Research Article

Fixed Point Theorems for Generalized Set-Valued Weak θ -Contractions in Complete Metric Spaces

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In this paper, the notion of generalized set-valued weak θ -contractions is introduced and a new fixed point theorem for such contractions is established in the setting metric spaces. The main result is a generalization of fixed point theorems in the literature. An example and an application to generalized differential equation are given to support the validity of the main theorem.

1. Introduction and Preliminaries

Jleli and Samet [1] introduced the notion of θ -contractions in Branciari metric spaces [2] and obtained a generalization of the Banach contraction principle, where $\theta : (0, \infty) \rightarrow (1, \infty)$ is a function satisfying the following conditions:

($\theta 1$) θ is nondecreasing

($\theta 2$) For any sequence $\{t_n\} \subset (0, \infty)$,

$$\lim_{n \rightarrow \infty} \theta(t_n) = 1 \Leftrightarrow \lim_{n \rightarrow \infty} t_n = 0 \quad (1)$$

($\theta 3$) There exist $r \in (0, 1)$ and $l \in (0, \infty)$ such that

$$\lim_{t \rightarrow 0^+} \frac{\theta(t) - 1}{t^r} = l. \quad (2)$$

Consider the following conditions:

($\theta 4$) θ is continuous on $(0, \infty)$

($\theta 5$) θ is strictly increasing

($\theta 6$) For each finite sequence $\{t_k > 0 : k = 1, 2, \dots, n\}$,

$$\theta\left(\sum_{k=1}^n t_k\right) \leq \prod_{k=1}^n \theta(t_k) \quad (3)$$

($\theta 7$) For any $A \subset (0, \infty)$ with $\inf A > 0$,

$$\inf \theta(A) = \theta(\inf A) \quad (4)$$

Denote by Ξ the class of all functions $\theta : (0, \infty) \rightarrow (1, \infty)$ satisfying conditions

$$(\theta 1), (\theta 2), (\theta 3), \text{ and } (\theta 7). \quad (5)$$

Let Γ be the class of all functions $\theta : (0, \infty) \rightarrow (1, \infty)$ satisfying conditions

$$(\theta 1), (\theta 2), (\theta 4), \text{ and } (\theta 6), \quad (6)$$

and Ω the family of all functions $\theta : (0, \infty) \rightarrow (1, \infty)$ satisfying conditions

$$(\theta 2), (\theta 4), (\theta 5), \text{ and } (\theta 6), \quad (7)$$

and let Λ be the family of all functions $\theta : (0, \infty) \rightarrow (1, \infty)$ satisfying conditions

$$(\theta 1), (\theta 2), (\theta 4), (\theta 6), \text{ and } (\theta 7). \quad (8)$$

Indeed, Λ is also an extension of the family Θ in [3], which satisfies conditions

$$(\theta_1), (\theta_2), (\theta_3), \text{ and } (\theta_4). \quad (9)$$

Jleli et al. [3] obtained a generalization of the result of [1] with a function $\theta \in \Theta$, and Ahmad et al. [4] gave an extension of the result of Jleli and Samet [1] to metric spaces with a simple condition (θ_4) instead of (θ_3) .

Since then, Al-Sulami et al. [5] introduced the concept of θ_R -contractions by using a binary relation and obtained a generalization of the result of [1]. Also, Ahmad et al. [6] gave α -fuzzy fixed point results by introducing the concept of generalized θ -contraction fuzzy mappings in metric spaces.

Let (X, d) be a metric space and $CB(X)$ be the family of nonempty closed and bounded subsets of X , and let $H(\cdot, \cdot)$ be the Pompeiu-Hausdorff distance on $CB(X)$, i.e., for all $A, B \in CB(X)$,

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}, \quad (10)$$

where $d(a, B) = \inf \{d(a, b) : b \in B\}$ is the distance from the point a to the subset B .

For $A, B \in CB(X)$, let $D(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$.

Then, we have $D(A, B) \leq H(A, B)$ for all $A, B \in CB(X)$.

Recall that the metric space $(CB(X), H)$ is complete if and only if (X, d) is complete.

Nadler [7] extended the Banach contraction principle to set-valued mappings by using Pompeiu-Hausdorff distance.

Since then, many researchers (for example, [8–18] and reference therein) have been studying the fixed point results of set-valued maps in not only metric spaces but also various abstract spaces.

