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Research Article

The Split Feasibility Problem with Some Projection Methods in Banach Spaces

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In this paper, we study the split feasibility problem in Banach spaces. At first, we prove that a solution of this problem is a solution of the equivalent equation defined by using a metric projection, a generalized projection, and sunny generalized nonexpansive retraction, respectively. Then, using the hybrid method with these projections, we prove strong convergence theorems in mathematical programing in order to find a solution of the split feasibility problem in Banach spaces.

Dedicated to the late Professor Wataru Takahashi with our respect

1. Introduction

Bregman proposed a generalization for the cyclic metric projection method of computing points in the intersection of linear closed subspaces of a Hilbert space in [1], invented by von Neumann [2]. Alber and Butnariu achieve distinction of the study of this Bregman projection and the result of the properties. They used this cyclic Bregman projection method for finding the solution of the consistent convex feasibility problem of computing a common point of the closed convex subspaces in a reflexive Banach space [3]. Some fruitful results have been introduced with respect to the sequential algorithm with successive Bregman projection for computing a solution of the convex feasibility problem [4, 5] and so on. Ibaraki and Takahashi studied the properties of a generalized projection which is a special case of Bregman projection and a sunny generalized nonexpansive retraction in Banach spaces [6].

Alsulami, Latif, and Takahashi treated with the following convex feasibility problem [7]: Let H be a Hilbert space; let E be a strictly convex, reflexive, and smooth Banach space; let A be a bounded linear operator from H into E; let C and D be convex and closed subsets of H and E, respectively. Then, find a point $z \in C \cap A^{-1}(D)$. In particular, such a problem is called the split feasibility problem. Using the methods with metric projections in mathematical programing, they showed strong convergence theorems for finding a solution of the split feasibility problem. In the case of finite dimensional spaces, Byrne treated with the iterative algorithm [8]: $x_{n+1} = P_C(x_n + rA^T(P_D - I)Ax_n)$, where $n \in \mathbb{N}$ and a linear operator A is represented as a matrix which can be selected to impose consistency with measured data. With respect to examples in this case, there are results by Landweber [9] and Gordon, Bender, and Herman [10]. In [11], Takahashi treated with this problem of a linear bounded operator A from E into F, where E and F are uniformly convex and smooth Banach spaces. In that paper, it is shown that $z \in C \cap A^{-1}(D)$ is equivalent to

$$z = P_C\left(I_E - rJ_E^{-1}A^*J_F(I_F - P_D)A\right)z, \tag{1}$$

where P_C and P_D are metric projections on subsets C of E and D of F, respectively; I_E and I_F are the identity mappings on E

and F, respectively; J_E and J_F are duality mappings on E and F, respectively; $r \in (0, \infty)$. Furthermore, the following convergence theorem is proved by the hybrid method with metric projections: Let E and F be uniformly convex and smooth Banach spaces; let C and D be nonempty, closed, and convex subsets of E and F, respectively; let J_E and J_F be duality mappings on E and F, respectively; let A be a bounded linear operator from E into F with $A \neq 0$; let A^* be the adjoint operator of A; let $r \in (0, \infty)$. Suppose that $C \cap A^{-1}(D) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = \left(I_E - rJ_E^{-1}A^*J_F(I_F - P_D)A\right)x_n, \\ C_n = \{z \in C \mid \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ Q_n = \{z \in C \mid \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1=}P_{C_n \cap Q_n}x_1 \end{cases} \quad (2)$$

for any $n \in \mathbb{N}$. Then, $\{x_n\}$ is strongly convergent to a point $z_0 \in C \cap A^{-1}(D)$ for any $r \in (0, \|A\|^{-2})$, where $z_0 = P_{C \cap A^{-1}(D)}x_1$.

In this paper, for uniformly convex and smooth Banach spaces E and F, we study the split feasibility problem of a bounded linear operator A from E to F. First, we give the diversity of equivalent equations regarding equation (1) with respect to metric projections, generalized projections, and sunny generalized nonexpansive retractions, respectively. Then, using the hybrid methods with these projections, we prove the strong convergence theorems in mathematical programing in order to find a solution of the split feasibility problem in Banach spaces.

2. Preliminaries

We know that the following hold; for instance, see [12–14].

(T1) Let E be a Banach space, let E^* be the topological dual space of E, and let J_E be the duality mapping on E defined by

$$J_E x = \left\{x^* \in E^* \mid \|x\|^2 = \langle x, x^*\rangle = \|x^*\|^2\right\} \quad (3)$$

for any $x \in E$. Then, E is strictly convex if and only if J_E is injective; that is, $x \neq y$ implies $J_E x \cap J_E y = \emptyset$.

(T2) Let E be a Banach space, let E^* be the topological dual space of E, and let J_E be the duality mapping on E. Then, E is reflexive if and only if J_E is surjective; that is, $\bigcup_{x \in E} J_E x = E^*$.

(T3) Let E be a Banach space and let J_E be the duality mapping on E. Then, E is smooth if and only if J_E is single-valued.

(T4) Let E be a Banach space and let J_E be the duality mapping on E. Then, E is strictly convex if and only if

$$1 - \langle x, y^*\rangle > 0 \quad (4)$$

for any $x, y \in E$ with $x \neq y$ and $\|x\| = \|y\| = 1$ and for any $y^* \in J_E y$.

(T5) Let E be a Banach space and let E^* be the topological dual space of E. Then, E is reflexive if and only if E^* is reflexive.

(T6) Let E be a Banach space and let E^* be the topological dual space of E. If E^* is strictly convex, then E is smooth. Conversely, if E is reflexive and smooth, then E^* is strictly convex.

(T7) Let E be a Banach space and let E^* be the topological dual space of E. If E^* is smooth, then E is strictly convex. Conversely, if E is reflexive and strictly convex, then, E^* is smooth.

(T8) If E is uniformly convex, that is, for any $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$ implies $\|(x + y)/2\| \leq 1 - \delta$, then E is reflexive.

(T9) Let E be a Banach space, let E^* be the topological dual space of E, and let J_E be the duality mapping on E. If E has a Fréchet differentiable norm, then J_E is norm-to-norm continuous.

(T10) Let E be a Banach space and let E^* be the topological dual space of E. Then, E is uniformly smooth, that is, E has a uniformly Fréchet differentiable norm, if and only if E^* is uniformly convex.

Definition 1. Let E be a smooth Banach space, let J_E be the duality mapping on E, and let V_E be the mapping from $E \times E$ into $[0, \infty)$ defined by

$$V_E(x, y) = \|x\|^2 - 2\langle x, J_E y\rangle + \|y\|^2 \quad (5)$$

for any $x, y \in E$.

Since by (T3) J_E is single-valued, V_E is well-defined. It is obvious that $x = y$ implies $V_E(x, y) = 0$. Conversely, by (T4),

(T11) If E is also strictly convex, then $V_E(x, y) = 0$ implies $x = y$.

Let E be a strictly convex and smooth Banach space. By (T1) and (T3), J_E is a bijective mapping from E onto $J_E(E)$. In particular, if E is also reflexive, then by (T2), J_E is a bijective mapping from E onto E^*. If E is strictly convex, reflexive, and smooth, then by (T5), (T6) and (T7) E^* is also strictly convex, reflexive, and smooth. Furthermore, since E is reflexive, $E^{**} = E$ holds and the duality mapping on E^* is J_E^{-1}.

We use the following lemmas in this paper. The following is shown in [14].

Lemma 2. *Let E be a Banach space and let J_E be the duality mapping on E. Then, $\langle x - y, x^* - y^*\rangle \geq 0$ for any $x, y \in E$, for any $x^* \in J(x)$, and for any $y^* \in J_E y$. Furthermore, if E is strictly convex and smooth, then $\langle x - y, x^* - y^*\rangle = 0$ if and only if $x = y$.*

Definition 3. Let E be a strictly convex, reflexive, and smooth Banach space and let C be a nonempty, closed, and convex subset of E. We know that for any $x \in E$ there exists a unique element $z \in C$ such that $\|x - z\| = \min_{y \in C}\|x - y\|$. Such a z is denoted by $P_C x$, and P_C is called the metric projection of E onto C.

The following holds.

