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Research Article

A New Iterative Algorithm for Pseudomonotone Equilibrium Problem and a Finite Family of Demicontractive Mappings

F. U. Ogbuisi [1,2] **and F. O. Isiogugu** [1,2,3]

[1] *School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa*
[2] *Department of Mathematics, University of Nigeria, Nsukka, Nigeria*
[3] *DST-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), Pretoria, South Africa*

Correspondence should be addressed to F. U. Ogbuisi; ferdinard.ogbuisi@unn.edu.ng

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In this paper, we introduce a new iterative method in a real Hilbert space for approximating a point in the solution set of a pseudomonotone equilibrium problem which is a common fixed point of a finite family of demicontractive mappings. Our result does not require that we impose the condition that the sum of the control sequences used in the finite convex combination is equal to 1. Furthermore, we state and prove a strong convergence result and give some numerical experiments to demonstrate the efficiency and applicability of our iterative method.

1. Introduction

In this paper, we will always take C to be a nonempty closed and convex subset of a real Hilbert space H endowed with inner product $\langle \cdot,\cdot \rangle$ and induced norm $\|\cdot\|$, and $F(T)$ denotes the set of fixed points of a mapping $T: C \longrightarrow C$, that is, $F(T) := \{x \in C: x = Tx\}$.

Definition 1. A mapping $T: C \longrightarrow C$ is said to be

(1) Nonspreading [1] if

$$\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C, \tag{1}$$

or equivalently

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C. \tag{2}$$

(2) κ-Strictly pseudononspreading [1] if there exists $\kappa \in [0, 1)$ such that

$$\begin{aligned}\|Tx - Ty\|^2 \leq{}& \|x - y\|^2 + \kappa\|x - Tx - (y - Ty)\|^2 \\ &+ 2\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C.\end{aligned} \tag{3}$$

(3) β-Strictly pseudocontractive [2] if there exists $\beta \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \beta\|x - Tx - (y - Ty)\|^2 \quad \forall x, y \in C. \tag{4}$$

(4) ρ-Demicontractive [3] if there exists $\rho \in [0, 1)$ such that

$$\|Tx - y\|^2 \leq \|x - y\|^2 + \rho\|x - Tx\|^2 \quad \forall x \in C, y \in F(T). \tag{5}$$

Obviously, the class of demicontractive mappings is more general than the class of quasi-nonexpansive mappings. Moreover, If T is κ-strictly pseudononspreading (or κ-strictly pseudocontractive) and $F(T) \neq \varnothing$, then T is κ-demicontractive.

Definition 2. A bifunction $f: C \times C \longrightarrow \mathbb{R}$ is

(i) Strongly monotone on C with modulus $\beta > 0$ (β-strongly monotone on C) if and only if

$$f(x, y) + f(y, x) \leq -\beta \|y - x\|^2, \quad \forall x, y \in C; \tag{6}$$

(ii) monotone on C if and only if

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in C; \tag{7}$$

(iii) pseudomonotone on C if and if

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0, \quad \forall x, y \in C. \tag{8}$$

Let $f: C \times C \longrightarrow \mathbb{R}$ be a bifunction such that $f(x, x) = 0$, for all $x \in C$. We consider the equilibrium problem (EP) in the sense of Blum and Oettli [4], which is to find

$$x^* \in C \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in C. \tag{9}$$

Let $\mathrm{EP}(C, f)$ denote the set of solutions of EP (9). If $f(x, y) = \langle A(x), y - x \rangle \, \forall x, y \in C$, where $A: C \longrightarrow H$, then EP (9) reduces to the variational inequality problem:

$$\text{find } x^* \in C \text{ such that } \langle A(x), y - x \rangle \geq 0, \quad \forall y \in C. \tag{10}$$

EPs form a very important area of research and have recently been considered in many research papers. EP (9) is applied in solving problems from optimization, variational inequality, Kakutani fixed point, Nash equilibria in non-cooperative game theory, and minimax problems [4, 5].

A popular method that has been applied to solve EP (9) is the subgradient projection method which is developed from the steepest descent projection method in smooth optimization. If bifunction f is convex, subdifferentiable with respect to the second argument, Lipschitz, and strongly monotone on C, then regularization parameters can be chosen such that the subgradient projection method is linearly convergent [6]. However, when f is only monotone, the subgradient projection method may not be convergent. In order to get a method that guarantees convergence for pseudomonotone equilibrium problems (that is, equilibrium problems for pseudomonotone bifunctions) the extragradient (or double projection) method developed by Korpelevich [7] was extended to equilibrium problems. However, the extragradient algorithms involve two projections on the admissible set C, which may be costly to compute if the nature of the admissible set C is complicated. In the light of the need to obtain a more efficient algorithm, the inexact subgradient algorithms using only one projection [8, 9] has been proposed for solving equilibrium problems with paramonotone equilibrium bifunctions. Some other methods that have been utilized to solve equilibrium problems include the auxiliary problem principle method [10], gap function method [11], and the Tikhonov and proximal point regularization methods [12–15].

Recently, the problem of finding a common point in $\mathrm{EP}(C, f)$ and the set of fixed points of mappings has become an attractive and interesting subject [16–22]. This interest is because of the possible application of these problems to mathematical models whose constraints can be present as fixed points of mappings and/or (EP). Such a problem occurs, in particular, in the practical problems as signal processing, network resource allocation, image recovery (see [23, 24]).

In 2007, Tada and Takahashi [22] proposed the following iterative algorithm for approximating a common element of the set of solutions of equilibrium problem for monotone bifunctions and the set of fixed points of a nonexpansive mapping T.

Algorithm 1

$$z_k \in C \text{ such that } f(z_k, y) + \frac{1}{\lambda_k} \langle y - z_k, z_k - x_k \rangle \geq 0, \quad \forall y \in C,$$

$$\begin{cases} w_k = \alpha_k x_k + (1 - \alpha_k) T(z_k), \\ C_k = \left\{ z \in H: \left\| w_k - z \right\| \leq \left\| x_k - z \right\| \right\}, \\ D_k = \left\{ z \in H: \langle x_k - z, x_0 - x_k \rangle \geq 0 \right\}, \\ x_{k+1} = P_{C_k \cap D_k}(x_0), \end{cases} \tag{11}$$

where $\lambda_k > 0$ is the regularization parameter at iteration $k, x_0 \in C$ and P_C is the metric projection onto C. They assume that f is a monotone bifunction and obtained a strong convergence result.

Recently, Anh and Muu [25] proposed a new type of algorithm which uses only one projection and does not require any Lipschitz condition for the bifunctions for finding a common point in the solution set of the class of pseudomonotone equilibrium problems and the set of fixed points of nonexpansive mappings. More precisely, they gave an iteration scheme generated as follows.