In particular, Hançer et al. [19] extended the result of [1] to set-valued mappings defined on metric spaces. They proved the following theorem.

Theorem 1 [19]. *Let (X, d) be a complete metric space and let $T : X \rightarrow CB(X)$ be a set-valued map. Assume that there exists $k \in (0, 1)$ such that for all $x, y \in X$ with $H(Tx, Ty) > 0$,*

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^k, \quad (11)$$

where $\theta \in \Xi$.

Then, T has a fixed point.

Recently, Durmaz [20] generalized the result of Hançer et al. [19] as follows.

Theorem 2 [20]. *Let (X, d) be a complete metric space and let $T : X \rightarrow CB(X)$ be a set-valued map. Assume that there exists $k \in (0, 1)$ such that for all $x, y \in X$ with $H(Tx, Ty) > 0$,*

$$\theta(H(Tx, Ty)) \leq [\theta(m(x, y))]^k, \quad (12)$$

where $m(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty), 1/2\{d(x, Ty) + d(y, Tx)\}\}$ and $\theta \in \Xi$.

Then, T has a fixed point.

On the other hand, Khojasteh et al. [21] gave an extension of Banach contraction principle to Θ -metric spaces by introducing the concept of Θ -metrics, where $\Theta : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is a continuous mapping with respect to each variable.

The difference between the concept of Θ -contraction in the sense of Khojasteh et al. and the concept of θ -contraction in the sense of Jleli and Samet is that the concept of Θ -contraction in the sense of Khojasteh is defined in Θ -metric space and the concept of θ -contraction in the sense of Jleli and Samet is defined in metric space.

In the paper, we introduce the concept of generalized set-valued weak θ -contraction maps, and we establish a new fixed point theorem for such set-valued contraction maps in the setting of metric spaces.

Lemma 3. *Let (X, d) be a metric space, and let $B \in CB(X)$ and $a \in X$. If $0 < d(a, B) < c$, then there exists $b \in B$ such that*

$$0 < d(a, b) < c. \quad (13)$$

Proof. Let $\varepsilon = c - d(a, B)$.

Since $d(a, B) < c$, there exists $b \in B$ such that $0 < d(a, b) < d(a, B) + \varepsilon$, by the definition of infimum. Hence, $0 < d(a, b) < c$.

2. Fixed Point Theorems

A function $f : (X, d) \rightarrow [0, \infty)$, where (X, d) is metric space, is called *lower semicontinuous function* if for each $x \in X$ and each sequence $\{x_n\}$ with $\lim_{n \rightarrow \infty} d(x, x_n) = 0$, we have

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n). \quad (14)$$

Let $\phi : [1, \infty) \rightarrow [1, \infty)$ is a strictly increasing and lower semicontinuous function such that

$$(\phi 1)\phi^{-1}(\{1\}) = 1$$

$$(\phi 2) \text{ For any sequence } \{t_n\} \subset (0, \infty),$$

$$\prod_{n=0}^{\infty} \phi(\theta(t_n)) < \infty \text{ implies } \prod_{n=0}^{\infty} \theta(t_n) < \infty \quad (15)$$

Let (X, d) be a metric space.

A set-valued mapping $T : X \rightarrow CB(X)$ is called *set-valued weak θ -contraction* if there exists a function $\theta : (0, \infty) \rightarrow (1, \infty)$ such that for all $x, y \in X$ with $d(x, y) > 0$ and for any $u \in Tx$, there exists $v \in Ty$ with $d(u, v) > 0$ such that

$$\theta(d(u, v)) \leq \frac{\theta(d(x, y))}{\phi(\theta(d(x, y)))}. \quad (16)$$

A set-valued mapping $T : X \rightarrow CB(X)$ is called *generalized set-valued weak θ -contraction* if there exists a function $\theta : (0, \infty) \rightarrow (1, \infty)$ such that for all $x, y \in X$ with $d(x, y) > 0$ and for any $u \in Tx$, there exists $v \in Ty$ with $d(u, v) > 0$ such that

$$\theta(d(u, v)) \leq \frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))}, \quad (17)$$

where $m(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty), 1/2\{d(x, Ty) + d(y, Tx)\}\}$.