Lemma 4. *Let E be a strictly convex and smooth Banach space, let C be a nonempty closed subset of E, and let J_E be duality mapping on E. Then, for any $(x, z) \in E \times C$, $z = P_C x$ if and only if $\langle z - y, J_E(x - z)\rangle \geq 0$ for any $y \in C$.*

Definition 5. Let E be a strictly convex, reflexive, and smooth Banach space and let C be a nonempty, closed, and convex subset of E. We know that for any $x \in E$, there exists a unique element $z \in C$ such that $V_E(z, x) = \min_{y \in C} V_E(y, x)$. Such a z is denoted by $\Pi_C x$, and Π_C is called the generalized projection of E onto C.

The following is shown in [15].

Lemma 6. *Let E be a strictly convex and smooth Banach space; let C be a nonempty, closed, and convex subset of E; let J_E be the duality mapping on E. Then, the following hold.*

(i) For any $(x, z) \in E \times C$, $z = \Pi_C x$ if and only if $\langle z - y, J_E x - J_E z\rangle \geq 0$ for any $y \in C$;

(ii) $V_E(y, \Pi_C x) + V_E(\Pi_C x, x) \leq V_E(y, x)$ for any $x \in E$ and for any $y \in C$.

Definition 7. Let C be a nonempty subset of a smooth Banach space E. A mapping T from C into E is said to be generalized nonexpansive [6] if the set of all fixed points of T is nonempty and

$$V_E(Tx, y) \leq V_E(x, y) \tag{6}$$

for any $x \in C$ and for any fixed point y of T. Let C be a nonempty subset of a Banach space E. A mapping R from E onto C is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx \tag{7}$$

for any $x \in E$ and for any $t \in [0, \infty)$. A mapping R from E onto C is called a retraction or a projection if $Rx = x$ for any $x \in C$.

The following are shown in [16].

Lemma 8. *Let E be a strictly convex, reflexive, and smooth Banach space and let C be a nonempty and closed subset of E. Then, the following are equivalent:*

(i) There exists a sunny generalized nonexpansive retraction of E onto C;

(ii) There exists a generalized nonexpansive retraction of E onto C;

(iii) $J_E(C)$ is closed and convex.

Lemma 9. *Let E be a strictly convex, reflexive, and smooth Banach space, let C be a nonempty and closed subset of E, and $(x, z) \in E \times C$. Suppose that there exists a sunny generalized nonexpansive retraction R_C of E onto C. Then, the following are equivalent:*

(i) $z = R_C x$;

(ii) $V_E(x, z) = \min_{y \in C} V_E(x, y)$.

The following are shown in [6].

Lemma 10. *Let E be a strictly convex and smooth Banach space and let C be a nonempty and closed subset of E. Suppose that there exists a sunny generalized nonexpansive retraction of E onto C. Then, the sunny generalized nonexpansive retraction is uniquely determined.*

Lemma 11. *Let E be a strictly convex and smooth Banach space, let C be a nonempty and closed subset of E, and let J_E be the duality mapping on E. Suppose that there exists a sunny generalized nonexpansive retraction R_C of E onto C. Then, the following hold.*

(i) For any $(x, z) \in E \times C$, $z = R_C x$ if and only if $\langle x - z, J_E z - J_E y\rangle \geq 0$ for any $y \in C$.

(ii) $V_E(R_C x, y) + V_E(x, R_C x) \leq V_E(x, y)$ for any $x \in E$ and for any $y \in C$.

Definition 12. Let $p \in (1, \infty)$. Define a mapping J_p from E into E^* by

$$J_p x = \left\{x^* \in E^* \mid \langle x, x^*\rangle = \|x\|^p \text{ and } \|x^*\| = \|x\|^{p-1}\right\} \tag{8}$$

for any $x \in E$. Then, J_p is called the generalized duality mapping on E. In particular, $J_2 = J_E$.

The following are shown in [17].

Lemma 13. *Let E be a Banach space. Then, the following are equivalent:*

(i) E is uniformly convex;

(ii) For any $p \in (1, \infty)$ and for any $\rho \in (0, \infty)$, there exists a continuous, strictly increasing, and convex function $g_{p,\rho}$ from $[0, \infty)$ into $[0, \infty)$ such that $g_{p,\rho}(0) = 0$ and

$$\|x + y\|^p \geq \|x\|^p + p\langle y, x^*\rangle + g_{p,\rho}(\|y\|) \tag{9}$$

for any $x, y \in B_\rho(E) =^{def} \{z \in E \mid \|z\| \leq \rho\}$ and for any $x^* \in J_p x$;

(iii) For any $p \in (1, \infty)$ and for any $\rho \in (0, \infty)$, there exists a continuous, strictly increasing, and convex

function $g_{p,\rho}$ from $[0,\infty)$ into $[0,\infty)$ such that $g_{p,\rho}(0)=0$ and

$$\langle x-y, x^*-y^*\rangle \geq g_{p,\rho}(\|x-y\|) \tag{10}$$

for any $x, y \in B_\rho(E)$, for any $x^* \in J_p x$, and for any $y^* \in J_p y$.

Lemma 14. *Let E be a smooth Banach space. Then, the following are equivalent:*

(i) E is uniformly smooth;

(ii) For any $q \in (1,\infty)$ and for any $\rho \in (0,\infty)$, there exists a continuous, strictly increasing, and convex function $g^*_{q,\rho}$ from $[0,\infty)$ into $[0,\infty)$ such that $g^*_{q,\rho}(0)=0$ and

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q x\rangle + g^*_{q,\rho}(\|y\|) \tag{11}$$

for any $x, y \in B_\rho(E)$;

(iii) For any $q \in (1,\infty)$ and for any $\rho \in (0,\infty)$, there exists a continuous, strictly increasing, and convex function $g^*_{q,\rho}$ from $[0,\infty)$ into $[0,\infty)$ such that $g^*_{q,\rho}(0)=0$ and

$$\langle x-y, J_q x - J_q y\rangle \leq g^*_{q,\rho}(\|x-y\|) \tag{12}$$

for any $x, y \in B_\rho(E)$.

The following is shown in [18].

Lemma 15. *Let E be a uniformly convex and smooth Banach space and let $\rho \in (0,\infty)$. Then, there exists a continuous, strictly increasing, and convex function g_ρ from $[0,\infty)$ into $[0,\infty)$ such that $g_\rho(0)=0$ and*

$$g_\rho(\|x-y\|) \leq V_E(x,y) \tag{13}$$

for any $x, y \in B_\rho(E)$.

3. Equivalent Conditions to the Existence of Solutions

In this section, we consider equivalent conditions to the existence of solutions of the split feasibility problem.

Theorem 16. *Let E and F be strictly convex, reflexive, and smooth Banach spaces; let I_E and I_F be the identity mappings on E and F, respectively; let J_E and J_F be duality mappings on E and F, respectively; let C and D be nonempty, closed, and convex subsets of E and F, respectively; let A be a bounded linear operator from E into F; let A^* be the adjoint operator of A; let $r \in (0,\infty)$. Suppose that $C \cap A^{-1}(D) \neq \varnothing$. Consider the following condition:*

(i) $z \in C \cap A^{-1}(D)$.

The following are equivalent to (i):

(ii) $z = P_C(I_E - rJ_E^{-1}A^*J_F(I_F - P_D)A)z$;

(iii) $z = P_C(I_E - rJ_E^{-1}A^*(J_F - J_F\Pi_D)A)z$;

(iv) $z = \Pi_C J_E^{-1}(J_E - rA^*J_F(I_F - P_D)A)z$;

(v) $z = \Pi_C J_E^{-1}(J_E - rA^*(J_F - J_F\Pi_D)A)z$.

Proof. The equivalence of (i) and (ii) is shown in [11, Lemma 3.1]. We show the rest.

Suppose that (i) holds. Since $Az \in D$, $P_D Az = \Pi_D Az = Az$ holds. Therefore,

$$J_F(I_F - P_D)Az = (J_F - J_F\Pi_D)Az = 0, \tag{14}$$

and hence,

$$\begin{gathered}\text{the right side of (iii)} = P_C z,\\ \text{the right sides of (iv) and (v)} = \Pi_C z.\end{gathered} \tag{15}$$

Since $z \in C$, we obtain

$$\text{the right sides of (iii), (iv), and (v)} = z. \tag{16}$$

Conversely, suppose that (iii), (iv), or (v) holds. Since these equations have the form of $z = P_C x$ or $z = \Pi_C x$, $z \in C$ holds. We show $z \in A^{-1}(D)$.