Algorithm 2. Pick $x_1 \in C$. At each iteration $k = 1, 2, \ldots$, do the following:

$$\begin{cases} \text{compute } w_k = \partial_{\varepsilon_k} f(x_k, \cdot)(x_k), \\ \text{take } \gamma_k := \max\left\{ \lambda_k, \left\| w_k \right\| \right\}, \\ \alpha_k := \dfrac{\beta_k}{\gamma_k}, \\ y_k = P_C(x_k - \alpha_k w_k), \\ x_{k+1} = \delta_k x_k + (1 - \delta_k) T(y_k). \end{cases} \tag{12}$$

Inspired by Anh and Muu [25], Wangkeeree et al. [26] presented an iterative method for finding hierarchically an element in $F(T) \cap EP(C, f)$ with respect to a nonexpansive mapping. Precisely, they considered the following problems:

$$\text{find } \overline{x} \in \Gamma \coloneqq F(T) \cap \mathrm{EP}(C, f) \text{ such that} \quad \langle \overline{x} - S(\overline{x}), \overline{x} - x \rangle \leq 0, \quad \forall x \in \Gamma, \tag{13}$$

where T and S are nonexpansive mappings.

Other authors have also considered different algorithms which involve either projection mapping or projection mapping and the construction of sequences of sets $\{C_n\}$ and $\{Q_n\}$ for approximating a common solution of pseudomonotone equilibrium problems and fixed point problems of nonexpansive mappings (see, for example, [27–29]). Those methods are tasking and difficult to compute.

In 2018, Thong and Hieu [30] proposed the following iterative algorithm for the approximation of a common fixed point of a finite family of demicontractive operators. Let $\{x_n\}$ be a sequence in H defined by

$$\begin{aligned} &x_0 \in H, \\ &x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n \sum_{i=1}^{N} w_i T_i x_n. \end{aligned} \tag{14}$$

Among other standard assumptions, they assumed that $\{w_i\}_{i=1}^N$ is a finite sequence of positive numbers such that

$$\sum_{i=1}^{N} w_i = 1. \tag{15}$$

Let K be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $\{T_i\}_{i=1}^N$, $N \geq 2$ is a countable finite family of mappings T_i: $K \longrightarrow K$. In [31], the authors consider the horizontal iteration process generated from an arbitrary x_1 for the finite family of mappings $\{T_i\}_{i=1}^N$, using a finite family of th$\{T^i\}_{i=1}^N$ control sequences $\{\{\alpha_n^i\}_{n=1}^\infty\}_{i=1}^N$ as follows.

For $N = 2$,

$$x_{n+1} = \alpha_n^1 x_n + \left(1 - \alpha_n^1\right)\left[\alpha_n^2 T_1 x_n + \left(1 - \alpha_n^2\right) T_2 x_n\right]. \tag{16}$$

For $N = 3$,

$$\begin{aligned} x_{n+1} = {} & \alpha_n^1 x_n + \left(1 - \alpha_n^1\right)\left[\alpha_n^2 T_1 x_n + \left(1 - \alpha_n^2\right)\left[\alpha_n^3 T_2 x_n \right.\right. \\ & \left.\left. + \left(1 - \alpha_n^3\right) T_3 x_n\right]\right]. \end{aligned} \tag{17}$$

For an arbitrary but finite $N \geq 2$,

$$\begin{aligned} x_{n+1} &= \alpha_n^1 x_n + \left(1 - \alpha_n^1\right)\left[\alpha_n^2 T_1 x_n + \left(1 - \alpha_n^2\right)\left[\alpha_n^3 T_2 x_n \right.\right. \\ &\quad \left.\left. + \left(1 - \alpha_n^3\right)\left[\ldots \alpha_N T_{N-1} x_n + (1 - \alpha_N) T_N x_n \ldots\right]\right]\right] \\ &= \alpha_n^1 x_n + \sum_{i=2}^{N} \alpha_n^i \prod_{j=1}^{i-1}\left(1 - \alpha_n^j\right) T_{i-1} x_n + \prod_{j=1}^{N}\left(1 - \alpha_n^j\right) T_N x_n, \\ &\qquad n \geq 1. \end{aligned} \tag{18}$$

1.1. Question. Is it possible to give an iterative algorithm and obtain a strong convergence result for finding a common element in the set of fixed points of a finite family of demicontractive mappings which also solves equilibrium problems for pseudomonotone bifunctions without imposing the type of condition in (15) on the control sequences?

In this paper, motivated by the works of Anh and Muu [25] and Wangkeeree et al. [26], we propose an iterative algorithm for finding a common element in the set of fixed points of a finite family of demicontractive mappings, which also solves equilibrium problems for pseudomonotone bifunctions and prove a strong convergence result which does not require such condition as in (15) on the control sequences. We further give a numerical experiment to demonstrate the performance of our iterative algorithm.

2. Preliminaries

In the sequel, we shall need the following definitions and lemmas. Let H be a real Hilbert space, and C a nonempty, closed, and convex subset of H. By P_C, we denote the metric projection operator onto C, that is,

$$P_C(x) \in C: \left\|x - P_C(x)\right\| \leq \|x - y\|, \quad \forall y \in C. \tag{19}$$

Lemma 1. *Suppose that C is a nonempty, closed, and convex subset in H. Then, P_C has the following properties:*

$$\|x - y\|^2 \geq \left\|x - P_C(x)\right\|^2 + \left\|y - P_C(x)\right\|^2, \quad \forall x \in H,\ y \in C. \tag{20}$$

Lemma 2 (see [32]). *Let $\{a_k\}$ and $\{b_k\}$ be two nonnegative real sequences satisfying the following conditions:*

$$\begin{aligned} &a_{k+1} \leq a_k + b_k \ \forall k \geq 0, \\ &\sum_{k=1}^{\infty} b_k < +\infty. \end{aligned} \tag{21}$$

Then, $\lim_{k \longrightarrow \infty} a_k$ exists.

Lemma 3 (see [33]). *Let H be a real Hilbert space, C a closed convex subset of H, and let T: $C \longrightarrow C$ be a continuous pseudocontractive mapping, then*

(i) $F(T)$ is closed convex subset of C

(ii) $(I - T)$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \longrightarrow 0$, as $n \longrightarrow \infty$, then $x = T(x)$

Lemma 4 (see [1]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, and let T: $C \longrightarrow C$ be a ρ-strictly pseudononspreading mapping. If $F(T) \neq \emptyset$, then it is closed and convex.*

Lemma 5 (see [1]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, and let $T: C \longrightarrow C$ be a ρ-strictly pseudononspreading mapping. Then, $(I-T)$ is demiclosed at 0.*

Definition 3. Let C be a nonempty closed and convex subset of a Hilbert space E. Let $f: C\times C \longrightarrow \mathbb{R}$ be a bifunction where $f(x,\cdot)$ is a convex function for each $x \in C$. Then, the ϵ-subdifferential (ϵ-diagonal subdifferential) of f at x denoted by $\partial_\epsilon f(x,\cdot)(x)$ is given by

$$\partial_\epsilon f(x,\cdot)(x) = \{w \in H: f(x,y) - f(x,x) + \epsilon \geq \langle w, y-x\rangle, \forall y \in C\}. \tag{22}$$

Lemma 6. *Let $\{\alpha_i\}_{i=1}^N \subseteq \mathbb{R}$ be a countable subset of the set of real numbers $\mathbb{R}$, where $N \geq 2$ is an arbitrary integer. Then, the following holds:*

$$\alpha_1 + \sum_{i=2}^{N} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j) + \prod_{j=1}^{N}(1-\alpha_j) = 1. \tag{23}$$