Remark 4. For $l > 0$, $\phi(\theta(l)) > 1$. In fact, if $l > 0$, then $\theta(l) > 1$. Because ϕ is strictly increasing, $\phi(\theta(l)) > \phi(1)$. It follows from $(\phi 1)$ that $\phi(\theta(l)) > 1$.

Now, we prove our main result.

Theorem 5. Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a generalized set-valued weak θ -contraction. If $\theta \in \Gamma$, then T has a fixed point.

Proof. Let $x_0 \in X$ be a point, and let $x_1 \in Tx_0$ be such that $d(x_0, x_1) > 0$.

Then, it follows from (17) that there exists $x_2 \in Tx_1$ with $d(x_1, x_2) > 0$ such that

$$\theta(d(x_1, x_2)) \leq \frac{\theta(m(x_0, x_1))}{\phi(\theta(m(x_0, x_1)))}, \quad (18)$$

where $m(x_0, x_1) = \max \{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), 1/2\{d(x_0, Tx_1) + d(x_1, Tx_0)\}\}$.

We deduce that $m(x_0, x_1) = \max \{d(x_0, x_1), d(x_1, Tx_1)\}$.

If $m(x_0, x_1) = d(x_1, Tx_1)$, then it follows from (18) and Remark 4 that

$$\theta(d(x_1, x_2)) \leq \frac{\theta(d(x_1, Tx_1))}{\phi(\theta(d(x_1, Tx_1)))} < \theta(d(x_1, Tx_1)) \leq \theta(d(x_1, x_2)), \quad (19)$$

which is a contradiction.

Thus, we have $m(x_0, x_1) = d(x_0, x_1)$, and so,

$$\theta(d(x_1, x_2)) \leq \frac{\theta(d(x_0, x_1))}{\phi(\theta(d(x_0, x_1)))}. \quad (20)$$

Since $d(x_0, x_1) > 0$, it follows from Remark 4 that $\phi(\theta(d(x_0, x_1))) > 1$.

Hence,

$$\theta(d(x_1, x_2)) \leq \frac{\theta(d(x_0, x_1))}{\phi(\theta(d(x_0, x_1)))} < \theta(d(x_0, x_1)). \quad (21)$$

Since θ is nondecreasing,

$$d(x_1, x_2) < d(x_0, x_1). \quad (22)$$

Again, there exists $x_3 \in Tx_2$ with $d(x_2, x_3) > 0$ such that

$$\theta(d(x_2, x_3)) \leq \frac{\theta(m(x_1, x_2))}{\phi(\theta(m(x_1, x_2)))}, \quad (23)$$

where $m(x_1, x_2) = \max \{d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), 1/2\{d(x_1, Tx_2) + d(x_2, Tx_1)\}\}$.

We obtain that $m(x_1, x_2) = \max \{d(x_1, x_2), d(x_2, Tx_2)\}$.

Assume that $m(x_1, x_2) = d(x_2, Tx_2)$.

Then,

$$\theta(d(x_2, x_3)) \leq \frac{\theta(d(x_2, Tx_2))}{\phi(\theta(d(x_2, Tx_2)))} < \theta(d(x_2, Tx_2)) \leq \theta(d(x_2, x_3)), \quad (24)$$

which is a contradiction.

Thus, we have $m(x_1, x_2) = d(x_1, x_2)$, and so,

$$\theta(d(x_2, x_3)) \leq \frac{\theta(d(x_1, x_2))}{\phi(\theta(d(x_1, x_2)))}, \quad (25)$$

which implies

$$\theta(d(x_2, x_3)) \leq \frac{\theta(d(x_1, x_2))}{\phi(\theta(d(x_1, x_2)))} < \theta(d(x_1, x_2)). \quad (26)$$

Since θ is nondecreasing,

$$d(x_2, x_3) < d(x_1, x_2). \quad (27)$$

Inductively, we have a sequence $\{x_n\} \subset X$ such that, for all $n = 1, 2, 3, \dots$,

$$\begin{aligned} x_n \in Tx_{n-1}, 0 < d(x_n, x_{n+1}) &< d(x_{n-1}, x_n)\theta(d(x_n, x_{n+1})) \\ &\leq \frac{\theta(d(x_{n-1}, x_n))}{\phi(\theta(d(x_{n-1}, x_n)))}. \end{aligned} \quad (28)$$

Since $\{d(x_{n-1}, x_n)\}$ is a decreasing sequence, there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_{n-1}, x_n) = r. \quad (29)$$

We now show that $r = 0$.

