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Research Article

Solution of Integral Differential Equations by New Double Integral Transform (Laplace–Sumudu Transform)

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The primary purpose of this research is to demonstrate an efficient replacement double transform named the Laplace–Sumudu transform (DLST) to unravel integral differential equations. The theorems handling fashionable properties of the Laplace–Sumudu transform are proved; the convolution theorem with an evidence is mentioned; then, via the usage of these outcomes, the solution of integral differential equations is built.

1. Introduction

Double integral transform and their characteristics and theories are nevertheless new and below studies [1–3], in which the preceding research treated some components of them along with definitions, simple theories, and the answer of normal and partial differential equations [4–16]; additionally, some researchers addressed these transforms and combine them with exclusive mathematical method such as differential transform approach, homotopy perturbation technique, Adomian decomposition method, and variational iteration method [7–16] so that we can solve the linear and nonlinear fractional differential equations.

In this paper, we are ready to spotlight the way during which the Laplace–Sumudu transform is blend to solve the integral differential equations.

A wide range of linear integral differential equations are considered which include the Volterra integral equation (Section 3.1), the Volterra integro-partial differential equation (Section 3.2), and the partial integro-differential equation (Section 3.3).

Definition 1. The double Laplace–Sumudu transform of the function $\phi(x, t)$ of two variables $x > 0$ and $t > 0$ is denoted by $L_x S_t[\phi(x, t)] = \overline{\phi}(\rho, \sigma)$ and defined as

$$L_x S_t[\phi(x, t)] = \overline{\phi}(\rho, \sigma) = \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x - t/\sigma}\phi(x, t)dxdt. \quad (1)$$

Clearly, double Laplace–Sumudu transform is a linear integral transformation as shown below:

$$\begin{aligned}
&L_x S_t[\gamma\phi(x, t) + \eta\varphi(x, t)] \\
&\quad = \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x - t/\sigma}[\gamma\phi(x, t) + \eta\varphi(x, t)]dxdt \\
&\quad = \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x - t/\sigma}\gamma\phi(x, t)dxdt \\
&\qquad + \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x - t/\sigma}\eta\varphi(x, t)dxdt \\
&\quad = \frac{\gamma}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x - t/\sigma}\phi(x, t)dxdt \\
&\qquad + \frac{\eta}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x - t/\sigma}\varphi(x, t)dxdt \\
&\quad = \gamma L_x S_t[\phi(x, t)] + \eta L_x S_t[\varphi(x, t)],
\end{aligned} \quad (2)$$

where γ and η are constants.

Definition 2. The inverse double Laplace–Sumudu transform $L_x^{-1}S_t^{-1}[\overline{\phi}(\rho,\sigma)] = \phi(x,t)$ is defined by the following form:

$$L_x^{-1}S_t^{-1}\left[\overline{\phi}(\rho,\sigma)\right] = \phi(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\rho x}\,d\rho \cdot \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{1}{\sigma} e^{t/\sigma}\,\overline{\phi}(\rho,\sigma)\,d\sigma. \quad (3)$$

2. Double Laplace–Sumudu Transform of Basic Functions

(1) Let $\phi(x,t) = 1$, $x > 0$, $t > 0$, then

$$\begin{aligned}\overline{\phi}(\rho,\sigma) = L_xS_t[1] &= \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x - t/\sigma}dxdt \\ &= \int_0^\infty e^{-\rho x}dx\int_0^\infty \frac{1}{\sigma}e^{-t/\sigma}dt = \frac{1}{\rho}.\end{aligned} \quad (4)$$

(2) Let $\phi(x,t) = x^c t^d$, $x > 0$, $t > 0$, then

$$\begin{aligned}\overline{\phi}(\rho,\sigma) = L_xS_t\left[x^ct^d\right] &= \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x - t/\sigma}x^ct^d dxdt \\ &= \int_0^\infty e^{-\rho x}x^c dx\int_0^\infty \frac{1}{\sigma}e^{-t/\sigma}t^d dt \\ &= \frac{\Gamma[c+1]}{\rho^{c+1}}\sigma^d\Gamma[d+1], \operatorname{Re}[c] > -1, \operatorname{Re}[d] > -1.\end{aligned} \quad (5)$$