Proof. This result has been proved in [31], but for the sake of completeness, we present the proof again here. For $N=2$,

$$\begin{aligned}\alpha_1 &+ \sum_{i=2}^{2} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j) + \prod_{j=1}^{2}(1-\alpha_j)\\ &= \alpha_1 + \alpha_2(1-\alpha_1) + (1-\alpha_1)(1-\alpha_2)\\ &= \alpha_1 + (1-\alpha_1)[\alpha_2 + (1-\alpha_2)]\\ &= \alpha_1 + (1-\alpha_1) = 1.\end{aligned} \tag{24}$$

We assume it is true for N and prove for $N+1$.

$$\begin{aligned}\alpha_1 + \sum_{i=2}^{N+1} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j) + \prod_{j=1}^{N+1}(1-\alpha_j) &= \alpha_1 + \sum_{i=2}^{N} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j) + \alpha_{N+1}\prod_{j=1}^{N}(1-\alpha_j) + \prod_{j=1}^{N+1}(1-\alpha_j)\\ &= \alpha_1 + \sum_{i=2}^{N} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j) + \prod_{j=1}^{N}(1-\alpha_j)[\alpha_{N+1} + (1-\alpha_{N+1})]\\ &= \alpha_1 + \sum_{i=2}^{N} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j) + \prod_{j=1}^{N+1}(1-\alpha_j) = 1.\end{aligned} \tag{25}$$

□

Remark 1. Lemma 6 holds if $\{\alpha_i\}_{i=1}^N$ is replaced with $\{\alpha_i\}_{i=0}^N$.

Lemma 7 (see also [31]). *Let t and u be arbitrary elements of a real Hilbert space H, and let $N \in \mathbb{N}$ be such that $2 \leq N$. Let $\{v^i\}_{i=1}^N \subseteq H$ and $\{\alpha_i\}_{i=1}^N \subseteq [0,1]$ be a countable finite subset of H and $\mathbb{R}$, respectively. Define*

$$y = \alpha_1 t + \sum_{i=2}^{N} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j)v_{i-1} + \prod_{j=1}^{N}(1-\alpha_j)v_N. \tag{26}$$

Then,

$$\begin{aligned}\|y-u\|^2 \leq{}& \alpha_1\|t-u\|^2 + \sum_{i=2}^{N} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j)\|v_{i-1}-u\|^2 + \prod_{j=1}^{N}(1-\alpha_j)\|v_N-u\|^2\\ &- \alpha_1\left[\sum_{i=2}^{N} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j)\|t-v_{i-1}\|^2 + \prod_{j=1}^{N}(1-\alpha_j)\|t-v_N\|^2\right].\end{aligned} \tag{27}$$

Proof. Let $w_k = \sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha^j)v_{i-1} + \prod_{j=k}^{N}(1-\alpha_j)v_N$, $k=1,2,\ldots,N-1$ and $w_N = (1-\alpha_N)v_N$. Observe that, for $k \leq N-1$, $w_k = (1-\alpha_k)[\alpha_{k+1}v_k + w_{k+1}]$. Using the well-known identity

$$\|\beta x + (1-\beta)y\|^2 = \beta\|x\|^2 + (1-\beta)\|y\|^2 - \beta(1-\beta)\|x-y\|^2, \tag{28}$$