Assume that $r \neq 0$.

By letting $n \rightarrow \infty$ in (28), and using lower semi-continuity of ϕ and continuity of θ , we have

$$\theta(r) \leq \frac{\theta(r)}{\phi(\theta(r))}. \quad (30)$$

Since $r > 0$, it follows from Remark 4 that $\phi(\theta(r)) > 1$.

Hence, we have

$$\theta(r) \leq \frac{\theta(r)}{\phi(\theta(r))} < \theta(r), \quad (31)$$

which is a contradiction.

Thus, we have

$$\lim_{n \rightarrow \infty} d(x_{n-1}, x_n) = 0, \quad (32)$$

and so,

$$\lim_{n \rightarrow \infty} \theta(d(x_{n-1}, x_n)) = 1. \quad (33)$$

It follows from (28) that

$$\left[\prod_{k=0}^n \phi(\theta(d(x_k, x_{k+1}))) \right] \theta(d(x_{n+1}, x_{n+2})) \leq \theta(d(x_0, x_1)). \quad (34)$$

Hence,

$$\prod_{n=0}^{\infty} \phi(\theta(d(x_n, x_{n+1}))) \leq \theta(d(x_0, x_1)) < \infty, \quad (35)$$

which implies

$$\prod_{n=0}^{\infty} \theta(d(x_n, x_{n+1})) < \infty. \quad (36)$$

Given $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n \geq n_0$,

$$\prod_{k=n}^{\infty} \theta(d(x_k, x_{k+1})) < \theta(\varepsilon). \quad (37)$$

Hence, we have that for all $m > n \geq n_0$,

$$\begin{aligned} \theta(d(x_n, x_m)) &\leq \theta(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m)) \\ &\leq \theta(d(x_n, x_{n+1})) \times \theta(d(x_{n+1}, x_{n+2})) \times \dots \\ &\quad \times \theta(d(x_{m-1}, x_m)) \leq \prod_{k=n}^{\infty} \theta(d(x_k, x_{k+1})) < \theta(\varepsilon), \end{aligned} \quad (38)$$

which implies

$$\forall m > n \geq n_0, \quad d(x_n, x_m) < \varepsilon. \quad (39)$$

Hence, $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists a point $x_* \in X$ such that

$$\lim_{n \rightarrow \infty} d(x_*, x_n) = 0. \quad (40)$$

Since $d(x_n, x_*) > 0$ and $x_{n+1} \in Tx_n \forall n = 1, 2, 3, \dots$, it follows from (17) that there exists $u \in Tx_*$ such that

$$\theta(d(x_{n+1}, u)) \leq \frac{\theta(m(x_n, x_*))}{\phi(\theta(m(x_n, x_*)))}, \quad (41)$$

where $m(x_n, x_*) = \max \{d(x_n, x_*), d(x_n, Tx_n), d(x_*, Tx_*), 1/2\{d(x_n, Tx_*) + d(x_*, Tx_n)\}\}$.

We infer that

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) \leq \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0, \quad (42)$$

and so,

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0. \quad (43)$$

Let $\alpha = \lim_{n \rightarrow \infty} 1/2\{d(x_n, Tx_*) + d(x_*, Tx_n)\}$. Then,

$$\alpha \leq \lim_{n \rightarrow \infty} \frac{1}{2} \{d(x_n, Tx_*) + d(x_*, x_{n+1})\} \leq \frac{1}{2} \{d(x_*, Tx_*)\}. \quad (44)$$

Hence,

$$\lim_{n \rightarrow \infty} m(x_n, x_*) = \max \{0, 0, d(x_*, Tx_*), \alpha\} = d(x_*, Tx_*). \quad (45)$$

Letting $n \rightarrow \infty$ in (41), and using continuity of θ and lower semicontinuity of ϕ , we have

$$\theta(d(x_*, u)) \leq \frac{\theta(d(x_*, Tx_*))}{\phi(\theta(d(x_*, Tx_*)))}. \quad (46)$$

Assume that $x_* \notin Tx_*$.