If c and d are positive integral, then

$$L_xS_t\left[x^ct^d\right] = \frac{c!d!}{\rho^{c+1}}\sigma^d. \quad (6)$$

(3) Let $\phi(x,t) = e^{cx+dt}$, then

$$\begin{aligned}\overline{\phi}(\rho,\sigma) = L_xS_t\left[e^{cx+dt}\right] &= \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x - t/\sigma}e^{cx+dt}dxdt \\ &= dx\int_0^\infty \frac{1}{\sigma}e^{-(t/\sigma - d)t}dt = \frac{1}{(\rho-c)(1-d\sigma)}.\end{aligned} \quad (7)$$

Similarly,

$$\begin{aligned}L_xS_t\left[e^{i(cx+dt)}\right] &= \frac{1}{(\rho-ic)(1-id\sigma)} \\ &= \frac{(\rho-cd\sigma)+i(c+d\sigma\rho)}{(\rho^2+c^2)(1+d^2\sigma^2)}.\end{aligned} \quad (8)$$

Consequently,

$$\begin{aligned}L_xS_t[\sin(cx+dt)] &= \frac{c+d\sigma\rho}{(\rho^2+c^2)(1+d^2\sigma^2)}, \\ L_xS_t[\cos(cx+dt)] &= \frac{\rho-cd\sigma}{(\rho^2+c^2)(1+d^2\sigma^2)}.\end{aligned} \quad (9)$$

(4) Let $\phi(x,t) = \sinh(cx+dt)$ or $\cosh(cx+dt)$. Recall that

$$\sinh(cx+dt) = \frac{e^{cx+dt}-e^{-(cx+dt)}}{2}, \cosh(cx+dt) = \frac{e^{cx+dt}+e^{-(cx+dt)}}{2}. \quad (10)$$

Therefore,

$$\begin{aligned}L_xS_t[\sinh(cx+dt)] &= \frac{c+d\sigma\rho}{(\rho^2-c^2)(1-d^2\sigma^2)}, \\ L_xS_t[\cosh(cx+dt)] &= \frac{\rho+cd\sigma}{(\rho^2-c^2)(1-d^2\sigma^2)}.\end{aligned} \quad (11)$$

(5) Let $\phi(x,t) = J_0(c\sqrt{xt})$, then

$$\begin{aligned}L_xS_t\left[J_0\left(c\sqrt{xt}\right)\right] &= \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x - t/\sigma}J_0\left(c\sqrt{xt}\right)dxdt \\ &= \int_0^\infty \frac{1}{\sigma}e^{-t/\sigma}dt\int_0^\infty e^{-\rho x}J_0\left(c\sqrt{xt}\right)dx \\ &= \frac{1}{\rho}S\left[e^{-c^2t/4\rho}\right] = \frac{4}{4\rho+\sigma c^2},\end{aligned} \quad (12)$$

where $J_0(x)$ is the modified Bessel function of order zero.
(6) Let $\phi(x,t) = f(x)\,g(t)$, then

$$\begin{aligned}\overline{\phi}(\rho,\sigma) = L_xS_t[f(x)g(t)] &= \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x - t/\sigma}f(x)g(t)dxdt \\ &= \left[\int_0^\infty e^{-\rho x}f(x)dx\right]\left[\int_0^\infty \frac{1}{\sigma}e^{-t/\sigma}g(t)dt\right] \\ &= L_x[f(x)]S_t[g(t)].\end{aligned} \quad (13)$$

2.1. Existence Condition for the Double Laplace–Sumudu Transform. If $\phi(x,t)$ is an exponential order, then c and d as $x \to \infty$, $t \to \infty$., and if $\exists$ a positive constant K such that $\forall x > X$, $t > T$, then

$$|\phi(x,t)| = Ke^{cx+dt}, \quad (14)$$

and we write $\phi(x,t) = Oe^{cx+dt}$ as $x \to \infty$, $t \to \infty$. Or, equivalently,

$$\lim_{x\to\infty,t\to\infty} e^{-\rho x-t/\sigma}|\phi(x,t)| = K \lim_{x\to\infty,t\to\infty} e^{-(\rho-c)x-(t/\sigma-d)t} = 0, \rho > c, \frac{1}{\sigma} > d. \tag{15}$$

The function $\phi(x,t)$ is called an exponential order as $x \to \infty, t \to \infty$, and clearly, it does not grow faster than Ke^{cx+dt} as $x \to \infty, t \to \infty$.