which holds for all $x, y \in H$ and for all $t \in [0,1]$, we have

$$
\begin{aligned}
\|y-u\|^2 &= \left\|\alpha_1 t + \sum_{i=2}^{N} \alpha_i \prod_{j=1}^{i-1}\left(1-\alpha_j\right) v_{i-1} + \prod_{j=1}^{N}\left(1-\alpha_j\right) v_N - u\right\|^2 \\
&= \left\|\alpha_k t + w_1 - u\right\|^2 \\
&= \left\|\alpha_1 t + (1-\alpha_1)\left[\alpha_2 v_1 + w_2\right] - u]\right\|^2 \\
&= \alpha_1 \|t-u\|^2 + (1-\alpha_1)\left\|\alpha_2 v_1 + w_2 - u\right\|^2 - \alpha_1(1-\alpha_1)\left\|t - \left[\alpha_2 v_1 + w_2\right]\right\|^2 \\
&= \alpha_1 ||t-u||^2 + (1-\alpha_1)\Big[\alpha_2\left\|v_1-u\right\|^2 + (1-\alpha_2)\left\|\alpha_3 v_2 + w_3 - u\right\|^2 - \alpha_2(1-\alpha_2)\left\|v_1 - \left[\alpha_3 v_2 + w_3\right]\right\|^2\Big] \\
&\quad - \alpha_1(1-\alpha_1)\Big[\alpha_2\left\|t-v_1\right\|^2 + (1-\alpha_2)\left\|t - \left[\alpha_3 v_2 + w_3\right]\right\|^2 - \alpha_2(1-\alpha_2)\left\|v_1 - \left[\alpha_3 v_2 + w_3\right]\right\|^2\Big], \\
&= \alpha_1\|t-u\|^2 + (1-\alpha_1)\alpha_2\left\|v_1-u\right\|^2 + (1-\alpha_1)(1-\alpha_2)\left\|\alpha_3 v_2 + w_3 - u\right\|^2 - (1-\alpha_1)\alpha_2(1-\alpha_2)\left\|v_1 - \left[\alpha_3 v_2 + w_3\right]\right\|^2 \\
&\quad - \alpha_1(1-\alpha_1)\alpha_2\left\|t-v_1\right\|^2 - \alpha_1(1-\alpha_1)(1-\alpha_2)\left\|t - \left[\alpha_3 v_2 + w_3\right]\right\|^2 + \alpha_1(1-\alpha_1)\alpha_2(1-\alpha_2)\left\|v_1 - \left[\alpha_3 v_2 + w_3\right]\right\|^2\Big] \\
&= \alpha_1\|t-u\|^2 + (1-\alpha_1)\alpha_2\left\|v_1-u\right\|^2 - \alpha_1(1-\alpha_1)\alpha_2\left\|t-v_1\right\|^2 + (1-\alpha_1)(1-\alpha_2)\left\|\alpha_3 v_2 + w_3 - u\right\|^2 \\
&\quad - \alpha_1(1-\alpha_1)(1-\alpha_2)\left\|t - \left[\alpha_3 v_2 + w_3\right\|^2 - \alpha_2(1-\alpha_1)(1-\alpha_2)(1-\alpha_1)\left\|v_1 - \left[\alpha_3 v_2 + w_3^2\right\|\right., \\
(13) &= \alpha_1\|t-u\|^2 + (1-\alpha_1)\alpha_2\left\|v_1-u\right\|^2 - \alpha_1(1-\alpha_1)\alpha_2\left\|t-v_1\right\|^2 - \alpha_2(1-\alpha_1)(1-\alpha_2)(1-\alpha_1)\left\|v_1 - \left[\alpha_3 v_2 + w_3\right\|^2 \\
&\quad + (1-\alpha_1)(1-\alpha_2)\left\|\alpha_3 v_2 + w_3 - u\right\|^2 - \alpha_1(1-\alpha_1)(1-\alpha_2)\left\|t - \left[\alpha_3 v_2 + w_3\right\|^2 \\
&= \alpha_1\|t-u\|^2 + (1-\alpha_1)\alpha_2\left\|v_1-u\right\|^2 - \alpha_1(1-\alpha_1)\alpha_2\left\|t-v_1\right\|^2 - \alpha_2(1-\alpha_1)(1-\alpha_2)(1-\alpha_1)\left\|v_1 - \left[\alpha_3 v_2 + w_3\right]\right\|^2 \\
&\quad + (1-\alpha_1)(1-\alpha_2)\left\|\alpha_3 v_2 + (1-\alpha_3)\left[\alpha_4 v_3 + w_4\right] - u\right\|^2 - \alpha_1(1-\alpha_1)(1-\alpha_2)\left\|\alpha_3 v_2 + (1-\alpha_3)\left[\alpha_4 v_3 + w_4\right] - t\right\|^2 \\
&= \alpha_1\|t-u\|^2 + (1-\alpha_1)\alpha_2\left\|v_1-u\right\|^2 - \alpha_1(1-\alpha_1)\alpha_2\left\|t-v^k\right\|^2 + (1-\alpha_1)(1-\alpha_2)\alpha_3\left\|v_2-u\right\|^2 \\
&\quad + (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\left\|\alpha_4 v_3 + w_4 - u\right\|^2 - (1-\alpha_1)(1-\alpha_2)\alpha_3(1-\alpha_3)\left\|v_2 - \left[\alpha_4 v_3 + w_4\right]\right\|^2 \\
&\quad - \alpha_1(1-\alpha_1)(1-\alpha_2)\alpha_3\left\|v_2-t\right\|^2 \\
&\quad - \alpha_1(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\left\|\alpha_4 v_3 + w_4 - t\right\|^2 + \alpha_1(1-\alpha_1)(1-\alpha_2)\alpha_3(1-\alpha_3\left\|v_2 - \alpha_4 v_3 + w_4\right\|^2 \\
&\quad - \alpha_2(1-\alpha_1)(1-\alpha_2)(1-\alpha_1)\left\|v_1 - \left[\alpha_3 v_2 + w_3\right]\right\|^2, \\
&= \alpha_1\|t-u\|^2 + (1-\alpha_1)\alpha_2\left\|v_1-u\right\|^2 - \alpha_1(1-\alpha_1)\alpha_2\left\|t-v_1\right\|^2 + (1-\alpha_1)(1-\alpha_2)\alpha_3\left\|v_2-u\right\|^2 \\
&\quad + (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\left\|\alpha_4 v_3 + w_4 - u\right\|^2 - \alpha_1(1-\alpha_1)(1-\alpha_2)\alpha_3\left\|v_2-t\right\|^2 - \alpha_1(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\left\|\alpha_4 v_3 + w_4 - t\right\|^2 \\
&\quad - \alpha_2(1-\alpha_1)(1-\alpha_2)(1-\alpha_1)\left\|v_1 - \left[\alpha_3 v_2 + w_3\right]\right\|^2 - (1-\alpha_1)(1-\alpha_2)\alpha_3(1-\alpha_3)\left\|v_2 - \left[\alpha_4 v_3 + w_4\right]\right\|^2, \\
&= \alpha_1||t-u||^2 + \sum_{i=2}^{3}\alpha_i\prod_{j=1}^{i-1}\left(1-\alpha_j\right)\left\|v_{i-1}-u\right\|^2 - \alpha_1\left[\sum_{i=2}^{3}\alpha_i\prod_{j=1}^{i-1}\left(1-\alpha_j\right)\left\|t-v_{i-1}\right\|^2\right] \\
&\quad - (1-\alpha_1)\left[\sum_{i=2}^{3}\alpha_i\prod_{j=1}^{i}\left(1-\alpha_j\right)\left\|v_{i-1} - \left[\alpha_{i+1}v_i + w_{i+1}\right]\right\|^2\right. \\
&\quad \left. + \prod_{j=1}^{3}\left(1-\alpha_j\right)\left\|\left[\alpha_4 v_3 + w_4\right] - u\right\|^2 - \alpha_1\prod_{j=1}^{3}\left(1-\alpha_j\right)\left\|t - \left[\alpha_4 v_3 + w_4\right]\right\|^2 + \alpha_3\prod_{j=1}^{3}\left(1-\alpha_j\right)\left\|u-v_2\right\|^2\right] \\
&\;\;\vdots \\
&= \alpha_1\|t-u\|^2 + \sum_{i=2}^{N}\alpha_i\prod_{j=1}^{i-1}\left(1-\alpha_j\right)\left\|v_{i-1}-u\right\|^2 + \coprod_{j=1}^{N}\left(1-\alpha_j\right)\left\|v_N-u\right\|^2 - \alpha_1\left[\sum_{i=2}^{N}\alpha_i\prod_{j=1}^{i-1}\left(1-\alpha_j\right)\left\|t-v_{i-1}\right\|^2 + \prod_{j=1}^{N}\left(1-\alpha_j\right)\left\|t-v_N\right\|^2\right] \\
&\quad - (1-\alpha_1)\left[\sum_{i=2}^{N-1}\alpha_i\prod_{j=1}^{i}\left(1-\alpha_j\right)\left\|v_{i-1} - \left[\alpha_{i+1}v_i + w_{i+1}\right]\right\|^2 + \alpha_N\prod_{j-1}^{N}\left(1-\alpha_j\right)\left\|v-v_{N-1}\right\|^2\right].
\end{aligned}
\tag{29}
$$

□

3. Main Results

We now give a strong convergence result for the approximation of a solution of a pseudomonotone equilibrium problem which is also a common fixed point of a finite family of demicontractive mappings.

Let $f\colon C\times C\longrightarrow\mathbb{R}$ be a bifunction that satisfies the following conditions: (B1) f is pseudomonotone on C, $f(x,x)=0$, and $f(x,\cdot)$ is lower semicontinuous for all $x\in C$; (B2) $f(\cdot,y)$ is sequentially weakly upper semicontinuous on C for each fixed point $y\in C$, i.e., if $\{x_n\}\subset C$ is a sequence converging weakly to $x\in C$, then $\limsup_{n\longrightarrow\infty} f(x_n,y)\le f(x,y)$; (B3) $f(x,\cdot)$ is convex and subdifferentiable on E for every fixed $x\in C$; (B4) If $\{x_k\}\subseteq C$ is bounded $\epsilon_k\downarrow 0$ as $k\longrightarrow\infty$, then the sequence $\{y_k\}$ with $y_k\in\partial_{\epsilon_k} f(x_k,\cdot)(x_k)$ is bounded; (B5) f is pseudomonotone on C with respect to every $x\in \mathrm{EP}(C,f)$ and satisfies the strict paramonotonicity property, i.e.,

$$\begin{gathered} x\in \mathrm{EP}(C,f),\\ y\in C,\\ f(y,x)=0\Longrightarrow y\in \mathrm{EP}(C,f). \end{gathered}\tag{30}$$

It has been proved that under the conditions (B1)–(B3), the solution set $\mathrm{EP}(f,C)$ of EP (9) is closed and convex [34].

