = 0$.

Proof. Let $f_0 \in C_T$ we define $T_1(f) = (1/2)f_0 + (1/2)T(f), f \in L_\rho$, as seen in the proof of Proposition 2 T_1 is a G-monotone ρ-contraction mapping, $T_1(f)$ is ρ-closed for every $f \in L_\rho$ then T_1 has a fixed point f_1 such that $f_1 \in [f_0]_G$ as we have already seen in Theorem 2. Then, there exists $F_1 \in T(f_1)$ such that

$$f_1 = \frac{1}{2}f_0 + \frac{1}{2}F_1, \quad (23)$$

thus, it comes clearly that $F_1 \in [f_1]_G$. In addition:

$$\rho(f_1 - F_1) = \rho\left(\frac{1}{2}(f_0 - F_1)\right) \le \frac{1}{2}\rho(f_1 - F_1) \le \frac{1}{2}\delta(C), \quad (24)$$

as ρ is convex.

Clearly $f_1 \in C_T$. Let define $T_2(f) = (1/3)f_1 + (2/3)T(f)$, for all $f \in L_\rho$. As above T_2 admits a fixed point f_2 such that $f_2 \in [f_1]_G$ and then there exists $F_2 \in T(f_2)$ such that $f_2 = (1/3)f_1 + (2/3)F_2$, thus $F_2 \in [f_2]_G$ and $\rho(f_2 - F_2) \le (1/3)\,\delta(C)$.

By induction on $n \ge 1$, if we define the set-valued mapping T_n to be

$$T_n(f) = \frac{1}{n+1}f_n + \left(1 - \frac{1}{n+1}\right)T(f), \quad \forall f \in L_\rho, \quad (25)$$

we get the existence of a fixed point f_n of T_n such that $f_n \in [f_{n-1}]_G$, thus there exists $F_n \in T(f_n)$ such that $f_n = (1/(n+1))f_{n+1} + (1 - 1/(n+1)F_n)$ and then $F_n \in [f_n]_G$ and $\rho(f_n - F_n) \le (1/(n+1))\delta(C)$. Since C is ρ-bounded we get the result. □

We then apply Lemma 2 to get a new fixed point result for G-monotone nonexpansive mapping.

Theorem 3. *Let $\rho \in \mathscr{R}$ be convex, satisfies the Δ_2-type condition. Let $C \subset L_\rho$ be a nonempty convex, ρ-compact, and ρ-bounded subset, G a reflexive digraph compatible with the vector structure of L_ρ with the (P)-property. Let $T : C \to \mathscr{K}(C)$ be a G-monotone ρ-nonexpansive mapping, if $C_T := \{f \in C : f \in [g]_G$ for some $g \in T(f)\}$ is nonempty, then T has a fixed point.*

Proof. The preceding lemma (Lemma 2) establish the existence of sequences $(f_n)_n$ and $(F_n)_n$ such that $F_n \in T(f_n)$, $f_{n+1} \in [f_n]_G$, $F_n \in [f_n]_G$ for every n, and $\lim_n \rho(F_n - f_n) = 0$.

Since C is ρ-compact, there exists a sub-sequence $(f_{k_n})_n$ of $(f_n)_n$ that ρ-converges to some $f \in C$, but as $\lim_n \rho\left(F_{k_n} - f_{k_n}\right) = 0$ and the ρ-convergence is equivalent to the convergence in $\left(L_\rho, \|.\|_\rho\right)$ (due to the Δ_2-type condition, see [11, Proposition 3.13]); then $\left(F_{k_n}\right)_n$ ρ-converges to f. Thus one can suppose that $(f_n)_n$ and $(F_n)_n$ both ρ-converge to f, and as $f_{n+1} \in [f_n]_G$ for every n, using the (P)-property we get $f \in [f_n]_G$ for every n.

Now, since T is G-monotone ρ-nonexpansive mapping, we have for all n there exists $g_n \in T(f)$ such that $g_n \in [F_n]_G$ and $\rho(F_n - g_n) \le \rho(f_n - f)$ and thus $g_n \to f(\rho)$ as $n \to \infty$. Since $T(f)$ is ρ-closed because it is ρ-compact, we obtain $f \in T(f)$, i.e., f is a fixed point for T. □

Example 2. Consider the modular function space $L_\rho = L^2(]-1,1[)$ the space of all square-integrable functions over $]-1,1[$, and let

$$\rho(f) = \int_{-1}^{1} |f(x)|^2 dx, \quad (26)$$

for every $f \in L_\rho$. It is clear that ρ is convex modular function satisfying the Δ_2-type condition.