Theorem 3. *If a function $\phi(x,t)$ is a continuous function in every finite interval $(0,X)$ and $(0,T)$ of exponential order e^{cx+dt}, then the double Laplace–Sumudu transform of $\phi(x,t)$ exists for all ρ and $1/\sigma$ provided* Re $[\rho] > c$ *and* Re $[1/\sigma] > d$.

Proof. From the Definition 1., we have

$$\begin{aligned}|\overline{\phi}(\rho,\sigma)| &= \left|\frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x-t/\sigma}\phi(x,t)dxdt\right| \\ &\le K\int_0^\infty e^{-(\rho-c)x}dx\int_0^\infty \frac{1}{\sigma}e^{-(t/\sigma-d)t}dt \\ &= \frac{K}{(\rho-c)(1-d\sigma)}, \mathrm{Re}\,[\rho] > c, \mathrm{Re}\left[\frac{1}{\sigma}\right] > d.\end{aligned} \tag{16}$$

Then, from Eq. (16) we have $\lim_{x\to\infty,t\to\infty}|\overline{\phi}(\rho,\sigma)| = 0$, or $\lim_{x\to\infty,t\to\infty}\overline{\phi}(\rho,\sigma) = 0$.

2.2. Basic Derivative Properties of the Double Laplace–Sumudu Transform. If $\overline{\phi}(\rho,\sigma) = L_xS_t[\phi(x,t)]$, then

$$\text{(I)}\quad L_xS_t\left[\frac{\partial\phi(x,t)}{\partial x}\right] = \rho\overline{\phi}(\rho,\sigma) - S[\phi(0,t)]. \tag{17}$$

Proof.

$$\begin{aligned}L_xS_t\left[\frac{\partial\phi(x,t)}{\partial x}\right] &= \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x-t/\sigma}\frac{\partial\phi(x,t)}{\partial x}dxdt \\ &= \frac{1}{\sigma}\int_0^\infty e^{-t/\sigma}dt\int_0^\infty e^{-\rho x}\frac{\partial\phi(x,t)}{\partial x}dx.\end{aligned} \tag{18}$$

Using integration by parts, let $u = e^{-\rho x}$, $dv = ((\partial\phi(x,t))/\partial x)dx$, then

$$\begin{aligned}L_xS_t\left[\frac{\partial\phi(x,t)}{\partial x}\right] &= \frac{1}{\sigma}\int_0^\infty e^{-t/\sigma}dt\left\{-\phi(0,t) + \rho\int_0^\infty e^{-\rho x}\phi(x,t)dx\right\} \\ &= \rho\overline{\phi}(\rho,\sigma) - S(\phi(0,t)).\end{aligned} \tag{19}$$

$$\text{(II)}\quad L_xS_t\left[\frac{\partial\phi(x,t)}{\partial t}\right] = \frac{1}{\sigma}\overline{\phi}(\rho,\sigma) - \frac{1}{\sigma}L(\phi(x,0)). \tag{20}$$

Proof.

$$\begin{aligned}L_xS_t\left[\frac{\partial\phi(x,t)}{\partial t}\right] &= \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x-t/\sigma}\frac{\partial\phi(x,t)}{\partial t}dxdt \\ &= \frac{1}{\sigma}\int_0^\infty e^{-\rho x}dx\int_0^\infty e^{-\frac{t}{\sigma}}\frac{\partial\phi(x,t)}{\partial t}dt.\end{aligned} \tag{21}$$

Using integration by parts, let $u = e^{-t/\sigma}$, $dv = ((\partial\phi(x,t))/\partial t)dt$, then $L_xS_t[\partial\phi(x,t)/\partial t] = 1/\sigma\int_0^\infty e^{-\rho x}dx\{-\phi(x,0) + 1/\sigma\int_0^\infty e^{-t/\sigma}\phi(x,t)dt\} = 1/\sigma\overline{\phi}(\rho,\sigma) - 1/\sigma L(\phi(x,0))$.