Then, $d(x_*, Tx_*) > 0$, and so, $\phi(\theta(d(x_*, Tx_*))) > 1$ by Remark 4. Thus, we have

$$\theta(d(x_*, Tx_*)) \leq \theta(d(x_*, u)) \leq \frac{\theta(d(x_*, Tx_*))}{\phi(\theta(d(x_*, Tx_*)))} < \theta(d(x_*, Tx_*)), \quad (47)$$

which is a contradiction.

Hence, we have $x_* \in Tx_*$.

Corollary 6. Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a set-valued map such that

$$\begin{aligned} &\text{for all } x, y \in X \text{ with } \inf_{a \in Tx} d(a, Ty) > 0, \\ &\theta(D(Tx, Ty)) < \frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))}. \end{aligned} \quad (48)$$

If either $\theta \in \Omega$ or $\theta \in \Lambda$, then T has a fixed point.

Proof. Suppose that $\theta \in \Omega$.

Since θ is strictly increasing, there exists the inverse function $\theta^{-1} : [1, \infty) \rightarrow [1, \infty)$ such that

$$D(Tx, Ty) < \theta^{-1} \left(\frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))} \right). \quad (49)$$

Let $u \in Tx$.

Then, it follows from the condition $\inf_{a \in Tx} d(a, Ty) > 0$, $\forall x, y \in X$ that

$$0 < d(u, Ty) < \theta^{-1} \left(\frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))} \right). \quad (50)$$

By Lemma 3, there exists $v \in Ty$ such that

$$0 < d(u, v) < \theta^{-1} \left(\frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))} \right). \quad (51)$$

Hence,

$$\theta(d(u, v)) < \frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))}. \quad (52)$$

By Theorem 5, T has a fixed point.

Assume that $\theta \in \Lambda$.

Let $u \in Tx$.

It follows from (θ7) that

$$\inf_{z \in Ty} \theta(d(u, z)) = \theta \left(\inf_{z \in Ty} d(u, z) \right) = \theta(d(u, Ty)). \quad (53)$$

Because θ is nondecreasing,

$$\inf_{z \in Ty} \theta(d(u, z)) \leq \theta(D(Tx, Ty)), \quad (54)$$

and so,

$$\inf_{z \in Ty} \theta(d(u, z)) < \frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))}. \quad (55)$$

Let

$$\varepsilon = \frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))} - \inf_{z \in Ty} \theta(d(u, z)). \quad (56)$$

By the definition of infimum, there exists $v \in Ty$ such that

$$\theta(d(u, v)) < \inf_{z \in Ty} \theta(d(u, z)) + \varepsilon. \quad (57)$$

Thus,

$$\theta(d(u, v)) < \frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))}. \quad (58)$$

By Theorem 5, T has a fixed point.

Corollary 7. Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a set-valued map such that

$$\begin{aligned} & \text{for all } x, y \in X \text{ with } \inf_{a \in Tx} d(a, Ty) > 0, \\ & \theta(H(Tx, Ty)) < \frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))}. \end{aligned} \quad (59)$$

If either $\theta \in \Omega$ or $\theta \in \Lambda$, then T has a fixed point.

Corollary 8. Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a set-valued map satisfying the following condition: there exists $\theta \in \Gamma$ such that for all $x, y \in X$ with $d(x, y) > 0$ and for any $u \in Tx$, there exists $v \in Ty$ with $d(u, v) > 0$

such that

$$\theta(d(u, v)) \leq [\theta(m(x, y))]^k, \quad (60)$$

where $k \in (0, 1)$.

Then, T has a fixed point.

Proof. We define a function $\phi : [1, \infty) \rightarrow [1, \infty)$ by

$$\phi(t) = t^{1-k}. \quad (61)$$

Then, ϕ is nondecreasing and lower semicontinuous such that (φ1) and (φ2) hold. It follows from (60) that for all $x, y \in X$ with $d(x, y) > 0$ and for any $u \in Tx$, there exists $v \in Ty$ with $d(u, v) > 0$ with

$$\theta(d(u, v)) \leq [\theta(m(x, y))]^k = \frac{\theta(m(x, y))}{[\theta(m(x, y))]^{1-k}} = \frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))}. \quad (62)$$

By Theorem 5, T has a fixed point.