In the case of (iii): By Lemma 4, we obtain

$$\begin{aligned}0 &\leq \left\langle z-y, J_E\left(\left(I_E - rJ_E^{-1}A^*(J_F - J_F\Pi_D)A\right)z - z\right)\right\rangle\\ &= -r\langle z-y, A^*(J_F - J_F\Pi_D)Az\rangle\\ &= -r\langle Az-Ay, (J_F - J_F\Pi_D)Az\rangle\end{aligned} \tag{17}$$

for any $y \in C$. Therefore,

$$\langle Az-Ay, (J_F - J_F\Pi_D)Az\rangle \leq 0. \tag{18}$$

On the other hand, by Lemma 6, we obtain

$$\langle \Pi_D Az - v, J_F Az - J_F\Pi_D Az\rangle \geq 0 \tag{19}$$

for any $v \in D$. Since $C \cap A^{-1}(D) \neq \varnothing$, there exists $z_0 \in C \cap A^{-1}(D)$. Putting $y = z_0$ and $v = Az_0$, $y \in C$ and $v \in D$ hold. Therefore,

$$\begin{gathered}\langle Az - Az_0, (J_F - J_F\Pi_D)Az\rangle \leq 0,\\ \langle \Pi_D Az - Az_0, J_F Az - J_F\Pi_D Az\rangle \geq 0,\end{gathered} \tag{20}$$

and hence,

$$\langle Az - \Pi_D Az, J_F Az - J_F \Pi_D Az\rangle \le 0. \quad (21)$$

By Lemma 2, we obtain $\Pi_D Az = Az$, and hence, $Az \in D$; that is, $z \in A^{-1}(D)$.

In the case of (iv): By Lemma 6, we obtain

$$\begin{aligned} 0 &\le \langle z - y, (J_E - rA^* J_F(I_F - P_D)A)z - J_E z\rangle \\ &= -r\langle z - y, A^* J_F(I_F - P_D)Az\rangle \\ &= -r\langle Az - Ay, J_F(I_F - P_D)Az\rangle, \end{aligned} \quad (22)$$

for any $y \in C$. Therefore,

$$\langle Az - Ay, J_F(I_F - P_D)Az\rangle \le 0. \quad (23)$$

On the other hand, by Lemma 4, we obtain

$$\langle P_D Az - v, J_F(Az - P_D Az)\rangle \ge 0 \quad (24)$$

for any $v \in D$. Since $C \cap A^{-1}(D) \ne \varnothing$, there exists $z_0 \in C \cap A^{-1}(D)$. Putting $y = z_0$ and $v = Az_0$, $y \in C$ and $v \in D$ hold. Therefore,

$$\begin{aligned} \langle Az - Az_0, J_F(I_F - P_D)Az\rangle &\le 0, \\ \langle P_D Az - Az_0, J_F(Az - P_D Az)\rangle &\ge 0, \end{aligned} \quad (25)$$

and hence,

$$\langle Az - P_D Az, J_F(Az - P_D Az)\rangle = \|Az - P_D Az\|^2 \le 0. \quad (26)$$

Therefore, we obtain $P_D Az = Az$, and hence, $Az \in D$; that is, $z \in A^{-1}(D)$.

In the case of (v): By Lemma 6, we obtain

$$\begin{aligned} 0 &\le \left\langle z - y, J_E J_E^{-1}(J_E - rA^*(J_F - J_F \Pi_D)A)z - J_E z\right\rangle \\ &= -r\langle z - y, A^*(J_F - J_F \Pi_D)Az\rangle \\ &= -r\langle Az - Ay, (J_F - J_F \Pi_D)Az\rangle, \end{aligned} \quad (27)$$

for any $y \in C$. Therefore,

$$\langle Az - Ay, (J_F - J_F \Pi_D)Az\rangle \le 0. \quad (28)$$

On the other hand, by Lemma 6, we obtain

$$\langle \Pi_D Az - v, J_F Az - J_F \Pi_D Az\rangle \ge 0 \quad (29)$$

for any $v \in D$. Since $C \cap A^{-1}(D) \ne \varnothing$, there exists $z_0 \in C \cap A^{-1}(D)$. Putting $y = z_0$ and $v = Az_0$, $y \in C$ and $v \in D$ hold. Therefore,

$$\begin{aligned} \langle Az - Az_0, (J_F - J_F \Pi_D)Az\rangle &\le 0, \\ \langle \Pi_D Az - Az_0, J_F Az - J_F \Pi_D Az\rangle &\ge 0, \end{aligned} \quad (30)$$

and hence,

$$\langle Az - \Pi_D Az, J_F Az - J_F \Pi_D Az\rangle \le 0. \quad (31)$$

By Lemma 2, we obtain $\Pi_D Az = Az$, and hence, $Az \in D$; that is, $z \in A^{-1}(D)$.

Remark 17. Since in [11, Lemma 3.1] only the metric projection was used, only Lemma 4 was used for proving the equivalence between (i) and (ii). In Theorem 16, both of the metric projection and the generalize projection are used. Therefore, we have to use both of Lemmas 4 and 6 for proving the equivalence between (i) and (iii), and (iv) and (iv).

Theorem 18. *Let E and F be strictly convex, reflexive, and smooth Banach spaces; let F^* be the dual space of F, let I_E and I_{F^*} be the identity mappings on E and F^*, respectively; let J_E and J_F be duality mappings on E and F, respectively; let C and D be nonempty, closed, and convex subsets of E and F, respectively; let A be a bounded linear operator from E into F; let A^* be the adjoint operator of A; let $r \in (0,\infty)$. Suppose that $C \cap A^{-1}(D) \ne \varnothing$. Consider the following condition:*

(i) $z \in C \cap A^{-1}(D)$.

If $J_F(D)$ is closed, then the following are equivalent to (i):

(vi) $z = P_C(I_E - rJ_E^{-1}A^*(I_{F^*} - R_{J_F(D)})J_F A)z$;

(vii) $z = \Pi_C J_E^{-1}(J_E - rA^*(I_{F^*} - R_{J_F(D)})J_F A)z$.

If $J_E(C)$ is closed, then the following are equivalent to (i):

(viii) $z = J_E^{-1}R_{J_E(C)}(J_E - rA^* J_F(I_F - P_D)A)z$;

(ix) $z = J_E^{-1}R_{J_E(C)}(J_E - rA^*(J_F - J_F \Pi_D)A)z$.

If $J_E(C)$ and $J_F(D)$ are closed, then the following is equivalent to (i):

(x) $z = J_E^{-1}R_{J_E(C)}(J_E - rA^*(I_{F^*} - R_{J_F(D)})J_F A)z$.

Proof. Suppose that (i) holds. Since $Az \in D$,

$$P_D Az = \Pi_D Az = Az \text{ and } R_{J_F(D)} J_F Az = J_F Az \quad (32)$$

hold. Therefore,

$$J_F(I_F - P_D)Az = (J_F - J_F \Pi_D)Az = \left(I_{F^*} - R_{J_F(D)}\right)J_F Az = 0, \quad (33)$$

and hence,

$$\begin{gathered} \text{the right side of (vi)} = P_C z, \\ \text{the right side of (vii)} = \Pi_C z, \\ \text{the right sides of (viii), (ix), and (x)} = J_E^{-1}R_{J_E(C)}J_E z. \end{gathered} \quad (34)$$

Since $z \in C$, we obtain

$$\text{the right sides of (vi), (vii), (viii), (ix), and (x)} = z. \tag{35}$$

Conversely, suppose that (vi), (vii), (viii), (ix), or (x) holds. Since these equations have the form of $z = P_C x$, $z = \Pi_C x$, or $z = J_E^{-1} R_{J_E(C)} x$, $z \in C$ holds. We show $z \in A^{-1}(D)$.