Similarly, we can prove

$$\begin{aligned}&L_xS_t\left[\frac{\partial^2\phi(x,t)}{\partial x^2}\right] = \rho^2\overline{\phi}(\rho,\sigma) - \rho S(\phi(0,t)) - S(\phi_x(0,t)). \\ &\text{(III)}\quad L_xS_t\left[\frac{\partial^2\phi(x,t)}{\partial t^2}\right] = \frac{1}{\sigma^2}\overline{\phi}(\rho,\sigma) - \frac{1}{\sigma^2}L(\phi(x,0)) - \frac{1}{\sigma}L(\phi_t(x,0)). \\ &L_xS_t\left[\frac{\partial^2\phi(x,t)}{\partial x\partial t}\right] = \frac{\rho}{\sigma}\overline{\phi}(\rho,\sigma) - \frac{\rho}{\sigma}L(\phi(x,0))\end{aligned} \tag{22}$$

Theorem 4. *If $\overline{\phi}(\rho,\sigma) = L_xS_t[\phi(x,t)]$, then*

$$L_xS_t[\phi(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] = e^{-\rho\delta-\varepsilon/\sigma}\overline{\phi}(\rho,\sigma), \tag{23}$$

where $H(x,t)$ is the Heaviside unit step function defined by

$$H(x-\delta,t-\varepsilon) = \left\{\begin{matrix}1, & x > \delta, t > \varepsilon \\ 0, & \text{otherwise}\end{matrix}\right\}. \tag{24}$$

Proof. We have, by Definition 1.,

$$\begin{aligned}&L_xS_t[\phi(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] \\ &\quad= \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x-t/\sigma}\phi(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)dxdt \\ &\quad= \frac{1}{\sigma}\int_\delta^\infty\int_\varepsilon^\infty e^{-\rho x-t/\sigma}\phi(x-\delta,t-\varepsilon)dxd,\end{aligned} \tag{25}$$

that is, by putting $x-\delta = q$, $t-\varepsilon = w$

$$\begin{aligned}&= e^{-\rho\delta-\varepsilon/\sigma}\frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho q-w/\sigma}\phi(q,w)\,dq\,dw \\ &= e^{-\rho\delta-\varepsilon/\sigma}\overline{\phi}(\rho,\sigma).\end{aligned} \tag{26}$$

2.3. Convolution Theorem of Double Laplace–Sumudu Transform

Definition 5. The convolution of $\phi(x,t)$ and $\psi(x,t)$ is denoted by $(\phi**\psi)(x,t)$ and defined by

$$(\phi**\psi)(x,t)=\int_0^x\int_0^t\phi(x-\delta,t-\varepsilon)\psi(\delta,\varepsilon)d\delta d\varepsilon. \tag{27}$$

Theorem 6. *(convolution theorem) If $L_xS_t[\phi(x,t)]=\overline{\phi}(\rho,\sigma)$ and $L_xS_t[\psi(x,t)]=\bar{\psi}(\rho,\sigma)$, then*

$$L_xS_t[(\phi**\psi)(x,t)]=\sigma\overline{\phi}(\rho,\sigma)\bar{\psi}(\rho,\sigma). \tag{28}$$

Proof. From the definition 1., we have

$$\begin{aligned}L_xS_t&[(\phi**\psi)(x,t)]\\&=\frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x-t/\sigma}(\phi**\psi)(x,t)dxdt\\&=\frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x-t/\sigma}\left\{\int_0^x\int_0^t\phi(x-\delta,t-\varepsilon)\psi(\delta,\varepsilon)d\delta d\varepsilon\right\}dxdt,\end{aligned} \tag{29}$$

which is, using the Heaviside unit step function,

$$\begin{aligned}&=\frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x-t/\sigma}\left\{\int_0^\infty\int_0^\infty\phi(x-\delta,t-\varepsilon)H\right.\\&\quad\left.\cdot(x-\delta,t-\varepsilon)\psi(\delta,\varepsilon)d\delta d\varepsilon\right\}dxdt\\&=\int_0^\infty\int_0^\infty\psi(\delta,\varepsilon)d\delta d\varepsilon\left\{\frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x-t/\sigma}\phi(x-\delta,t-\varepsilon)H\right.\\&\quad\left.\cdot(x-\delta,t-\varepsilon)dxdt\right\},\end{aligned} \tag{30}$$