Corollary 9. Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a set-valued map such that

$$\begin{aligned} & \text{for all } x, y \in X \text{ with } \inf_{a \in Tx} d(a, Ty) > 0, \\ & \theta(D(Tx, Ty)) < [\theta(m(x, y))]^k, \end{aligned} \quad (63)$$

where $k \in (0, 1)$.

If either $\theta \in \Omega$ or $\theta \in \Lambda$, then T has a fixed point.

Corollary 10. Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a set-valued map such that

$$\begin{aligned} & \text{for all } x, y \in X \text{ with } \inf_{a \in Tx} d(a, Ty) > 0, \\ & \theta(H(Tx, Ty)) < [\theta(m(x, y))]^k, \end{aligned} \quad (64)$$

where $k \in (0, 1)$.

If either $\theta \in \Omega$ or $\theta \in \Lambda$, then T has a fixed point.

Corollary 11. Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a set-valued weak θ -contraction. If $\theta \in \Gamma$, then T has a fixed point.

Corollary 12. Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a set-valued map such that

$$\begin{aligned} & \text{for all } x, y \in X \text{ with } \inf_{a \in Tx} d(a, Ty) > 0, \\ & \theta(D(Tx, Ty)) < \frac{\theta(d(x, y))}{\phi(\theta(d(x, y)))}. \end{aligned} \quad (65)$$

If either $\theta \in \Omega$ or $\theta \in \Lambda$, then T has a fixed point.

Corollary 13. Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a set-valued map such that

$$\text{for all } x, y \in X \text{ with } \inf_{a \in Tx} d(a, Ty) > 0, \quad (66)$$

$$\theta(H(Tx, Ty)) < \frac{\theta(d(x, y))}{\phi(\theta(d(x, y)))}.$$

If either $\theta \in \Omega$ or $\theta \in \Lambda$, then T has a fixed point.

Corollary 14. Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a set-valued map satisfying the following condition: there exist $\theta \in \Gamma$ such that for all $x, y \in X$ with $d(x, y) > 0$ and for any $u \in Tx$, there exists $v \in Ty$ with $d(u, v) > 0$ such that

$$\theta(d(u, v)) \leq [\theta(d(x, y))]^k, \quad (67)$$

where $k \in (0, 1)$.

Then, T has a fixed point.

Corollary 15. Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a set-valued map such that for all $x, y \in X$ with $m(x, y) > 0$ and for any $u \in Tx$, there exists $v \in Ty$ such that

$$d(u, v) \leq m(x, y) - \varphi(m(x, y)), \quad (68)$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing and lower semicontinuous such that $\varphi^{-1}(\{0\}) = 0$. Then, T has a fixed point.

Proof. Let $\theta(t) = e^t, \forall t > 0$.

From (68), we have that for all $x, y \in X$ with $m(x, y) > 0$ and for any $u \in Tx$, there exists $v \in Ty$ such that

$$\theta(d(u, v)) = e^{d(u, v)} \leq e^{m(x, y) - \varphi(m(x, y))} = \frac{e^{m(x, y)}}{e^{\varphi(m(x, y))}}. \quad (69)$$

Let $\varphi(t) = \ln(\phi(\theta(t))), \forall t \geq 0$, where $\phi : [1, \infty) \rightarrow [1, \infty)$ is nondecreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$.

Then, φ is nondecreasing and lower semicontinuous, and

$$\varphi(t) = 0 \Leftrightarrow \phi(\theta(t)) = 1 \Leftrightarrow \theta(t) = e^t = 1 \Leftrightarrow t = 0. \quad (70)$$

It follows from (69) that for all $x, y \in X$ with $m(x, y) > 0$ and for any $u \in Tx$, there exists $v \in Ty$ such that

$$\theta(d(u, v)) \leq \frac{\theta(m(x, y))}{e^{\ln(\phi(\theta(m(x, y))))}} = \frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))}. \quad (71)$$

By Theorem 5, T has a fixed point.

Remark 16. Corollary 15 is an extension of Theorem 1 of [22] to set-valued maps and is a modification of Theorem 18 of [23].

By taking $\theta(t) = 2 - (2/\pi) \arctan(1/t^\alpha)$, where $\alpha \in (0, 1)$, $t > 0$ in Theorem 5, we obtain the following result.

