

Research Article

WEB-Spline Finite Elements for the Approximation of Navier-Lamé System with $C_{A,B}$ Boundary Condition

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The objective of this article is to discuss the existence and the uniqueness of a weighted extended B-spline- (WEB-spline-) based discrete solution for the 2D Navier-Lamé equation of linear elasticity with a different type of mixed boundary condition called $C_{A,B}$ boundary condition. Along with the usual weak mixed formulation, we give existence and uniqueness results for weak solution. Then, we illustrate the performance of Ritz–Galerkin schemes for a model problem and applications in linear elasticity. Finally, we discuss several implementation aspects. The numerical tests confirm that, due to the new integration routines, the weighted B-spline solvers have become considerably more efficient.

1. Introduction

The finite element method has become the method of choice for solving many types of partial differential equations in engineering and physical sciences. Important applications include structural mechanics, fluid flow, thermodynamics, and electromagnetic fields [1], using the approximation of Lagrange [2]. This type of approximation has experienced a great restriction in the level of the geometric domain, especially in the case of complicated boundaries such as that in the form of curvilinear graphs. A new way of approximation came a few years ago called B-spline that will solve this problem with excellence.

B-splines have become standard tools in approximation, computer graphics, design, and manufacturing. Recently, they have also been used to construct basis functions for finite element methods. The resulting techniques combine the computational efficiency of regular grids with the geometric flexibility of classical finite elements. Some key advantages are the free choice of order and smoothness, a simple data structure with one parameter per grid point, and the exact representation of boundary conditions. A type of B-spline is called weighted extended B-spline (WEB-spline)

[3]. As will be described in the next section, a weight function, ω , is being multiplied to the B-spline functions to ensure that all WEB-splines satisfy the boundary conditions exactly. The books [4, 5] provide a comprehensive treatment of the relevant theory. Which technique is best suited for a particular problem depends to some extent on the topological form of the simulation domain and its representation. Weighted methods are a good choice for problems with natural (Neumann) boundary conditions or if the part of the boundary, where essential (Dirichlet) boundary conditions are prescribed, has a convenient implicit description. Isogeometric methods can handle domains well which are parametrized over rectangles or cuboids or which can be expressed as union of few such parametrizations, e.g., NURBS representations for CAD/CAM applications. There are also problems where a combination of both techniques might be useful [6].

In this project, we use the implementation of weighted finite element methods to solve the Navier-Lamé system with a different type of mixed boundary condition $C_{A,B}$ [7] that generalizes the well-known basis conditions, especially the Dirichlet and the Neumann conditions. This boundary condition helps us to implement a single MATLAB code that summarizes any kind of boundary conditions that we can

meet. At the programming level of the method, we can reap several programs in one. The outline of this article is as follows. In the next section, we introduce the modeling of the Navier-Lamé equation and we will give the variational problem that corresponds to this equation. We review the preliminary aspects on WEB-spline-based methods for Navier-Lamé equations. In Section 3, we prove inf-sup conditions for the variational problem in two-dimensional case using WEB-spline-based mesh-free method, which are necessary for the well posedness of the problem.

2. Governing Equation

Linear elasticity is the mathematical study of how each point of the solid object is displaced. The latter produces a deformation while the object becomes subjected to internal stresses due to the prescribed loading conditions, knowing that the linear elasticity models materials in a continuous state. Linear elasticity is a simple case of the nonlinear elasticity theory and is a branch of continuous domain mechanics. The basic theory of linear elasticity is based on the following: the strains (or stress) are infinitesimal or small and the relationships between the components of the stress and the strain are linear, and the linear elasticity is valid only for the states of stress which do not produce a yield. These assumptions cited above are reasonable for the analysis of many engineering materials and in technical design. It is often that this analysis is done using the finite element method.