Algorithm 3

Step 1. Let $\{\lambda_k\},\{\beta_k\}$ and $\{\delta_k\}$ be sequences of nonnegative real numbers and let T_i, $i=1,2,\dots,N$, be ρ_i-demicontractive mappings on C. Suppose that the following conditions are satisfied:

(1) $0<\eta\le\lambda_k<\lambda, \rho=\max_{1\le i\le N}\rho_i$ and $0\le\rho<a<\delta_{k,i}<b<1$, $\lim_{k\longrightarrow\infty}\delta_{k,1}=(1/2)$.
(2) $\beta_k>0$, $\sum_{k=1}^{\infty}\beta_k=+\infty$ and $\sum_{k=1}^{\infty}\beta_k^2<+\infty$.
(3) $\sum_{k=1}^{\infty}\beta_k\epsilon_k<+\infty$.

Step 2

$$\begin{cases} x_1\in C;\\ y_k\in\partial_{\epsilon_k} f(x_k,\cdot)(x_k);\\ \gamma_k:=\max\{\lambda_k,\|y_k\|\} \text{ and } \alpha_k:=\dfrac{\beta_k}{\gamma_k};\\ w_k=P_C(x_k-\alpha_k y_k);\\ x_{k+1}=\delta_{k,1}w_k+\displaystyle\sum_{i=2}^{N}\delta_{k,i}\prod_{j=1}^{i-1}\left(1-\delta_{k,j}\right)T_{i-1}w_k+\prod_{j=1}^{N}\left(1-\delta_{k,j}\right)T_N w_k. \end{cases}\tag{31}$$

Theorem 1. *Let C be a closed and convex subset of a real Hilbert space H, and let $f\colon C\times C\longrightarrow\mathbb{R}$ be a bifunction that satisfies conditions (B1)–(B5). Let $T_i\colon C\longrightarrow C$, $i=1,2,\dots,N$ be a finite family of ρ_i-demicontractive mappings such that $(I-T_i)$ is demiclosed at 0 for each i and $\Omega:=EP(C,f)\cap(\cap_{i=1}^{N}F(T_i))\ne\emptyset$. Then, the sequences $\{x_k\}$ and $\{w_k\}$ generated by Algorithm 3 converges strongly to a point $p\in\Omega$, where $p=\lim_{k\longrightarrow\infty}P_\Omega(x_k)$.*

Proof. Let $x^*\in\Omega$. From $w_k\in P_C(x_k-\alpha_k y_k)$ and $x^*\in C$, we have

$$\langle x_k-\alpha_k y_k-w_k, w_k-x^*\rangle\ge 0.\tag{32}$$

Therefore,

$$\begin{aligned}\langle x^*-w_k, x_k-w_k\rangle &\le \alpha_k\langle y_k, x^*-w_k\rangle\\ &=\alpha_k\langle y_k, x^*-x_k\rangle+\alpha_k\langle y_k, x_k-w_k\rangle\\ &\le \alpha_k\langle y_k, x^*-x_k\rangle+\alpha_k\|y_k\|\|x_k-w_k\|.\end{aligned}\tag{33}$$

Since $x_k\in C$, we also have

$$\langle x_k-\alpha_k y_k-w_k, w_k-x_k\rangle\ge 0.\tag{34}$$

From (33) and (34), we have

$$\begin{aligned}\langle x_k-w_k, x_k-w_k\rangle &=\|x_k-w_k\|^2\\ &\le\alpha_k\langle y_k, x_k-w_k\rangle\\ &\le\alpha_k\|y_k\|\|x_k-w_k\|.\end{aligned}\tag{35}$$

That is,

$$\|x_k-w_k\|\le\alpha_k\|y_k\|.\tag{36}$$

Therefore,

$$\begin{aligned}\alpha_k\|y_k\|\|x_k-w_k\| &\le\left(\alpha_k\|y_k\|\right)^2=\left(\frac{\beta_k\|y_k\|}{\gamma_k}\right)^2\\ &=\beta_k^2\left(\frac{\|y_k\|}{\max\{\lambda_k,\|y_k\|\}}\right)^2\le\beta_k^2.\end{aligned}\tag{37}$$

Moreover, since $x_k\in C$ and $y_k\in\partial_{\epsilon_k} f(x_k,\cdot)(x_k)$, we have

$$\begin{aligned} f(x_k,x^*)+\epsilon_k &= f(x_k,x^*)-f(x_k,x_k)+\epsilon_k\\ &\ge\langle y_k, x^*-x_k\rangle.\end{aligned}\tag{38}$$

It then follows from the definitions of α_k and γ_k that

$$\alpha_k=\frac{\beta_k}{\gamma_k}=\frac{\beta_k}{\max\{\lambda_k,\|y_k\|\}}\le\frac{\beta_k}{\lambda_k}.\tag{39}$$

Thus, from (33)–(39), we obtain

$$\langle x^*-w_k, x_k-w_k\rangle\le\alpha_k f(x_k,x^*)+\frac{\beta_k\epsilon_k}{\lambda_k}+\beta_k^2.\tag{40}$$

But

$$2\langle x^*-w_k, x_k-w_k\rangle=\|w_k-x^*\|^2+\|x_k-w_k\|^2-\|x_k-x^*\|^2.\tag{41}$$

Therefore, from (40) and (41), we get

$$\left\|w_k - x^*\right\|^2 \leq \left\|x_k - x^*\right\|^2 - \left\|x_k - y_k\right\|^2 + 2\alpha_k f\left(x_k, x^*\right) + \frac{2\beta_k \epsilon_k}{\lambda_k} + 2\beta_k^2. \tag{42}$$

Now, from Proposition 2.9, $x^* \in F(T_i)$, and T_i is ρ_i-demicontractive, for $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}
\left\|x_{k+1} - x^*\right\|^2 \leq{}& \delta_{k,1}\left\|w_k - x^*\right\|^2 + \sum_{i=2}^{N}\delta_{k,i}\prod_{j=1}^{i-1}\left(1-\delta_{k,j}\right)\left\|T_{i-1}w_k - x^*\right\|^2 + \prod_{j=1}^{N}\left(1-\delta_{k,j}\right)\left\|T_N w_k - x^*\right\|^2 \\
&- \delta_{k,1}\left[\sum_{i=2}^{N}\delta_{k,i}\prod_{j=1}^{i-1}\left(1-\delta_{k,j}\right)\left\|w_k - T_{i-1}w_k\right\|^2 + \prod_{j=1}^{N}\left(1-\delta_{k,j}\right)\left\|w_k - T_N w_k\right\|^2\right] \\
\leq{}& \delta_{k,1}\left\|w_k - x^*\right\|^2 + \sum_{i=2}^{N}\delta_{k,i}\prod_{j=1}^{i-1}\left(1-\delta_{k,j}\right)\left[\left\|w_k - x^*\right\|^2 + \rho_{i-1}\left\|w_k - T_{i-1}w_k\right\|^2\right] \\
&+ \prod_{j=1}^{N}\left(1-\delta_{k,j}\right)\left[\left\|w_k - x^*\right\| + \rho_N\left\|T_N w_k - w_k\right\|^2\right] \\
&- \delta_{k,1}\left[\sum_{i=2}^{N}\delta_{k,i}\prod_{j=1}^{i-1}\left(1-\delta_{k,j}\right)\left\|w_k - T_{i-1}w_k\right\|^2 + \prod_{j=1}^{N}\left(1-\delta_{k,j}\right)\left\|w_k - T_N w_k\right\|^2\right] \\
\leq{}& \left\|w_k - x^*\right\|^2 - D_k,
\end{aligned} \tag{43}$$

where

$$D_k = \left(\delta_{k,1} - \rho\right)\left[\sum_{i=2}^{N}\delta_{k,i}\prod_{j=1}^{i-1}\left(1-\delta_{k,j}\right)\left\|w_k - T_{i-1}w_k\right\|^2 + \prod_{j=1}^{N}\left(1-\delta_{k,j}\right)\left\|w_k - T_N w_k\right\|^2\right]. \tag{44}$$