In the case of (vi): By Lemma 4, we obtain

$$\begin{aligned} 0 &\le \left\langle z - y, J_E\left(\left(I_E - rJ_E^{-1}A^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A\right)z - z\right)\right\rangle \\ &= -r\left\langle z - y, A^*\left(I_{F^*} - R_{J_F(D)}\right)J_F Az\right\rangle \\ &= -r\left\langle Az - Ay, \left(I_{F^*} - R_{J_F(D)}\right)J_F Az\right\rangle, \end{aligned} \tag{36}$$

for any $y \in C$. Therefore,

$$\left\langle Az - Ay, \left(I_{F^*} - R_{J_F(D)}\right)J_F Az\right\rangle \le 0. \tag{37}$$

On the other hand, by Lemma 11, we obtain

$$\left\langle J_F^{-1} R_{J_F(D)} J_F Az - J_F^{-1} v, J_F Az - R_{J_F(D)} J_F Az\right\rangle \ge 0 \tag{38}$$

for any $v \in J_F(D)$. Since $C \cap A^{-1}(D) \ne \emptyset$, there exists $z_0 \in C \cap A^{-1}(D)$. Putting $y = z_0$ and $v = J_F Az_0$, $y \in C$ and $v \in J_F(D)$ hold. Therefore,

$$\begin{gathered} \left\langle Az - Az_0, \left(I_{F^*} - R_{J_F(D)}\right)J_F Az\right\rangle \le 0, \\ \left\langle J_F^{-1} R_{J_F(D)} J_F Az - Az_0, J_F Az - R_{J_F(D)} J_F Az\right\rangle \ge 0, \end{gathered} \tag{39}$$

and hence,

$$\left\langle Az - J_F^{-1} R_{J_F(D)} J_F Az, J_F Az - R_{J_F(D)} J_F Az\right\rangle \le 0. \tag{40}$$

By Lemma 2, we obtain $R_{J_F(D)} J_F Az = J_F Az$, and hence, $J_F Az \in J_F(D)$; that is, $z \in A^{-1}(D)$.

In the case of (vii): By Lemma 6, we obtain

$$\begin{aligned} 0 &\le \left\langle z - y, J_E J_E^{-1}\left(J_E - rA^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A\right)z - J_E z\right\rangle \\ &= -r\left\langle z - y, A^*\left(I_{F^*} - R_{J_F(D)}\right)J_F Az\right\rangle \\ &= -r\left\langle Az - Ay, \left(I_{F^*} - R_{J_F(D)}\right)J_F Az\right\rangle, \end{aligned} \tag{41}$$

for any $y \in C$. Therefore,

$$\left\langle Az - Ay, \left(I_{F^*} - R_{J_F(D)}\right)J_F Az\right\rangle \le 0. \tag{42}$$

On the other hand, by Lemma 11, we obtain

$$\left\langle J_F^{-1} R_{J_F(D)} J_F Az - J_F^{-1} v, J_F Az - R_{J_F(D)} J_F Az\right\rangle \ge 0 \tag{43}$$

for any $v \in J_F(D)$. Since $C \cap A^{-1}(D) \ne \emptyset$, there exists $z_0 \in C \cap A^{-1}(D)$. Putting $y = z_0$ and $v = J_F Az_0$, $y \in C$ and $v \in J_F(D)$ hold. Therefore,

$$\begin{gathered} \left\langle Az - Az_0, \left(I_{F^*} - R_{J_F(D)}\right)J_F Az\right\rangle \le 0, \\ \left\langle J_F^{-1} R_{J_F(D)} J_F Az - Az_0, J_F Az - R_{J_F(D)} J_F Az\right\rangle \ge 0, \end{gathered} \tag{44}$$

and hence,

$$\left\langle Az - J_F^{-1} R_{J_F(D)} J_F Az, J_F Az - R_{J_F(D)} J_F Az\right\rangle \le 0. \tag{45}$$

By Lemma 2, we obtain $R_{J_F(D)} J_F Az = J_F Az$, and hence, $J_F Az \in J_F(D)$; that is, $z \in A^{-1}(D)$.

In the case of (viii): By Lemma 11, we obtain

$$\begin{aligned} 0 &\le \langle J_E^{-1} J_E z - J_E^{-1} y, (J_E - rA^* J_F(I_E - P_D)A)z - J_E z\rangle \\ &= -r\langle z - J_E^{-1} y, A^* J_F(I_E - P_D)Az\rangle \\ &= -r\langle Az - AJ_E^{-1} y, J_F(I_E - P_D)Az\rangle, \end{aligned} \tag{46}$$

for any $y \in J_E(C)$. Therefore,

$$\langle Az - AJ_E^{-1} y, J_F(I_E - P_D)Az\rangle \le 0. \tag{47}$$

On the other hand, by Lemma 4, we obtain

$$\langle P_D Az - v, J_F(Az - P_D Az)\rangle \ge 0 \tag{48}$$

for any $v \in D$. Since $C \cap A^{-1}(D) \ne \emptyset$, there exists $z_0 \in C \cap A^{-1}(D)$. Putting $y = J_E z_0$ and $v = Az_0$, $y \in J_E(C)$ and $v \in D$ hold. Therefore,

$$\begin{gathered} \langle Az - Az_0, J_F(I_E - P_D)Az\rangle \le 0, \\ \langle P_D Az - Az_0, J_F(Az - P_D Az)\rangle \ge 0, \end{gathered} \tag{49}$$

and hence,

$$\langle Az - P_D Az, J_F(Az - P_D Az)\rangle = \|Az - P_D Az\|^2 \le 0. \tag{50}$$

Therefore, we obtain $P_D Az = Az$, and hence, $Az \in D$; that is, $z \in A^{-1}(D)$.

In the case of (ix): By Lemma 11, we obtain

$$\begin{aligned} 0 &\le \langle J_E^{-1} J_E z - J_E^{-1} y, (J_E - rA^*(J_F - J_F \Pi_D)A)z - J_E z\rangle \\ &= -r\langle z - J_E^{-1} y, A^*(J_F - J_F \Pi_D)Az\rangle \\ &= -r\langle Az - AJ_E^{-1} y, (J_F - J_F \Pi_D)Az\rangle, \end{aligned} \tag{51}$$

for any $y \in J_E(C)$. Therefore,

$$\left\langle Az - AJ_E^{-1}y, (J_F - J_F\Pi_D)Az \right\rangle \leq 0. \quad (52)$$

On the other hand, by Lemma 6, we obtain

$$\langle \Pi_D Az - v, J_F Az - J_F \Pi_D Az \rangle \geq 0 \quad (53)$$

for any $v \in D$. Since $C \cap A^{-1}(D) \neq \varnothing$, there exists $z_0 \in C \cap A^{-1}(D)$. Putting $y = J_E z_0$ and $v = Az_0$, $y \in J_E(C)$ and $v \in D$ hold. Therefore,

$$\begin{aligned} &\langle Az - Az_0, (J_F - J_F\Pi_D)Az \rangle \leq 0, \\ &\langle \Pi_D Az - Az_0, J_F Az - J_F \Pi_D Az \rangle \geq 0, \end{aligned} \quad (54)$$

and hence,

$$\langle Az - \Pi_D Az, J_F Az - J_F \Pi_D Az \rangle \leq 0. \quad (55)$$

By Lemma 2, we obtain $P_D Az = Az$, and hence, $Az \in D$; that is, $z \in A^{-1}(D)$.