Let us consider $\Omega \subset \mathbb{R}^2$ to be a bounded Lipschitz domain with boundary condition Γ which will be presented in a new form that generalizes the Neumann and Dirichlet boundary conditions. Given $f \in L^2(\Omega)$, $A, B \in L^\infty(\Gamma)^{2 \times 2}$, two matricial functions that are invertible or zero. $g \in H^1(\Gamma)$ as well as the positive parameters λ and μ . When μ the shear modulus is small, then this kind of problems is called singularly perturbed problems, where the uniform mesh or L2 norm-based error analysis does not converge to the original problem, and hence, one must use adaptive mesh with uniform error analysis. This numerical analysis is given in the papers [8–14].

When solid objects are subjected to external or internal loads, they deform and lead to stress. If the deformation of the solid is relatively small, linear relationships between the components of stress and strain are maintained. Consequently, linear elasticity theory is valid. In practice, linear elasticity theory is applicable to a wide range of natural and engineering materials and thus extensively used in structural analysis and engineering design. The equation of Navier-Lamé below is governed as follows.

A solid object is deformed under the action of forces applied. Any fixed point (x, y) of the domain and when surface forces are applied to the domain, this point moves to another point (X, Y) . Then, the vector $u = (u_1, u_2) = (X - x, Y - y)$ is called displacement. When the movement is small and the solid is elastic, then Hooke's law gives a relationship between the stress tensor and the strain tensor. $\sigma = \lambda \text{tr}(\varepsilon) I_2 + 2\mu \varepsilon$ is the stress tensor, $\varepsilon = 1/2(\nabla u + (\nabla u)^T)$ is the strain tensor, I_2 is the identity matrix, and μ is the shear modulus

(or rigidity), where λ is Lam's first parameter. The Navier-Lamé equation is given by the law of conservation moment $\rho a = \nabla \cdot \sigma$ where a is the acceleration and ρ is the density of the material. On the other hand,

$$\nabla \cdot \sigma = \lambda \nabla(\text{tr}(\varepsilon) I_2) + 2\mu \nabla \cdot \varepsilon. \quad (1)$$

Then, we have

$$\nabla \cdot \sigma = \lambda \nabla(\text{tr}(\varepsilon) I_2) + \mu \nabla(\nabla u) + \mu \nabla(\nabla u)^t. \quad (2)$$

With a simple calculation, we find that

$$\nabla(\text{tr}(\varepsilon) I_2) = \nabla(\nabla u)^t = \nabla(\nabla \cdot u). \quad (3)$$

Then, we get

$$\rho a = \mu \Delta u + (\lambda + \mu) \nabla(\nabla \cdot u). \quad (4)$$

If the solid is in dynamic equilibrium, then we have $\rho a + f = 0$; f are the external forces applied to the solid. Finally, we find out the equation

$$f = -\mu \Delta u - (\lambda + \mu) \nabla(\nabla \cdot u). \quad (5)$$

We refer the reader to [15, 16] for more information of the elasticity problems.

We create a new unknown $\psi = \nabla \cdot u = (\partial u_1 / \partial x) + (\partial u_2 / \partial y)$ that is equal to the divergence of the displacement. The equation of Navier-Lamé becomes

$$\begin{cases} -\mu \Delta u - (\lambda + \mu) \nabla \psi = f \text{ in } \Omega, \\ \psi - \nabla \cdot u = 0 \text{ in } \Omega, \\ Au + B \left(\mu \frac{\partial u}{\partial n} + \lambda \nabla \cdot u n \right) = g \text{ on } \Gamma. \end{cases} \quad (6)$$

Our mathematical model is the Navier-Lamé system with the boundary condition noted $C_{A,B}$ such that A is called the Dirichlet matrix and B is the Neumann matrix.

There are two strictly positive constants α and β , such that

$$\alpha u \cdot v < B^{-1} A u \cdot v < \beta u \cdot v \forall u, \quad v \in \mathbb{R}^2. \quad (7)$$

$\|\cdot\|$ is a matrix norm that will be defined below.

If $\|A\| \ll \|B\|$, then $C_{A,B}$ is the Neumann boundary; and if $\|B\| \ll \|A\|$, then $C_{A,B}$ is the Dirichlet boundary.