From (42) and (43), we have

$$\left\|x_{k+1} - x^*\right\|^2 \leq \left\|x_k - x^*\right\|^2 - \left\|x_k - w_k\right\|^2 + 2\alpha_k f\left(x_k, x^*\right) + \frac{2\beta_n \epsilon_k}{\lambda_k} + 2\beta_k^2 - D_k. \tag{45}$$

Furthermore, since $x^* \in \Omega$, then $f(x^*, x) \geq 0$ for all $x \in C$. Therefore, by the pseudomonotonicity of f with respect to x^*, we have $f(x, x^*) \leq 0$, for all $x \in C$. It then follows from $x_k \in C$ that $f(x_k, x^*) \leq 0$. Hence, we have from (45) that

$$\left\|x_{k+1} - x^*\right\|^2 \leq \left\|x_k - x^*\right\|^2 + \frac{2\beta_k \epsilon_k}{\lambda_k} + 2\beta_k^2. \tag{46}$$

Applying Lemma 2 to (46), we obtain the existence of

$$l := \lim_{k\longrightarrow\infty}\left\|x_k - x^*\right\|. \tag{47}$$

Since f is pseudomonotone on C and $f(x^*, x_k) \geq 0$, we have $-f(x_k, x^*) \geq 0$. Thus, from (45), we obtain

$$\begin{aligned}
0 &\leq D_k + 2\alpha_k\left[-f\left(x_k, x^*\right)\right] \\
&\leq \left\|x_k - x^*\right\|^2 - \left\|x_{k+1} - x^*\right\|^2 + \frac{2\beta_k \epsilon_k}{\lambda_k} + 2\beta_k^2.
\end{aligned} \tag{48}$$

Summing up the inequality (48), for every k, we have

$$\begin{aligned}
0 &\leq \sum_{k=1}^{\infty} D_k + \sum_{k=1}^{\infty} 2\alpha_k\left[-f\left(x_k, x^*\right)\right] \\
&\leq \left\|x_1 - x^*\right\|^2 + 2\sum_{k=1}^{\infty}\frac{\beta_k \epsilon_k}{\lambda_k} + 2\sum \beta_k^2.
\end{aligned} \tag{49}$$

Since the sequences $\{y_k\}$ and $\{\lambda_k\}$ are bounded, then there exists a constant $M > 0$ such that $\max\{\lambda_k, \|y_k\|\} \leq M$. Thus, we have

$$\alpha_k = \frac{\beta_k}{\gamma_k} = \frac{\beta_k}{\max\left\{\lambda_k, \left\|y_k\right\|\right\}} \geq \frac{\beta_k}{M}. \tag{50}$$

Thus, from (49), we obtain

$$\begin{aligned}
0 &\leq \sum_{k=1}^{\infty} D_k + \frac{2}{M}\sum_{k=1}^{\infty}\beta_k\left[-f\left(x_k, x^*\right)\right] \\
&\leq \sum_{k=1}^{\infty} D_k + 2\sum_{k=1}^{\infty}\alpha_k\left[-f\left(x_k, x^*\right)\right] \leq +\infty.
\end{aligned} \tag{51}$$

Thus, $\sum_{k=1}^{\infty} D_k < +\infty$ and $\sum_{k=1}^{\infty}\beta_k[-f(x_k, x^*)] < +\infty$. Since $\sum_{k=1}^{\infty}\beta_k = \infty$ and $-f(x_k, x^*) \geq 0$, we have that

$$\limsup_{k\longrightarrow\infty} f\left(x_k, x^*\right) = 0. \tag{52}$$

For any $x^* \in \Omega$, suppose that $\{x_{k_\tau}\}$ is the subsequence of $\{x_k\}$ such that

$$\limsup_{k \longrightarrow \infty} f(x_k, x^*) = \lim_{\tau \longrightarrow \infty} f(x_{k_\tau}, x^*), \tag{53}$$

and without loss of generality, we may assume that $x_{k_\tau} \rightharpoonup \overline{x}$ as $\tau \longrightarrow \infty$ for some $\overline{x} \in C$.

Next, we show that $\overline{x}$ is in $\mathrm{EP}(C, f)$. Observe that since $f(\cdot, x^*)$ is weakly upper semicontinuous, we have

$$\begin{aligned} f(\overline{x}, x^*) &\geq \limsup_{\tau \longrightarrow \infty} f(x_{k_\tau}, x^*) = \lim_{\tau \longrightarrow \infty} f(x_{k_\tau}, x^*) \\ &= \limsup_{k} f(x_k, x^*) = 0. \end{aligned} \tag{54}$$

But since f is pseudomonotone with respect to x^* and $f(x^*, \overline{x}) \geq 0$, we have

$$f(\overline{x}, x^*) \leq 0. \tag{55}$$

From (54) and (55), we conclude that $f(\overline{x}, x^*) = 0$. Thus, by condition B6, we obtain that $\overline{x}$ is in $\mathrm{EP}(C, f)$.

Since $\sum_{k=1}^{\infty} D_k < +\infty$, we have that

$$\|w_k - T_i w_k\| \longrightarrow 0, k \longrightarrow \infty, \quad i = 1, 2, \ldots, N. \tag{56}$$

Furthermore, with $w_k = P_C(x_k - \alpha_k y_k)$ and $x_k \in C$, we have

$$\begin{aligned} \|x_k - w_k\|^2 &\leq \alpha_k \langle y_k, x^* - w_k \rangle \\ &\leq \alpha_k \|y_k\| \|x_k - w_k\| \\ &= \frac{\beta_k}{\max\{\lambda_k, \|y_k\|\}} \|y_k\| \|x_k - w_k\| \\ &\leq \beta_k \|x_k - w_k\|, \end{aligned} \tag{57}$$

which implies that, and therefore

$$\|x_k - w_k\| \longrightarrow 0, k \longrightarrow \infty. \tag{58}$$

It then follows from $x_{k_\tau} \rightharpoonup \overline{x}$ that $w_{k_\tau} \rightharpoonup \overline{x}$. Since $(I - T_i), i = 1, 2, \ldots, N$ are demiclosed at 0, and $\|T_i w_k - w_k\| \longrightarrow 0, k \longrightarrow \infty, i = 1, 2, \ldots, N$. We have that $\overline{x} \in \cap_{i=1}^{N} F(T_i)$.