In the case of (x): By Lemma 11, we obtain

$$\begin{aligned} 0 &\leq \left\langle J_E^{-1}J_E z - J_E^{-1}y, \left(J_E - rA^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A\right)z - J_E z \right\rangle \\ &= -r\left\langle z - J_E^{-1}y, A^*\left(I_{F^*} - R_{J_F(D)}\right)J_F Az \right\rangle \\ &= -r\left\langle Az - AJ_E^{-1}y, \left(I_{F^*} - R_{J_F(D)}\right)J_F Az \right\rangle, \end{aligned} \quad (56)$$

for any $y \in J_E(C)$. Therefore,

$$\left\langle Az - AJ_E^{-1}y, \left(I_{F^*} - R_{J_F(D)}\right)J_F Az \right\rangle \leq 0. \quad (57)$$

On the other hand, by Lemma 11, we obtain

$$\left\langle J_F^{-1}R_{J_F(D)}J_F Az - J_F^{-1}v, J_F Az - R_{J_F(D)}J_F Az \right\rangle \geq 0 \quad (58)$$

for any $v \in J_F(D)$. Since $C \cap A^{-1}(D) \neq \varnothing$, there exists $z_0 \in C \cap A^{-1}(D)$. Putting $y = J_E z_0$ and $v = J_F Az_0$, $y \in J_E(C)$ and $v \in J_F(D)$ hold. Therefore,

$$\begin{aligned} &\left\langle Az - Az_0, \left(I_{F^*} - R_{J_F(D)}\right)J_F Az \right\rangle \leq 0, \\ &\left\langle J_F^{-1}R_{J_F(D)}J_F Az - Az_0, J_F Az - R_{J_F(D)}J_F Az \right\rangle \geq 0, \end{aligned} \quad (59)$$

and hence,

$$\left\langle Az - J_F^{-1}R_{J_F(D)}J_F Az, J_F Az - R_{J_F(D)}J_F Az \right\rangle \leq 0. \quad (60)$$

By Lemma 2, we obtain $R_{J_F(D)}J_F Az = J_F Az$, and hence, $J_F Az \in J_F(D)$; that is, $z \in A^{-1}(D)$.

4. Strong Convergence Theorems to Solutions

In this section, we consider strong convergence theorems to solutions of the split feasibility problem.

Lemma 19. *Let E be a uniformly convex and smooth Banach space and let $u, v \in E$. Put*

$$g(\rho) = \begin{cases} g_{2,\frac{\max\{\|u\|,\|v\|\}}{\|u-v\|}}(1) & \text{if } u = v \text{ and } \rho \in \left(0, \dfrac{\max\{\|u\|,\|v\|\}}{\|u-v\|}\right), \\ g_{2,\rho}(1) & \text{if } u = v \text{ or } \rho \in \left[\dfrac{\max\{\|u\|,\|v\|\}}{\|u-v\|}, \infty\right) \end{cases} \quad (61)$$

for any $\rho \in (0, \infty)$, where $g_{2,\rho}$ as in Lemma 13. Then,

$$\langle u - v, J_E u - J_E v \rangle \geq g(\rho)\|u - v\|^2 \quad (62)$$

holds for any $\rho \in (0, \infty)$.

Proof. If $u = v$, then for arbitrary $g(\rho)$, we obtain

$$\langle u - v, J_E u - J_E v \rangle = 0 = g(\rho)\|u - v\|^2. \quad (63)$$

In particular, the above equation holds for $g(\rho) = g_{2,\rho}(1)$. If $u \neq v$, then, since $(1/(\|u-v\|)u, (1/\|u-v\|)v \in B_\rho(E)$ for any $\rho \in [\max\{\|u\|,\|v\|\}/\|u-v\|, \infty)$, we obtain

$$\begin{aligned} &\left\langle \frac{1}{\|u-v\|}u - \frac{1}{\|u-v\|}v, \frac{1}{\|u-v\|}J_E u - \frac{1}{\|u-v\|}J_E v \right\rangle \\ &\quad \geq \begin{cases} g_{2,\max\{\|u\|,\|v\|\}/\|u-v\|}\left(\left\|\dfrac{1}{\|u-v\|}u - \dfrac{1}{\|u-v\|}v\right\|\right) & \text{if } \rho \in \left(0, \dfrac{\max\{\|u\|,\|v\|\}}{\|u-v\|}\right), \\ g_{2,\rho}\left(\left\|\dfrac{1}{\|u-v\|}u - \dfrac{1}{\|u-v\|}v\right\|\right) & \text{if } \rho \in \left[\dfrac{\max\{\|u\|,\|v\|\}}{\|u-v\|}, \infty\right) \end{cases} \\ &\quad = g(\rho). \end{aligned} \quad (64)$$

Therefore, we obtain

$$\langle u-v, J_E u - J_E v\rangle \geq g(\rho)\|u-v\|^2. \quad (65)$$

Lemma 20. *Let E be a uniformly smooth Banach space and let $u, v \in E$. Put*

$$g^*(r,\rho)=\begin{cases} g^*_{2,((\|u\|)/(\|v\|))+r}(r) & \text{if } v\neq 0 \text{ and } \rho\in\left(0,\dfrac{\|u\|}{\|v\|}+r\right), \\ g^*_{2,\rho}(r) & \text{if } v=0 \text{ or } \rho\in\left[\dfrac{\|u\|}{\|v\|}+r,\infty\right) \end{cases} \quad (66)$$

for any $r\in(0,\infty)$ and for any $\rho\in(0,\infty)$, where $g^*_{2,\rho}$ as in Lemma 14. Then,

$$\langle v, J_E(u-rv)-J_Eu\rangle \geq -\frac{g^*(r,\rho)}{r}\|v\|^2 \quad (67)$$

holds for any $r\in(0,\infty)$ and for any $\rho\in(0,\infty)$.

Proof. If $v=0$, then for arbitrary $g^*(r,\rho)$, we obtain

$$\langle v, J_E(u-rv)-J_Eu\rangle = 0 = -\frac{g^*(r,\rho)}{r}\|v\|^2. \quad (68)$$

In particular, the above equation holds for $g^*(r,\rho)=g^*_{2,\rho}(r)$. If $v\neq 0$, then, since $(1/\|v\|)(u-rv), (1/\|v\|)u \in B_{(\|u\|/\|v\|)+r}(E)$, we obtain

$$\begin{aligned}\left\langle -\frac{r}{\|v\|}v, \frac{1}{\|v\|}J_E(u-rv)-\frac{1}{\|v\|}J_Eu\right\rangle &\leq \begin{cases} g^*_{2,(\|u\|/\|v\|)+r}\left(\left\|-\dfrac{r}{\|v\|}v\right\|\right) & \text{if } \rho\in\left(0,\dfrac{\|u\|}{\|v\|}+r\right), \\ g^*_{2,\rho}\left(\left\|-\dfrac{r}{\|v\|}v\right\|\right) & \text{if } \rho\in\left[\dfrac{\|u\|}{\|v\|}+r,\infty\right) \end{cases} \\ &= g^*(r,\rho). \end{aligned} \quad (69)$$

Therefore, we obtain

$$\langle v, J_E(u-rv)-J_Eu\rangle \geq -\frac{g^*(r,\rho)}{r}\|v\|^2. \quad (70)$$

We consider the following condition for a uniformly convex and smooth Banach space:

($*$1) $\inf_{\rho\in(0,\infty)} g_{2,\rho}(1)>0$.

We consider the following condition for a uniformly smooth Banach space:

($*$2) There exist $r_0\in(0,\infty)$ and $\alpha\in(1,\infty)$ such that $\sup_{r\in(0,r_0),\rho\in(0,\infty)}(g^*_{2,\rho}(r)/r^\alpha)<\infty$.

Example 1. We consider each of examples satisfying the conditions ($*$1) and ($*$2), respectively.

Let E be a Banach space. The modulus of convexity δ_E of E is defined by

$$\delta_E(\varepsilon)=\inf\left\{1-\frac{1}{2}\|x+y\| \,\middle|\, \|x\|=\|y\|=1, \|x-y\|=\varepsilon\right\} \quad (71)$$

for any $\varepsilon\in[0,2]$. Let $p\in(1,\infty)$. E is said to be p-uniformly convex if there exists a constant $c\in(0,\infty)$ such that $\delta_E(\varepsilon)\geq c\varepsilon^p$ for any $\varepsilon\in[0,2]$. Furthermore, we know that E is uniformly convex if and only if $\delta_E(\varepsilon)>0$ for any $\varepsilon\in(0,2]$. Let J_p be the generalized duality mapping on a Banach space E. By [17, Corollary 1], E is p-uniformly convex if and only if there exists a constant $c\in(0,\infty)$ such that

$$\langle x-y, x^*-y^*\rangle \geq c\|x-y\|^p \quad (72)$$

for any $x,y\in E$, for any $x^*\in J_px$, and for any $y^*\in J_py$. Therefore, if E is 2-uniformly convex and smooth, then we can put $g_{2,\rho}(r)=cr^2$ for any $\rho\in(0,\infty)$. We obtain

$$\inf_{\rho\in(0,\infty)} g_{2,\rho}(1)=c>0. \quad (73)$$

The modulus of smoothness ρ_E of E is defined by

$$\rho_E(t)=\sup\left\{\frac{1}{2}(\|x+y\|+\|x-y\|)-1 \,\middle|\, \|x\|=1, \|y\|=t\right\} \quad (74)$$

for any$t\in(0,\infty)$. Let $q\in(1,\infty)$. E is said to be q-uniformly smooth if there exists a constant $c\in(0,\infty)$ such that $\rho_E(t)\leq ct^q$ for any $t\in(0,\infty)$. Furthermore, we know that E is uniformly smooth if and only if $\lim_{t\downarrow 0}(\rho_E(t)/t)=0$. Let

J_q be the generalized duality mapping on a smooth Banach space E.