Corollary 17. Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a set-valued map. If for all $x, y \in X$ with $d(x, y) > 0$ and for any $u \in Tx$, there exists $v \in Ty$ with $d(u, v) > 0$ such that

$$2 - \frac{2}{\pi} \arctan\left(\frac{1}{[d(u, v)]^\alpha}\right) \leq \frac{2 - (2/\pi) \arctan(1/[m(x, y)]^\alpha)}{\phi(2 - (2/\pi) \arctan(1/[m(x, y)]^\alpha))}, \quad (72)$$

where $\alpha \in (0, 1)$, then T has a fixed point.

We give an example to illustrate Theorem 5. This example shows that the main theorem is a genuine generalization of Nadler's contraction principle, Theorem 1 and Theorem 2.

Let $X = \{1/2^n : n = 1, 2, 3, \dots\} \cup \{0\}$, and let $d(x, y) = |x - y|$ for all $x, y \in X$.

Define a map $T : X \rightarrow CB(X)$ by

$$Tx = \begin{cases} \left\{1, \frac{1}{2^{n+1}}\right\}, & x = \frac{1}{2^n}, n = 1, 2, 3, \dots, \\ \{0\}, & x = 0, \end{cases} \quad (73)$$

and a function $\theta : (0, \infty) \rightarrow (1, \infty)$ by

$$\theta(t) = e^t. \quad (74)$$

Let $\phi(t) = t^{1/2}, \forall t > 1$.

Then, ϕ is strictly increasing and lower semicontinuous satisfying conditions $(\phi 1)$ and $(\phi 2)$.

We now show that T is generalized set-valued weak θ -contraction.

Let $x = 0, y = 1/2^n$.

We have that

$$m(x, y) = \max \left\{ d\left(0, \frac{1}{2^n}\right), d(0, T0), d\left(\frac{1}{2^n}, T\frac{1}{2^n}\right), \frac{1}{2} \left\{ d\left(0, T\frac{1}{2^n}\right) + d\left(\frac{1}{2^n}, T0\right) \right\} \right\} = \frac{1}{2^n} = d(x, y). \quad (75)$$

Then, for $u = 0 \in T0$, there exists $v = 1/2^{n+1} \in T(1/2^n)$ such that

$$\begin{aligned} \theta(d(u, v)) &= \theta\left(d\left(0, \frac{1}{2^{n+1}}\right)\right) = \theta\left(\frac{1}{2^{n+1}}\right) = e^{1/2^n - 1/2^{n+1}} \\ &= \frac{e^{1/2^n}}{[e^{1/2^n}]^{1/2}} = \frac{\theta(1/2^n)}{\phi(\theta(1/2^n))} \leq \frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))}. \end{aligned} \quad (76)$$

If $x = 1/2^n$ and $y = 1/2^m$ ($m > n$), then we have

$$m(x, y) = \max \left\{ d\left(\frac{1}{2^n}, \frac{1}{2^n}\right), d\left(\frac{1}{2^n}, T\frac{1}{2^n}\right), d\left(\frac{1}{2^m}, T\frac{1}{2^m}\right), \frac{1}{2} \left\{ d\left(\frac{1}{2^m}, T\frac{1}{2^n}\right) + d\left(\frac{1}{2^n}, T\frac{1}{2^m}\right) \right\} \right\} = \frac{1}{2^n} - \frac{1}{2^m}. \quad (77)$$

For $u = 1/2^{n+1} \in T(1/2^n)$, there exists $v = 1/2^{m+1} \in T(1/2^m)$ such that

$$\begin{aligned} \theta(d(u, v)) &= \theta\left(\frac{1}{2^{n+1}}, \frac{1}{2^{m+1}}\right) = e^{1/2^{n+1} - 1/2^{m+1}} \\ &= e^{1/2^n - 1/2^m - 1/2^{n+1} + 1/2^{m+1}} = \frac{e^{1/2^n - 1/2^m}}{[e^{1/2^n - 1/2^m}]^{1/2}} \\ &= \frac{\theta(1/2^n - 1/2^m)}{\phi(\theta(1/2^n - 1/2^m))} \leq \frac{\theta(m(x, y))}{\phi(\theta(m(x, y)))}. \end{aligned} \quad (78)$$

Hence, T is a generalized set-valued weak θ -contraction.

Thus, all hypotheses of Theorem 5 are satisfied, and T has a fixed point $x_* = 0$.