that is, by Theorem 4 gives

$$\begin{aligned}&=\int_0^\infty\int_0^\infty\psi(\delta,\varepsilon)d\delta d\varepsilon\left\{e^{-\rho\delta-\varepsilon/\sigma}\overline{\phi}(\rho,\sigma)\right\}\\&=\overline{\phi}(\rho,\sigma)\int_0^\infty\int_0^\infty e^{-\rho\delta-\varepsilon/\sigma}\psi(\delta,\varepsilon)d\delta d\varepsilon\\&=\sigma\overline{\phi}(\rho,\sigma)\bar{\psi}(\rho,\sigma).\end{aligned} \tag{31}$$

3. Application of Laplace–Sumudu Transform (DLST) of Integral Differential Equations

In this section, we apply the double Laplace–Sumudu transform (DLST) method to linear integral differential equations.

3.1. Volterra Integral Equation. Consider the linear Volterra integral equation as form

$$\phi(x,t)=g(x,t)+\lambda\int_0^x\int_0^t\phi(x-\delta,t-\varepsilon)\psi(\delta,\varepsilon)d\delta d\varepsilon, \tag{32}$$

where $\phi(x,t)$ is the unknown function, λ is a constant, and $g(x,t)$ and $\psi(x,t)$ are two known functions. Applying the double Laplace–Sumudu transform (DLST) with linearity to both sides of equation (32) and using Theorem 6 (convolution theorem), we get

$$\overline{\phi}(\rho,\sigma)=\bar{g}(\rho,\sigma)+\lambda\sigma\overline{\phi}(\rho,\sigma)\bar{\psi}(\rho,\sigma). \tag{33}$$

Consequently,

$$\overline{\phi}(\rho,\sigma)=\frac{\bar{g}(\rho,\sigma)}{1-\lambda\sigma\bar{\psi}(\rho,\sigma)}. \tag{34}$$

Taking $L_x^{-1}S_t^{-1}[\overline{\phi}(\rho,\sigma)]$ for equation (34), we obtain the solution $\phi(x,t)$ of equation (32).

$$\phi(x,t)=L_x^{-1}S_t^{-1}\left[\frac{\bar{g}(\rho,\sigma)}{1-\lambda\sigma\bar{\psi}(\rho,\sigma)}\right]. \tag{35}$$

We illustrate the above method by simple examples.

(a) Solve the equation

$$\phi(x,t)=a-\lambda\int_0^x\int_0^t\phi(\delta,\varepsilon)d\delta d\varepsilon. \tag{36}$$

where a and λ are constant.

Applying the double Laplace–Sumudu transform (DLST) of equation (36), we get

$$\overline{\phi}(\rho,\sigma)=\frac{a}{\rho}-\frac{\lambda\sigma}{\rho}\overline{\phi}(\rho,\sigma). \tag{37}$$

Consequently,

$$\overline{\phi}(\rho,\sigma)=\frac{a}{\rho+\lambda\sigma}. \tag{38}$$

Taking $L_x^{-1}S_t^{-1}$ for equation (38), we obtain the solution $\phi(x,t)$ of equation (36).

$$\phi(x,t)=L_x^{-1}S_t^{-1}\left[\frac{a}{\rho+\lambda\sigma}\right]=aJ_0\left(2\sqrt{\lambda xt}\right). \tag{39}$$

(b) Solve the equation

$$a^2t=\int_0^x\int_0^t\phi(x-\delta,t-\varepsilon)\phi(\delta,\varepsilon)d\delta d\varepsilon, \tag{40}$$

where a is a constant.