Now, observe that

$$\begin{aligned} \|x_{k+1} - w_k\|^2 \leq{} & \delta_{k,1} \|w_k - w_k\|^2 + \sum_{i=2}^{N} \delta_{k,i} \prod_{j=1}^{i-1} (1 - \delta_{k,j}) \\ & \cdot \|T_{i-1} w_k - w_k\|^2 \\ & + \prod_{j=1}^{N} (1 - \delta_{k,j}) \|T_N w_k - w_k\|^2 \longrightarrow 0, \quad k \longrightarrow \infty. \end{aligned} \tag{59}$$

Consequently,

$$\|x_{k+1} - x_k\| \leq \|x_{k+1} - w_k\| + \|w_k - x_k\| \longrightarrow 0, \quad k \longrightarrow \infty. \tag{60}$$

We now show that

$$\lim_{k \longrightarrow \infty} x_k = \lim_{k \longrightarrow \infty} w_k = \lim_{k \longrightarrow \infty} P_\Omega(x_k) = \overline{x}. \tag{61}$$

It follows from (46) that, for all $x^* \in \Omega$,

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 + \xi_k, \tag{62}$$

where $\xi_k := (2\beta_k \epsilon_k / \lambda_k) + 2\beta_k^2 > 0$, for all $k \geq 0$, and $\sum_{k=1}^{\infty} \xi_k < +\infty$.

Now, using Lemma 1, we have

$$\begin{aligned} \|x_{k+1} - P_\Omega(x_{k+1})\|^2 \leq{} & \|x_{k+1} - P_\Omega(w_k)\|^2 \\ \leq{} & \delta_{k,1} \|w_k - P_\Omega(w_k)\|^2 + \sum_{i=2}^{N} \delta_{k,i} \prod_{j=1}^{i-1} (1 - \delta_{k,j}) \|T_{i-1} w_k - P_\Omega(w_k)\|^2 \\ & + \prod_{j=1}^{N} (1 - \delta_{k,j}) \|T_N w_k - P_\Omega(w_k)\|^2 \\ \leq{} & \delta_{k,1} \|w_k - P_\Omega(w_k)\|^2 + \sum_{i=2}^{N} \delta_{k,i} \prod_{j=1}^{i-1} (1 - \delta_{k,j}) \|T_{i-1} w_k - w_k\|^2 \\ & - \sum_{i=2}^{N} \delta_{k,i} \prod_{j=1}^{i-1} (1 - \delta_{k,j}) \|w_k - P_\Omega(w_k)\|^2 \\ & + \prod_{j=1}^{N} (1 - \delta_{k,j}) \|T_N w_k - w_k\|^2 - \prod_{j=1}^{N} (1 - \delta_{k,j}) \|w_k - P_\Omega(w_k)\|^2. \end{aligned} \tag{63}$$

That is,

$$\begin{aligned}\left\|x_{k+1}-P_\Omega(x_{k+1})\right\|^2 &\le \delta_{k,1}\left\|w_k-P_\Omega(w_k)\right\|^2+\sum_{i=2}^{N}\delta_{k,i}\prod_{j=1}^{i-1}\left(1-\delta_{k,j}\right)\left\|T_{i-1}w_k-w_k\right\|^2\\ &\quad+\prod_{j=1}^{N}\left(1-\delta_{k,j}\right)\left\|T_Nw_k-w_k\right\|^2-\left(1-\delta_{k,1}\right)\left\|w_k-P_\Omega(w_k)\right\|^2\\ &\le\left(2\delta_{k,1}-1\right)\left\|w_k-P_\Omega(w_k)\right\|^2+\sum_{i=2}^{N}\delta_{k,i}\prod_{j=1}^{i-1}\left(1-\delta_{k,j}\right)\left\|T_{i-1}w_k-w_k\right\|^2\\ &\quad+\prod_{j=1}^{N}\left(1-\delta_{k,j}\right)\left\|T_Nw_k-w_k\right\|^2\longrightarrow 0,\quad k\longrightarrow\infty.\end{aligned}\tag{64}$$

For all $m>k$, since Ω is convex, we have $(1/2)(P_\Omega(x_m)+P_\Omega(x_k))\in\Omega$. Therefore,

$$\begin{aligned}\left\|P_\Omega(x_m)-P_\Omega(x_k)\right\|^2&=2\left\|x_m-P_\Omega(x_m)\right\|^2+2\left\|x_m-P_\Omega(x_k)\right\|^2-4\left\|x_m-\frac{1}{2}\left(P_\Omega(x_m)+P_\Omega(x_k)\right)\right\|^2\\ &\le 2\left\|x_m-P_\Omega(x_m)\right\|^2+2\left\|x_m-P_\Omega(x_k)\right\|^2-4\left\|x_m-P_\Omega(x_m)\right\|^2\\ &=2\left\|x_m-P_\Omega(x_k)\right\|^2-2\left\|x_m-P_\Omega(x_m)\right\|^2.\end{aligned}\tag{65}$$

Replacing x^* with $P_\Omega(x_k)$ in (62), we have that

$$\begin{aligned}\left\|x_m-P_\Omega(x_k)\right\|^2&\le\left\|x_{m-1}-P_\Omega(x_k)\right\|^2+\xi_{m-1}\\ &\le\left\|x_{m-2}-P_\Omega(x_k)\right\|^2+\xi_{m-1}+\xi_{m-2}\\ &\;\;\vdots\\ &\le\left\|x_k-P_\Omega(x_k)\right\|^2+\sum_{j=k}^{m-1}\xi_j.\end{aligned}\tag{66}$$

From (65) and (66), we have

$$\left\|P_\Omega(x_m)-P_\Omega(x_k)\right\|^2\le\left\|x_k-P_\Omega(x_k)\right\|^2-2\left\|x_m-P_\Omega(x_m)\right\|^2+\sum_{j=k}^{m-1}\xi_j.\tag{67}$$

Hence,

$$\lim_{m\longrightarrow\infty,k\longrightarrow\infty}\left\|P_\Omega(x_m)-P_\Omega(x_k)\right\|^2=0,\tag{68}$$

which implies that $\{P_\Omega(x_k)\}$ is a Cauchy sequence. Hence, $\{P_\Omega(x_k)\}$ strongly converges to some point $\bar{z}\in\Omega$. However, $\tau\longrightarrow\infty$, we obtain in the limit that

$$\bar{z}=\lim_{\tau\longrightarrow\infty}P_\Omega\left(x_{k_\tau}\right)=P_\Omega(\bar{x})=\bar{x}\in\Omega.\tag{69}$$

Thus, from (64), we conclude that $x_k\longrightarrow\infty$, and consequently $w_k\longrightarrow\bar{x}$. □

Corollary 1. *Let C be a closed and convex subset of a real Hilbert space H, and let $f\colon C\times C\longrightarrow\mathbb{R}$ be a bifunction that satisfies conditions (B1)–(B5). Let $T_i\colon C\longrightarrow C, i=1,2,\ldots,N$ be a finite family of ρ_i-strictly pseudononspreading mappings, such that $\Omega:=EP(C,f)\cap(\cap_{i=1}^{N}F(T_i))\neq\emptyset$. Then, the sequences $\{x_k\}$ and $\{w_k\}$ generated by Algorithm 3 converge strongly to a point $p\in\Omega$, where $p=\lim_{k\longrightarrow\infty}P_\Omega(x_k)$.*