By [17, Corollary 1 ;′], E is q-uniformly smooth if and only if there exists a constant $c \in (0, \infty)$ such that

$$\langle x - y, J_q x - J_q y \rangle \le c \|x - y\|^q \tag{75}$$

for any $x, y \in E$. Therefore, if E is 2-uniformly smooth, then we can put $g^*_{2,\rho}(r) = cr^2$ for any $\rho \in (0, \infty)$. Let $r_0 \in (0, \infty)$ and let $\alpha \in (1, 2]$. Then, we obtain

$$\sup_{r \in (0, r_0),\, \rho \in (0, \infty)} \frac{g^*_{2,\rho}(r)}{r^\alpha} = cr_0^{2-\alpha} < \infty. \tag{76}$$

Lemma 21. *Let E be a uniformly convex and smooth Banach space, let F be a uniformly smooth Banach space, and let $M \in (0, \infty)$. Suppose that E satisfies the condition $(*1)$ and F satisfies the condition $(*2)$. Then, there exists $r_M \in (0, \infty)$ such that $g(\rho) r - M g^*(r, \rho) > 0$ for any $r \in (0, r_M)$ and for any $\rho \in (0, \infty)$, where $g(\rho)$ as in Lemma 19 and $g^*(r, \rho)$ as in Lemma 20.*

Proof. We obtain

$$\begin{aligned} & g(\rho) r - M g^*(r, \rho) \\ & \ge \left(\inf_{\rho \in (0, \infty)} g_{2,\rho}(1) \right) r - M \left(\sup_{r \in (0, r_0),\, \rho \in (0, \infty)} \frac{g^*_{2,\rho}(r)}{r^\alpha} \right) r^\alpha \\ & = r \left(\inf_{\rho \in (0, \infty)} g_{2,\rho}(1) - M \left(\sup_{r \in (0, r_0),\, \rho \in (0, \infty)} \frac{g^*_{2,\rho}(r)}{r^\alpha} \right) r^{\alpha - 1} \right) \end{aligned} \tag{77}$$

for any $r \in (0, r_0)$. Put

$$r_M = \min \left\{ \left(\frac{\inf_{\rho \in (0, \infty)} g_{2,\rho}(1)}{M \sup_{r \in (0, r_0),\, \rho \in (0, \infty)} \left(g^*_{2,\rho}(r) / r^\alpha \right)} \right)^{1/(\alpha - 1)}, r_0 \right\}. \tag{78}$$

Then, we obtain $g(\rho) r - M g^*(r, \rho) > 0$ for any $r \in (0, r_M)$ and for any $\rho \in (0, \infty)$.

Theorem 22. *Let E be a uniformly convex and smooth Banach space; let F be a strictly convex, reflexive, and uniformly smooth Banach space; let J_E and J_F be duality mappings on E and F, respectively; let C and D be nonempty, closed, and convex subsets of E and F, respectively; let A be a bounded and linear operator from E into F with $A \ne 0$; let A^* be the adjoint operator of A; let $r \in (0, \infty)$. Suppose that E^* satisfies the condition $(*2)$, F^* satisfies the condition $(*1)$ and $C \cap A^{-1}(D) \ne \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1} (J_E - r A^* (J_F - J_F \Pi_D) A) x_n, \\ C_n = \{ z \in C \mid \langle z_n - z, J_E x_n - J_E z_n \rangle \ge 0 \}, \\ D_n = \{ z \in D_{n-1} \mid \langle x_n - z, J_E x_1 - J_E x_n \rangle \ge 0 \}, \\ x_{n+1} = \Pi_{C_n \cap D_n} x_1 \end{cases} \tag{79}$$

for any $n \in \mathbb{N}$, where $D_0 = C$. Then, there exists $r_{\|A\|^2} \in (0, \infty)$ such that $\{x_n\}$ is strongly convergent to a point $z_0 \in C \cap A^{-1}(D)$ for any $r \in (0, r_{\|A\|^2})$, where $z_0 = \Pi_{C \cap A^{-1}(D)} x_1$.

Proof. For the sake of the proof, we confirm the following facts. Since E is uniformly convex, by (T8) E is reflexive. Since E is uniformly convex and smooth, by (T10) E^* is uniformly smooth. Since F is reflexive and uniformly smooth, by (T10) F^* is uniformly convex. Since F is strictly convex and reflexive, by (T7) F^* is smooth.

It is obvious that $C_n \cap D_n$ is closed and convex for any $n \in \mathbb{N}$. We show that $C \cap A^{-1}(D) \subset C_n$ for any $n \in \mathbb{N}$. Let $z \in C \cap A^{-1}(D)$. If $Ax_n = 0$ or $Ax_n \in D$, then we obtain $z_n = x_n$, and hence,

$$\langle z_n - z, J_E x_n - J_E z_n \rangle = 0 ; \tag{80}$$

that is, $z \in C_n$. If $Ax_n \ne 0$ and $Ax_n \notin D$, then

$$\begin{aligned} & \langle z_n - z, J_E x_n - J_E z_n \rangle \\ & = r \big(\langle J_E^{-1} (J_E x_n - r A^* (J_F - J_F \Pi_D) A x_n) - x_n, \\ & \quad A^* (J_F - J_F \Pi_D) A x_n \rangle \\ & \quad + \langle x_n - z, A^* (J_F - J_F \Pi_D) A x_n \rangle \big) \\ & = r \big(\langle J_E^{-1} (J_E x_n - r A^* (J_F - J_F \Pi_D) A x_n) - x_n, \\ & \quad A^* (J_F - J_F \Pi_D) A x_n \rangle \\ & \quad + \langle A x_n - \Pi_D A x_n, (J_F - J_F \Pi_D) A x_n \rangle \\ & \quad + \langle \Pi_D A x_n - A z, (J_F - J_F \Pi_D) A x_n \rangle \big) \\ & = r \big(\langle J_E^{-1} (J_E x_n - r A^* (J_F - J_F \Pi_D) A x_n) - J_E^{-1} J_E x_n, \\ & \quad A^* (J_F - J_F \Pi_D) A x_n \rangle \\ & \quad + \langle J_F^{-1} J_F A x_n - J_F^{-1} J_F \Pi_D A x_n, J_F A x_n - J_F \Pi_D A x_n \rangle \\ & \quad + \langle \Pi_D A x_n - A z, J_F A x_n - J_F \Pi_D A x_n \rangle \big). \end{aligned} \tag{81}$$

By Lemma 21, there exists $r_{\|A\|^2} \in (0, \infty)$ such that $g(\rho) r - \|A\|^2 g^*(r, \rho) > 0$ for any $r \in (0, r_{\|A\|^2})$ and for any $\rho \in (0, \infty)$. Put

$$\rho_n \in \left[\max \left\{ \frac{\|x_n\|}{\|A^* (J_F - J_F \Pi_D) A x_n\|} + r, \frac{\|A x_n\|}{\|J_F A x_n - J_F \Pi_D A x_n\|}, \frac{\|\Pi_D A x_n\|}{\|J_F A x_n - J_F \Pi_D A x_n\|} \right\}, \infty \right). \tag{82}$$

Since E^* is uniformly smooth, by Lemma 20, we obtain

$$\begin{aligned}&\left\langle J_E^{-1}(J_E x_n - rA^*(J_F - J_F\Pi_D)Ax_n) - J_E^{-1}J_E x_n, A^*(J_F - J_F\Pi_D)Ax_n\right\rangle\\ &\quad\geq -\frac{g^*(r,\rho_n)}{r}\|A^*(J_F - J_F\Pi_D)Ax_n\|^2\\ &\quad\geq -\frac{\|A\|^2 g^*(r,\rho_n)}{r}\|(J_F - J_F\Pi_D)Ax_n\|^2.\end{aligned} \tag{83}$$