Applying (DLST) of equation (40), we get

$$\frac{a^2\sigma}{\rho}=\sigma\overline{\phi}^2(\rho,\sigma). \tag{41}$$

Or

$$\overline{\phi}(\rho,\sigma) = \frac{a}{\sqrt{\rho}}. \tag{42}$$

Taking $L_x^{-1}S_t^{-1}$ for equation (42), we obtain the solution $\phi(x,t)$ of equation (40).

$$\phi(x,t) = L_x^{-1}S_t^{-1}\left[\frac{a}{\sqrt{\rho}}\right] = \frac{a}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{x}}. \tag{43}$$

(c) Solve the equation

$$\int_0^x\int_0^t e^{\delta-\varepsilon}\phi(x-\delta,t-\varepsilon)d\delta d\varepsilon = xe^{x-t} - xe^x. \tag{44}$$

Applying (DLST) of equation (44), we get

$$\frac{\sigma\,\overline{\phi}(\rho,\sigma)}{(\rho-1)(1+\sigma)} = \frac{1}{(\rho-1)^2(1+\sigma)} - \frac{1}{(\rho-1)^2}. \tag{45}$$

Simplifying and taking $L_x^{-1}S_t^{-1}$ for equation (45), we obtain

$$\phi(x,t) = L_x^{-1}S_t^{-1}\left[\frac{-1}{(\rho-1)}\right] = -e^x. \tag{46}$$

3.2. Volterra Integro-Partial Differential Equations. Consider the linear Volterra integro-partial differential equation as form

$$\begin{aligned}&\frac{\partial\phi(x,t)}{\partial x} + \frac{\partial\phi(x,t)}{\partial t}\\ &\quad= g(x,t) + \lambda\int_0^x\int_0^t \phi(x-\delta,t-\varepsilon)\psi(\delta,\varepsilon)d\delta d\varepsilon,\end{aligned} \tag{47}$$

with the conditions:

$$\phi(x,0) = f_0(x),\ \phi(0,t) = h_0(t), \tag{48}$$

where $\phi(x,t)$ is the unknown function, λ is a constant, and $g(x,t)$ and $\psi(x,t)$ are two known functions.

Applying (DLST) to both sides of (47) and single (LT) and (ST) for equation (48) and simplification, we get

$$\overline{\phi}(\rho,\sigma) = \frac{\overline{f_0}(\rho) + \sigma\overline{h_0}(\sigma) + \sigma\overline{g}(\rho,\sigma)}{\left(1+\sigma\rho-\lambda\sigma^2\overline{\psi}(\rho,\sigma)\right)}. \tag{49}$$

Applying $L_x^{-1}S_t^{-1}$ to (49), we obtain the solution of (47) in the form

$$\begin{aligned}\phi(x,t) = L_x^{-1}S_t^{-1}\Bigg[&\frac{1}{\left(1+\sigma\rho-\lambda\sigma^2\overline{\psi}(\rho,\sigma)\right)}\\ &\cdot\left[\overline{f_0}(\rho) + \sigma\overline{h_0}(\sigma) + \sigma\overline{g}(\rho,\sigma)\right]\Bigg].\end{aligned} \tag{50}$$

We illustrate the above method by a simple example.

(d) Solve the equation

By substituting $\psi(\delta,\varepsilon) = 1$, $\lambda = 1$, $g(x,t) = -1 + e^x + e^t + e^{x+t}$ in (47), we have got

$$\begin{aligned}\frac{\partial\phi(x,t)}{\partial x} + \frac{\partial\phi(x,t)}{\partial t} &= -1 + e^x + e^t + e^{x+t}\\ &\quad+ \int_0^x\int_0^t \phi(x-\delta,t-\varepsilon)\,d\delta\,d\varepsilon,\end{aligned} \tag{51}$$

with the conditions:

$$\phi(x,0) = e^x = f_0(x),\ \phi(0,t) = e^t = h_0(t). \tag{52}$$

Substituting

$$\begin{aligned}\overline{f_0}(\rho) &= \frac{1}{\rho-1},\ \overline{h_0}(\sigma) = \frac{1}{1-\sigma},\ \overline{g}(\rho,\sigma)\\ &= \frac{-1}{\rho} + \frac{1}{\rho-1} + \frac{1}{\rho(1-\sigma)} + \frac{1}{(\rho-1)(1-\sigma)}\end{aligned} \tag{53}$$

in (50) and simplifying, we get the solution of (51)

$$\phi(x,t) = L_x^{-1}S_t^{-1}\left[\frac{1}{(\rho-1)(1-\sigma)}\right] = e^{x+t}. \tag{54}$$