Let C be the set of all constant maps f of L_ρ such that $\rho(f) \le 1$. Thus, C is ρ-bounded and convex subset of L_ρ. For the ρ-compactness it is enough to show that C satisfies the Riesz-Fréchet-Kolmogorov's conditions (see [15, Theorem IV.25]) which is obvious. Let $G = (C, E)$ be the directed graph defined by:

$$(f, g) \in E \iff (\exists c \in [0,1]), g(x) \le cf(x) \text{ a.e. in }]-1,1[. \quad (27)$$

It is clear that G is a reflexive digraph compatible with the vector structure of L_ρ. Moreover, it has the (P)-property. Indeed, let $(f_n)_n$ be a sequence that ρ-converges to f in L_ρ with the condition

$$f_{n+1} \in [f_n]_G \quad \forall n \in \mathbb{N}. \quad (28)$$

Then there exists a subsequence $\left(f_{n_k}\right)_k$ which is pointwise convergent almost everywhere to f. Thus,

$$f_{n_k}(x) \le f(x) \text{ a.e.} \quad (29)$$

Moreover, for each n there exists $c_n \in [0,1]$ such that

$$f_n(x) \le c_n f_{n+1}(x) \le f_{n+1}(x), \quad (30)$$

then for all n, $f_n(x) \le f(x)$ a.e., hence G has the (P)-property. Note that the digraph G is not even tranisitive.

Let T be the mapping defined by $Tf = f^+ = \max\{0, f\}$. It is clear that T is G-monotone ρ-nonexpansive mapping. Thus all the conditions of Theorem 3 hold, so T has a fixed point namely $T0 = 0$.

5. Fixed Point Theorem for ρ-A.E. Compact Subsets

The ρ-compactness assumption maybe relaxed using the weak concept of ρ-a.e. compactness. Of course, if a subset set C of L_ρ is ρ-compact then it is ρ-a.e. compact, see [11, Proposition 3.13] for more details.

Definition 9. (see [11]). Let $\rho \in \mathscr{R}$.

(i) A set $B \subset L_\rho$ is called ρ-a.e. compact, if for any sequence $(f_n)_n \in L_\rho$ there exists a subsequence $\left(f_{k_n}\right)_n$ and $f \in B$ such that f_{k_n} ρ-a.e. converges to f.

(ii) A set $B \subset L_\rho$ is called ρ-a.e. closed, if for any sequence $(f_n)_n \in B$, $f_n \to f$, ρ-a.e. implies $f \in B$.

Theorem 4. (see [11]). *Let $\rho \in \mathscr{R}$.*

(i) If $\rho(f_n - f) \to 0$ there exists $\left(f_{n_k}\right)_k$ subsequence of $(f_n)_n$ such that $f_{n_k} \to f$, ρ-a.e.

(ii) If $f_n \to f$ ρ-a.e, then $\rho(f) \le \liminf_{n\to+\infty} \rho(f_n)$ (The Fatou property).

We need the following definition of the growth function.

Definition 10. (see [11, 13]). Let ρ be a function modular, the function $\omega_\rho : [0,+\infty] \to [0,+\infty]$ defined by:

$$\omega_\rho(t) = \sup\left\{\frac{\rho(tf)}{\rho(f)} : f \in L_\rho \text{and } 0 < \rho(f) < \infty\right\}, \quad (31)$$

is called the growth of ρ.

The growth function has the following properties.

Proposition 3. (see [11, 16]). *Let $\rho \in \mathscr{R}$ that has the Δ_2-type condition, and ω_ρ its growth function, then:*

(i) $\omega_\rho(t) < \infty, \forall t \in [0, +\infty[$.
(ii) $\omega_\rho : [0, +\infty[\to [0, +\infty[$ is convex, and strictly increasing, it is then also continuous.

We then get the following lemma.

Lemma 3. *Let $\rho \in \mathscr{R}$ that has the Δ_2-type condition, and $(f_n)_n$ and $(g_n)_n$ be two sequences in L_ρ, such that $g_n \to 0(\rho)$, then:*

$$\liminf_n \rho(f_n + g_n) = \liminf_n \rho(f_n). \tag{32}$$

Proof. It is obvious that for $\alpha \in]0, +\infty[$ and $f \in L_\rho$ we have: $\rho(\alpha f) \leq \omega_\rho(\alpha)\rho(f)$. Let $\varepsilon \in]0, 1[$, then for every n we have:

$$\rho(f_n + g_n) \leq \rho\left(\frac{f_n}{1-\varepsilon}\right) + \rho\left(\frac{g_n}{\varepsilon}\right) \leq \omega_\rho\left(\frac{1}{1-\varepsilon}\right)\rho(f_n) + \omega_\rho\left(\frac{1}{\varepsilon}\right)\rho(g_n), \tag{33}$$

and then,

$$\liminf_n \rho(f_n + g_n) \leq \omega_\rho\left(\frac{1}{1-\varepsilon}\right) \liminf \rho(f_n), \tag{34}$$

as ω_ρ is continuous $\lim_{\varepsilon\to 0} \omega_\rho(1/1-\varepsilon) = \omega_\rho(1) = 1$ we then get: $\liminf_n \rho(f_n + g_n) \leq \liminf_n \rho(f_n)$ letting $\epsilon \to 0$.