Since F^* is uniformly convex and smooth, by Lemma 19, we obtain

$$\begin{aligned}&\left\langle J_F^{-1}J_F Ax_n - J_F^{-1}J_F\Pi_D Ax_n, J_F Ax_n - J_F\Pi_D Ax_n\right\rangle\\ &\quad\geq g(\rho_n)\|J_F Ax_n - J_F\Pi_D Ax_n\|^2.\end{aligned} \tag{84}$$

By Lemma 6, we obtain

$$\langle \Pi_D Ax_n - Az, J_F Ax_n - J_F\Pi_D Ax_n\rangle \geq 0. \tag{85}$$

Therefore, we obtain

$$\begin{aligned}\langle z_n - z, J_E x_n - J_E z_n\rangle &\geq \left(g(\rho_n)r - \|A\|^2 g^*(r,\rho_n)\right)\\ &\quad\cdot \|(J_F - J_F\Pi_D)Ax_n\|^2 \geq 0;\end{aligned} \tag{86}$$

that is, $z \in C_n$. We show that $C \cap A^{-1}(D) \subset D_n$ for any $n \in \mathbb{N}$. Since

$$D_1 = \{z \in C \mid \langle x_1 - z, J_E x_1 - J_E x_1\rangle \geq 0\} = C, \tag{87}$$

it is obvious that $C \cap A^{-1}(D) \subset D_1$. Suppose that there exists $k \in \mathbb{N}$ such that $C \cap A^{-1}(D) \subset D_k$. Then, $C \cap A^{-1}(D) \subset C_k \cap D_k$. Since $x_{k+1} = \Pi_{C_k \cap D_k} x_1$, by Lemma 6, we obtain

$$\langle x_{k+1} - z, J_E x_1 - J_E x_{k+1}\rangle \geq 0 \tag{88}$$

for any $z \in C_k \cap D_k$, and hence, the above inequality holds for any $z \in C \cap A^{-1}(D)$; that is, we obtain $C \cap A^{-1}(D) \subset D_{k+1}$. We obtain $C \cap A^{-1}(D) \subset D_n$ for any $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is well-defined.

It is obvious that $C \cap A^{-1}(D)$ is nonempty, closed, and convex. Therefore, $\Pi_{C\cap A^{-1}(D)}$ is well-defined.

By the definition of D_n, we obtain $x_n = \Pi_{D_n} x_1$. By Lemma 6,

$$V_E(z, x_n) + V_E(x_n, x_1) \leq V_E(z, x_1) \tag{89}$$

for any $z \in D_n$. Let $m \in \mathbb{N}$ with $m \geq n$. Since $x_m \in D_m \subset D_n$, we obtain

$$V_E(x_m, x_n) + V_E(x_n, x_1) \leq V_E(x_m, x_1), \tag{90}$$

and hence,

$$V_E(x_m, x_n) \leq V_E(x_m, x_1) - V_E(x_n, x_1). \tag{91}$$

Since $x_n = \Pi_{D_n} x_1$, we obtain

$$V_E(x_n, x_1) \leq V_E(z, x_1) \tag{92}$$

for any $z \in D_n$. Since $x_{n+1} = \Pi_{C_n \cap D_n} x_1$, we obtain $x_{n+1} \in C_n \cap D_n$. Since $x_{n+1} \in D_n$, we obtain

$$V_E(x_n, x_1) \leq V_E(x_{n+1}, x_1), \tag{93}$$

and hence, $\{V_E(x_n, x_1)\}$ is bounded and nondecreasing; that is, there exists the limit of $\{V_E(x_n, x_1)\}$. From (91), we obtain

$$\lim_{m,n\to\infty} V_E(x_m, x_n) = 0. \tag{94}$$

By Lemma 15, $\{x_n\}$ is a Cauchy sequence. Since E is complete, there exists $z_0 \in E$ such that $\lim_{n\to\infty} x_n = z_0$. Since $x_{n+1} = \Pi_{C_n\cap D_n} x_1$, by Lemma 6, we obtain

$$\langle x_{n+1} - z, J_E x_1 - J_E x_{n+1}\rangle \geq 0 \tag{95}$$

for any $z \in C_n \cap D_n$. Since $C \cap A^{-1}(D) \subset C_n \cap D_n$, the above inequality holds for any $z \in C \cap A^{-1}(D)$. Taking $n \longrightarrow \infty$, since by (T9) J_E is norm-to-norm continuous, we obtain

$$\langle z_0 - z, J_E x_1 - J_E z_0\rangle \geq 0 \tag{96}$$

for any $z \in C \cap A^{-1}(D)$. By Lemma 6, we obtain $z_0 = \Pi_{C\cap A^{-1}(D)} x_1$.

Theorem 23. *Let E be a uniformly convex and smooth Banach space; let F be a strictly convex, reflexive, and uniformly smooth Banach space; let I_{F^*} be the identity mapping on F^*; let J_E and J_F be duality mappings on E and F, respectively; let C and D be nonempty, closed, and convex subsets of E and F such that $J_F(D)$ is closed, respectively; let A be a bounded and linear operator from E into F with $A \neq 0$; let A^* be the adjoint operator of A; let $r \in (0,\infty)$. Suppose that E^* satisfies the condition $(*2)$, F^* satisfies the condition $(*1)$ and $C \cap A^{-1}(D) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\left(J_E - rA^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A\right)x_n, \\ C_n = \{z \in C \mid \langle z_n - z, J_E x_n - J_E z_n\rangle \geq 0\}, \\ D_n = \{z \in D_{n-1} \mid \langle x_n - z, J_E x_1 - J_E x_n\rangle \geq 0\}, \\ x_{n+1} = J_E^{-1} R_{J_E(C_n \cap D_n)} J_E x_1 \end{cases} \tag{97}$$

for any $n \in \mathbb{N}$, where $D_0 = C$. Then, there exists $r_{\|A\|^2} \in (0,\infty)$ such that $\{x_n\}$ is strongly convergent to a point $z_0 \in C \cap A^{-1}(D)$ for any $r \in (0, r_{\|A\|^2})$, where $z_0 = J_E^{-1} R_{J_E(C \cap A^{-1}(D))} J_E x_1$.

Proof. For the sake of the proof, we confirm the following facts. Since E is uniformly convex, by (T8), E is reflexive. Since E is uniformly convex and smooth, by (T10), E^* is uniformly smooth. Since F is reflexive and uniformly smooth, by (T10), F^* is uniformly convex. Since F is strictly convex and reflexive, by (T7), F^* is smooth. Since E^* and F^* have Fréchet differentiable norms, by (T9), J_E^{-1} and J_F^{-1} are norm-to-norm continuous. Therefore, $J_E(C)$ and $J_E(C_n \cap D_n)$ are closed.