3.3. Partial Integro-Differential Equation. Consider the linear partial integro-differential equation as form

$$\begin{aligned}&\frac{\partial^2\phi(x,t)}{\partial t^2} - \frac{\partial^2\phi(x,t)}{\partial x^2} + \phi(x,t)\\ &\quad+ \int_0^x\int_0^t \psi(x-\delta,t-\varepsilon)\,\phi(\delta,\varepsilon)\,d\delta\,d\varepsilon = g(x,t),\end{aligned} \tag{55}$$

with the conditions:

$$\begin{aligned}\phi(x,0) &= f_0(x),\ \frac{\partial\phi(x,0)}{\partial t} = f_1(x),\ \phi(0,t)\\ &= h_0(t),\ \frac{\partial\phi(0,t)}{\partial x} = h_1(t).\end{aligned} \tag{56}$$

Applying (DLST) to both sides of (55) and single (LT) and (ST) for equation (56) and simplification, we get

$$\overline{\phi}(\rho,\sigma) = \frac{\overline{f_0}(\rho) + \sigma\overline{f_1}(\rho) - \sigma^2\rho\,\overline{h_0}(\sigma) - \sigma^2\,\overline{h_1}(\sigma) + \sigma^2\,\overline{g}(\rho,\sigma)}{\left(1-\sigma^2\rho^2+\sigma^2+\sigma^3\,\overline{\psi}(\rho,\sigma)\right)}. \tag{57}$$

Applying $L_x^{-1}S_t^{-1}$ to (57), we obtain the solution of (55) in the form

$$\phi(x,t)=L_x^{-1}S_t^{-1}\left[\frac{\overline{f_0}(\rho)+\sigma\overline{f_1}(\rho)-\sigma^2\rho\,\overline{h_0}(\sigma)-\sigma^2\,\overline{h_1}(\sigma)+\sigma^2\,\bar{g}(\rho,\sigma)}{(1-\sigma^2\rho^2+\sigma^2+\sigma^3\,\bar{\psi}(\rho,\sigma))}\right]. \tag{58}$$

We illustrate the above method by a simple example.

(e) Solve the equation:

By substituting $\psi(x-\delta,t-\varepsilon)=e^{x-\delta+t-\varepsilon}$, $g(x,t)=e^{x+t}+xte^{x+t}$ in (55), we have got

$$\frac{\partial^2\phi(x,t)}{\partial t^2}-\frac{\partial^2\phi(x,t)}{\partial x^2}+\phi(x,t)+\int_0^x\int_0^t e^{x-\delta+t-\varepsilon}\,\phi(\delta,\varepsilon)\,d\delta\,d\varepsilon=e^{x+t}+x\,t\,e^{x+t}, \tag{59}$$

with the conditions:

$$\phi(x,0)=e^x=f_0(x),\ \frac{\partial\phi(x,0)}{\partial t}=e^x=f_1(x),\ \phi(0,t)=e^t=h_0(t),\ \frac{\partial\phi(0,t)}{\partial x}=e^t=h_1(t). \tag{60}$$

Substituting

$$\overline{f_0}(\rho)=\overline{f_1}(\rho)=\frac{1}{\rho-1},\ \overline{h_0}(\sigma)=\overline{h_1}(\sigma)=\frac{1}{1-\sigma},\ \bar{g}(\rho,\sigma)=\frac{1}{(\rho-1)(1-\sigma)}+\frac{\sigma}{(\rho-1)^2(1-\sigma)^2} \tag{61}$$

in (58) and simplifying, we get a solution of (59)

$$\phi(x,t)=L_x^{-1}S_t^{-1}\left[\frac{1}{(\rho-1)(1-\sigma)}\right]=e^{x+t}. \tag{62}$$

4. Conclusion

In this paper, the Laplace–Sumudu transform approach for solving integral differential equations is studied. We provided the theorems and popular properties for this new double transform and furnished some examples. The examples show that the Laplace–Sumudu transform approach is powerful in solving the equations of taken into consideration type, and a couple of advanced problems in linear and nonlinear partial differential equations and nonlinear integral differential equations could be discussed during a later paper.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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