The same arguments give: $\liminf_n \rho(f_n) = \liminf_n \rho(f_n + g_n - g_n) \leq \liminf_n \rho(f_n + g_n)$ which ends the proof. □

Definition 11. (Opial property). Let $\rho \in \mathscr{R}$, L_ρ is said satisfying the ρ-a.e. Opial property if for every $(f_n)_n \subset L_\rho$ which ρ-a.e. converges to 0, and such that $\sup_n \rho(\beta f_n) < +\infty$ for some $\beta > 1$, then we have:

$$\liminf_n \rho(f_n) < \liminf_n \rho(f_n + f), \tag{35}$$

for every $f \in L_\rho$ not equal to 0.

The following relaxed definition replace the (P)-property.

Definition 12. Let $C \subset L_\rho$ be a nonempty set and G a digraph such that $V(G) = C$.

If for any sequence $(f_n)_n$ in C which ρ-a.e. converges to f, and such that $f_{n+1} \in [f_n]_G$ for every n; we have $f \in [f_n]_G$ for every n; then we say that G has the (P')-property.

Remark 1. If we substitute "(P)-Property" by "(P')-Property" in the Theorem 2, Proposition 2, and Lemma 2; then all these results hold. This is due to the fact that for every ρ-convergence sequence $(f_n)_n$ there exists a sub-sequence $(f_{k_n})_n$ that ρ-a.e. converges.

We conclude this paper by the ρ-a.e. compact version of Theorem 3.

Theorem 5. *Let $\rho \in \mathscr{R}$ be convex, satisfies the Δ_2-type condition and the ρ-a.e. Opial property. Let $C \subset L_\rho$ be a nonempty convex, ρ-a.e. compact, and ρ-bounded subset, G a reflexive digraph compatible with the vector structure of L_ρ such that $V(G) = C$ and satisfies the (P')-property. Let $T : C \to \mathscr{K}(C)$ be a G-monotone ρ-nonexpansive mapping, if $C_T := \{f \in C : f \in [g]_G$ for some $g \in T(f)\}$ is nonempty, then T has a fixed point.*

Proof. By Lemma 2 and Remark 1, there exist two sequences $(f_n)_n$ and $(F_n)_n$ in C such that:

$$\begin{aligned} &F_n \in T(f_n), \\ &f_{n+1} \in [f_n]_G, \\ &F_n \in [f_n]_G, \end{aligned} \tag{36}$$

for every n, and $(F_n - f_n)_n$ ρ-a.e. converges to 0.

As C is ρ-a.e compact, there exists a sub-sequence $(f_{k_n})_n$ that ρ-a.e converges to some f in C. But then as $\lim_n \rho(f_{k_n} - F_{k_n}) = 0$, there exists a sub-sequence $(f_{l_n} - F_{l_n})_n$ of $(f_{k_n} - F_{k_n})_n$ that ρ-a.e. converges to 0. Thus both $(f_{l_n})_n$ and $(F_{l_n})_n$ ρ-a.e. converge to f. So, without loss of generality, we may suppose that both $(f_n)_n$ and $(F_n)_n$ ρ-a.e. converge to f.

Now the (P')-property gives $f \in [f_n]_G, \forall n$. Then for every n, there exists $h_n \in T(f)$ such that $h_n \in [F_n]_G$ and

$$\rho(h_n - F_n) \leq \rho(f_n - f). \tag{37}$$

Moreover, as $(h_n)_n \subset T(f)$ which is ρ-compact (since $Tf \in \mathscr{K}(C)$), it admits a sub-sequence that we will keep noting $(h_n)_n$ that ρ-converges to some $h \in C$. Applying Lemma 3 we get,

$$\begin{aligned} &\liminf_n \rho(f_n - h) = \liminf_n \rho \\ &(f_n - F_n + F_n - h_n + h_n - h) = \liminf_n \rho(F_n - h_n). \end{aligned} \tag{38}$$

So, by letting $n \to \infty$ in the inequality (37), we obtain

$$\liminf_n \rho(f_n - h) \leq \liminf_n \rho(f_n - f). \tag{39}$$

Then if $f \neq h$ the ρ-a.e. Opial property gives

$$\begin{aligned} \liminf_n \rho(f_n - f) &< \liminf_n \rho(f_n - f + f - h) \\ &= \liminf_n \rho(f_n - h), \end{aligned} \tag{40}$$

which is a contradiction. Then necessarily $f = h$ and as $(h_n)_n \subset T(f)$ which is ρ-compact, thus ρ-closed, hence $h = f \in T(f)$, i.e., f is a fixed point for T. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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