Note that Nadler's contraction principle is not satisfied. In fact, if $x = 0, y = 1/n$, then

$$H\left(T0, T\frac{1}{2^n}\right) \leq kd\left(0, \frac{1}{2^n}\right), \quad k \in (0, 1), \quad (79)$$

which implies

$$1 \leq k \frac{1}{2^n}. \quad (80)$$

Hence,

$$k \geq 2^n, \quad \forall n = 1, 2, 3, \dots, \quad (81)$$

and hence,

$$k > 1, \quad (82)$$

which is a contradiction. Thus, Nadler's contraction principle does not hold.

Also, note that Theorem 1 and Theorem 2 cannot be used here. In fact, if

$$\theta\left(H\left(T0, T\frac{1}{2^n}\right)\right) \leq \left[\theta\left(m\left(0, \frac{1}{2^n}\right)\right)\right]^k, \quad (83)$$

where $k \in (0, 1)$ and $\theta(t) = e^t, \forall t > 0$, then

$$e^1 \leq [e^{1/2^n}]^k = e^{k(1/2^n)}, \quad (84)$$

and so, $k > 1$. Hence, condition (12) is not satisfied. Since $m(0, 1/2^n) = d(0, 1/2^n)$, condition (11) does not hold. Thus, Theorem 1 and Theorem 2 are not applicable here.

3. Application to Initial Value Problem

Let $B \subset \mathbb{R}$ be an origin-centered closed ball, and let $CC(B)$ the family of nonempty closed convex subsets of B with Hausdorff metric D generated by the norm $\|\cdot\|$ of $\mathbb{R}^n$.

Let $R : [0, a] \times B \rightarrow CC(B)$ be a continuous map.

Let $X = C([0, a], \mathbb{R}^n)$ be the space of all continuous functions from $[0, a]$ into $\mathbb{R}^n$, and let

$$d(x, y) = \|x - y\| = \max_{t \in [0, a]} |x(t) - y(t)| \quad (85)$$

for all $x, y \in X$.

Then, (X, d) is a complete metric space.

Note that every continuous function from a compact space into a metric space is bounded.

Consider the following initial value problem:

$$\dot{x} \in R(t, x(t)), \quad x(0) = x_0. \quad (86)$$

The existence of solutions for generalized differential equation (86) under certain conditions on R is proved by Casting [24], Filippov [25], and Hermes [26].

We give an application of our fixed point result to the problem on the existence of solutions for generalized differential equation (86).

Theorem 18. Suppose that for each $x, y \in X$ and

$$u(t) = x_0 + \int_0^t z(s)ds, \quad z(x) \in R(s, x(s)), \quad (87)$$

there exists $w(s) \in R(s, y(s))$ with

$$v(t) = x_0 + \int_0^t w(s)ds, \quad (88)$$

and $|u - v| > 0$ such that

$$\theta(|u - v|) \phi(\theta(|x - y|)) \leq \theta(|x - y|). \quad (89)$$

Then, the initial value problem (86) has a solution.

Proof. Define a mapping $T : X \rightarrow CB(X)$ by

$$Tx = \{u \in X : u(t) = x_0 + \int_0^t z(s)ds, z(x) \in R(s, x(s))\}. \quad (90)$$

Let $x, y \in X$ and $u \in Tx$ be given.

By assumption, there exists $v \in Ty$ with $d(u, v) \geq |u - v| > 0$ such that

$$\theta(|u - v|) \phi(|x - y|) \leq \theta(|x - y|). \quad (91)$$

Hence,

$$\theta(|u - v|) \phi(|x - y|) \leq \theta(|x - y|) \leq \theta(\|x - y\|). \quad (92)$$

By taking maximum in $[0, a]$,

$$\theta(\|u - v\|)\phi(\|x - y\|) \leq \theta(\|x - y\|), \quad (93)$$

and so,

$$\theta(\|u - v\|) \leq \frac{\phi(\|x - y\|)}{\theta(\|x - y\|)}. \quad (94)$$

Thus, we have

$$\theta(d(u, v)) \leq \frac{\phi(d(x, y))}{\theta(d(x, y))}. \quad (95)$$

By Corollary 11, T has a fixed point, say x_* which is a solution of (86). That is, $\dot{x}_* \in R(t, x_*(t))$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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