It is obvious that $C_n \cap D_n$ is closed and convex for any $n \in \mathbb{N}$. We show that $C \cap A^{-1}(D) \subset C_n$ for any $n \in \mathbb{N}$. Let $z \in C \cap A^{-1}(D)$. If $Ax_n = 0$ or $Ax_n \in D$, then we obtain $z_n = x_n$, and hence,

$$\langle z_n - z, J_E x_n - J_E z_n\rangle = 0; \tag{98}$$

that is, $z \in C_n$. If $Ax_n \neq 0$ and $Ax_n \notin D$, then

$$\begin{aligned} &\langle z_n - z, J_E x_n - J_E z_n\rangle \\ &= r\Big(\Big\langle J_E^{-1}\left(J_E x_n - rA^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\right) \\ &\quad - x_n, A^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\Big\rangle \\ &\quad + \left\langle x_n - z, A^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\right\rangle\Big) \\ &= r\Big(\Big\langle J_E^{-1}\left(J_E x_n - rA^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\right) \\ &\quad - x_n, A^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\Big\rangle \\ &\quad + \left\langle A x_n - J_F^{-1} R_{J_F(D)} J_F A x_n, \left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\right\rangle \\ &\quad + \left\langle J_F^{-1} R_{J_F(D)} J_F A x_n - Az, \left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\right\rangle\Big) \\ &= r\Big(\Big\langle J_E^{-1}\left(J_E x_n - rA^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\right) \\ &\quad - J_E^{-1} J_E x_n, A^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\Big\rangle \\ &\quad + \Big\langle J_F^{-1} J_F A x_n - J_F^{-1} R_{J_F(D)} J_F A x_n, J_F A x_n \\ &\quad - R_{J_F(D)}, J_F A x_n\Big\rangle + \Big\langle J_F^{-1} R_{J_F(D)} J_F A x_n \\ &\quad - J_F^{-1} J_F A z, J_F A x_n - R_{J_F(D)}, J_F A x_n\Big\rangle\Big). \end{aligned} \tag{99}$$

By Lemma 21, there exists $r_{\|A\|^2} \in (0,\infty)$ such that $g(\rho)r - \|A\|^2 g^*(r,\rho) > 0$ for any $r \in (0, r_{\|A\|^2})$ and for any $\rho \in (0, \infty)$. Put

$$\rho_n \in \left[\max\left\{\frac{\|x_n\|}{\left\|A^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\right\|} + r, \frac{\|Ax_n\|}{\left\|J_F A x_n - R_{J_F(D)} J_F A x_n\right\|}, \frac{\|R_{J_F(D)} J_F A x_n\|}{\|J_F A x_n - R_{J_F(D)} J_F A x_n\|}\right\}, \infty\right). \tag{100}$$

Since E^* is uniformly smooth, by Lemma 20, we obtain

$$\begin{aligned} &\left\langle J_E^{-1}\left(J_E x_n - rA^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\right) - J_E^{-1} J_E x_n, A^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\right\rangle \\ &\geq -\frac{g^*(r,\rho_n)}{r}\left\|A^*\left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\right\|^2 \\ &\geq -\frac{\|A\|^2 g^*(r,\rho_n)}{r}\left\|\left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\right\|^2. \end{aligned} \tag{101}$$

Since F^* is uniformly convex and smooth, by Lemma 19, we obtain

$$\begin{aligned} &\left\langle J_F^{-1} J_F A x_n - J_F^{-1} R_{J_F(D)} J_F A x_n, J_F A x_n - R_{J_F(D)} J_F A x_n\right\rangle \\ &\quad \geq g(\rho_n)\left\|J_F A x_n - R_{J_F(D)} J_F A x_n\right\|^2. \end{aligned} \tag{102}$$

By Lemma 11, we obtain

$$\left\langle J_F^{-1} R_{J_F(D)} J_F A x_n - J_F^{-1} J_F A z, J_F A x_n - R_{J_F(D)} J_F A x_n\right\rangle \geq 0. \tag{103}$$

Therefore, we obtain

$$\begin{aligned} \langle z_n - z, J_E x_n - J_E z_n\rangle &\geq \left(g(\rho_n) r - \|A\|^2 g^*(r,\rho_n)\right) \\ &\quad \cdot \left\|\left(I_{F^*} - R_{J_F(D)}\right)J_F A x_n\right\|^2 \geq 0; \end{aligned} \tag{104}$$

that is, $z \in C_n$. We show that $C \cap A^{-1}(D) \subset D_n$ for any $n \in \mathbb{N}$. Since

$$D_1 = \{z \in C \mid \langle x_1 - z, J_E x_1 - J_E x_1\rangle \geq 0\} = C, \tag{105}$$

it is obvious that $C \cap A^{-1}(D) \subset D_1$. Suppose that there exists $k \in \mathbb{N}$ such that $C \cap A^{-1}(D) \subset D_k$. Then, $C \cap A^{-1}(D) \subset C_k \cap$

D_k. Since $x_{k+1} = J_E^{-1} R_{J_E(C_k \cap D_k)} J_E x_1$, by Lemma 11, we obtain

$$\left\langle x_{k+1} - J_E^{-1} z, J_E x_1 - J_E x_{k+1} \right\rangle \geq 0 \tag{106}$$

for any $z \in J_E(C_k \cap D_k)$, and hence, the above inequality holds for any $z \in J_E(C \cap A^{-1}(D))$; that is, we obtain $C \cap A^{-1}(D) \subset D_{k+1}$. We obtain $C \cap A^{-1}(D) \subset D_n$ for any $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is well-defined.

It is obvious that $J_E(C \cap A^{-1}(D))$ is nonempty, closed, and convex and $C \cap A^{-1}(D)$ is closed. Therefore, $R_{J_E(C \cap A^{-1}(D))}$ is well-defined.

By the definition of D_n, we obtain $x_n = J_E^{-1} R_{J_E(D_n)} J_E x_1$. By Lemma 11,

$$V_{E^*}(J_E x_n, z) + V_{E^*}(J_E x_1, J_E x_n) \leq V_{E^*}(J_E x_1, z) \tag{107}$$

for any $z \in J_E(D_n)$. Let $m \in \mathbb{N}$ with $m \geq n$. Since $x_m \in D_m \subset D_n$, we obtain

$$V_{E^*}(J_E x_n, J_E x_m) + V_{E^*}(J_E x_1, J_E x_n) \leq V_{E^*}(J_E x_1, J_E x_m), \tag{108}$$

and hence,

$$V_{E^*}(J_E x_n, J_E x_m) \leq V_{E^*}(J_E x_1, J_E x_m) - V_{E^*}(J_E x_1, J_E x_n). \tag{109}$$

Since $x_n = J_E^{-1} R_{J_E(D_n)} J_E x_1$, by Lemma 9, we obtain

$$V_{E^*}(J_E x_1, J_E x_n) \leq V_{E^*}(J_E x_1, z) \tag{110}$$

for any $z \in J_E(D_n)$. Since $x_{n+1} = J_E^{-1} R_{J_E(C_n \cap D_n)} J_E x_1$, we obtain $x_{n+1} \in C_n \cap D_n$. Since $x_{n+1} \in D_n$, we obtain

$$V_{E^*}(J_E x_1, J_E x_n) \leq V_{E^*}(J_E x_1, J_E x_{n+1}), \tag{111}$$

and hence, $\{V_{E^*}(J_E x_1, J_E x_n)\}$ is bounded and nondecreasing; that is, there exists the limit of $\{V_{E^*}(J_E x_1, J_E x_n)\}$. From (109), we obtain

$$\lim_{m,n \to \infty} V_{E^*}(J_E x_n, J_E x_m) = 0. \tag{112}$$

By Lemma 15, $\{J_E x_n\}$ is a Cauchy sequence. Since by (T9) J_E^{-1} is norm-to-norm continuous, $\{x_n\}$ is also a Cauchy sequence. Since E is complete, there exists $z_0 \in E$ such that $\lim_{n \to \infty} x_n = z_0$. Since $x_{n+1} = J_E^{-1} R_{J_E(C_n \cap D_n)} J_E x_1$, by Lemma 11, we obtain

$$\left\langle x_{n+1} - J_E^{-1} z, J_E x_1 - J_E x_{n+1} \right\rangle \geq 0 \tag{113}$$

for any $z \in J_E(C_n \cap D_n)$. Since $C \cap A^{-1}(D) \subset C_n \cap D_n$, the above inequality holds for any $z \in J_E(C \cap A^{-1}(D))$. Taking $n \longrightarrow \infty$, since by (T9) J_E is norm-to-norm continuous, we obtain

$$\left\langle z_0 - J_E^{-1} z, J_E x_1 - J_E z_0 \right\rangle \geq 0 \tag{114}$$

for any $z \in J_E(C \cap A^{-1}(D))$. By Lemma 11, we obtain $z_0 = J_E^{-1} R_{J_E(C \cap A^{-1}(D))} J_E x_1$.

Remark 24. In this paper, we consider only two strong convergence theorems with respect to (v) in Theorem 16 and (x) in Theorem 18. The strong convergence theorem with respect to (ii) is shown in [11, Theorem 3.2]. Of course, we can consider other strong convergence theorems with respect to (iii), (iv) in Theorem 16, and (vi), (vii), (viii), and (ix) in Theorem 18. They will be described in the next paper.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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