Corollary 2. *Let C be a closed and convex subset of a real Hilbert space H and let $f\colon C\times C\longrightarrow\mathbb{R}$ be a bifunction that satisfies conditions (B1)–(B5). Let $T_i\colon C\longrightarrow C, i=1,2,\ldots,N$ be a finite family of ρ_i-strictly pseudocontractive mappings, such that $\Omega:=EP(C,f)\cap(\cap_{i=1}^{N}F(T_i))\neq\emptyset$. Then, the sequences $\{x_k\}$ and $\{w_k\}$ generated by Algorithm 3 converge strongly to a point $p\in\Omega$, where $p=\lim_{k\longrightarrow\infty}P_\Omega(x_k)$.*

4. Numerical Example

We now give a numerical example to demonstrate the performance and efficiency of our proposed iterative algorithm. Let $H=\mathbb{R}$ and $C=[-3,1]$. Set $f(x,y)=2y-2x$ and define $T_i\colon C\longrightarrow C$, $i=1,2,\ldots,N$ by

$$T_i(x)=\begin{cases}x, & \text{if } x\in[-3,0];\\ \dfrac{-3ix}{1+i}, & \text{if } x\in[0,1].\end{cases}\tag{70}$$

Observe that $\Omega:=\mathrm{EP}(C,f)\cap(\cap_{i=1}^{N}F(T_i))=\{-3\}$. Moreover, $\rho_i=(2i-1/1+4i)$, and thus letting $N=5$, we

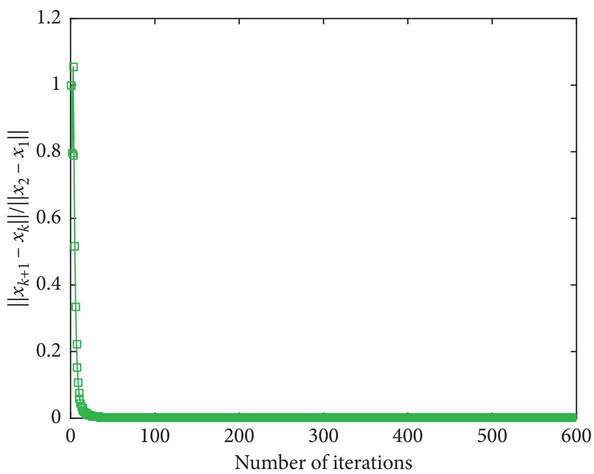

FIGURE 1: $x_1 = -1$.

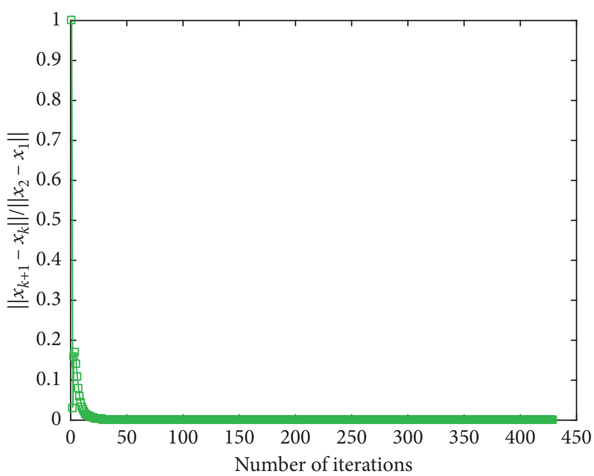

FIGURE 2: $x_1 = 0$.

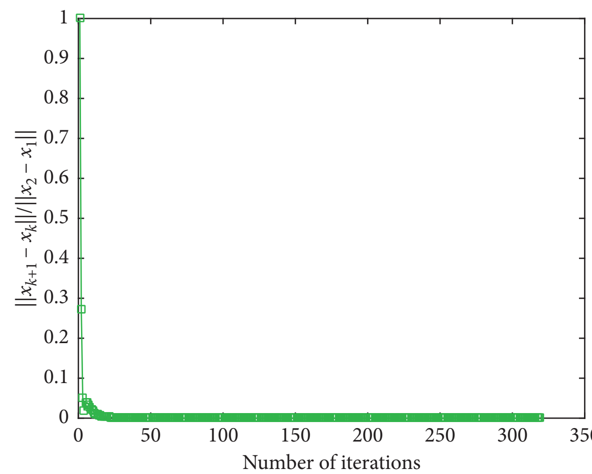

FIGURE 3: $x_1 = 1$.

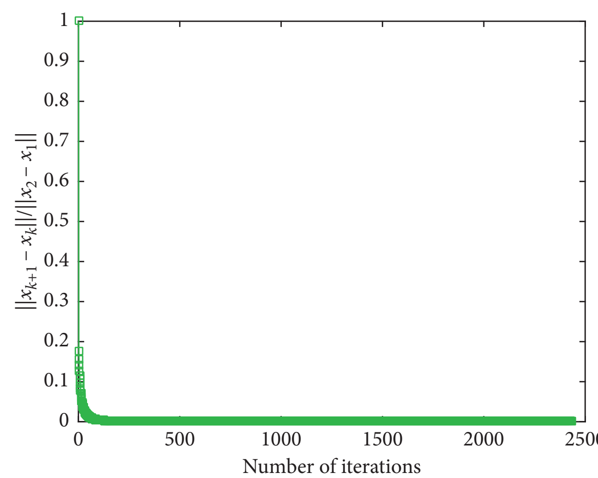

FIGURE 4: $x_1 = -3$.

have that $\rho = \max_{1\leq i\leq 5} \rho_i = (3/7)$. Take $\lambda_k = 1, \epsilon_k = 0, \beta_k = (1/k)$ and $\delta_{k,i} = (k + 4i/2k + 7i)$. Therefore, (32) is expressed as an iteration:

$$\begin{cases} x_1 \in C; \\ w_k = \begin{cases} -3 & \text{if } x_k - \beta_k < -3; \\ x_k - \beta_k & \text{if } x_k - \beta_k \in [-3, 1]; \\ 1, & \text{if } x_k - \beta_k > 1, \end{cases} \\ x_{k+1} = \delta_{k,1} w_k + \sum_{i=2}^{N} \delta_{k,i} \prod_{j=1}^{i-1} \left(1 - \delta_{k,j}\right) T_{i-1} w_k + \prod_{j=1}^{N} \left(1 - \delta_{k,j}\right) T_N w_k. \end{cases} \tag{71}$$

We make different choices of x_1 and use $(\|x_{n+1} - x_n\|/\|x_2 - x_1\|) < 0.00001$ for stopping criterion. Figures 1–4 are the graphs of the numerical computations of Algorithm 4.2 (71) corresponding, respectively, to $x_1 = -1$, $x_1 = 0$, $x_1 = 1$, and $x_1 = -3$.

Data Availability

The data used to support the findings of this study are included within the article.

Disclosure

Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the NRF.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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