We need functional spaces and norms

$$h^1(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \setminus u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^2(\Omega) \right\}, \quad (8)$$

$$V(\Omega) = H^1(\Omega) = \{h^1(\Omega)\}^2, \quad (9)$$

$$M(\Omega) = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \mid \int_{\Omega} q = 0 \right\}, \quad (10)$$

$$\|v\|_{1,\Omega} = \left\{ \int_{\Omega} \nabla v : \nabla v d\Omega + \int_{\Omega} v \cdot v d\Omega \right\}^{1/2}, \quad (11)$$

$$\|v\|_{0,\Omega} = \left\{ \int_{\Omega} v \cdot v d\Omega \right\}^{1/2}, \quad (12)$$

$$\| |A| \| = \max |a_{i,j}|, \quad i = 1,2, j = 1,2. \quad (13)$$

The variational formulation of the Navier-Lamé problem (6) is as follows.

Find $(u, \psi) \in V(\Omega) \times M(\Omega)$ such that

$$\begin{cases} \int_{\Omega} \mu \nabla u : \nabla v d\Omega + \int_{\Gamma} B^{-1} Au \cdot v d\Gamma, \\ + \int_{\Gamma} \mu \psi n \cdot v d\Gamma + \int_{\Omega} (\lambda + \mu) \psi \nabla \cdot v d\Omega, \\ = \int_{\Omega} f \cdot v d\Omega + \int_{\Gamma} B^{-1} g \cdot v d\Gamma, \\ \int_{\Omega} (\lambda + \mu) q \nabla \cdot u d\Omega - \int_{\Omega} (\lambda + \mu) \psi q d\Omega = 0. \end{cases} \quad (14)$$

The weak formulation (14) may be restated as follows.

Find $(u, \psi) \in V(\Omega) \times M(\Omega)$

$$\begin{cases} a(u, v) + b_{\Gamma}(v, \psi) = L(v) \forall v \in V(\Omega), \\ b(u, q) - d(\psi, q) = 0 \forall q \in M(\Omega). \end{cases} \quad (15)$$

With the bilinear forms

$$\begin{cases} a(u, v) = \int_{\Omega} \mu \nabla u : \nabla v d\Omega + \int_{\Gamma} B^{-1} Au \cdot v d\Gamma, \\ b(v, q) = \int_{\Omega} (\lambda + \mu) q \nabla \cdot v d\Omega, \\ b_{\Gamma}(v, q) = b(v, q) + \int_{\Gamma} \mu q n \cdot v d\Gamma, \\ d(\psi, q) = \int_{\Omega} (\lambda + \mu) \psi q d\Omega, \\ L(v) = \int_{\Omega} f \cdot v d\Omega + \int_{\Gamma} B^{-1} g \cdot v d\Gamma. \end{cases} \quad (16)$$

3. Solvability of the Generalized Saddle Point System

In this section, we introduce some existing saddle point theory. Let $V_0(\Omega)$ and $M_0(\Omega)$ be two finite- or infinite-dimensional Hilbert spaces equipped with the inner products $(\cdot, \cdot)_{V_0(\Omega)}$ and $(\cdot, \cdot)_{M_0(\Omega)}$ and the induced norms $\|\cdot\|_{1,\Omega}$ and $\|\cdot\|_{0,\Omega}$, respectively. Let $a(u, v)$, $b_1(v, q)$ and $b_2(v, q)$, $d(p, q)$ be bilinear forms on $V_0(\Omega) \times V_0(\Omega)$, $V_0(\Omega) \times M_0(\Omega)$ and $V_0(\Omega) \times M_0(\Omega)$, $M_0(\Omega) \times M_0(\Omega)$, respectively, which are bounded; i.e., there are positive constants $\|a\|$, $\|b_1\|$, $\|b_2\|$, and $\|d\|$ such that

$$\begin{aligned} |a(u, v)| &\leq \|a\| \|u\|_{1,\Omega} \|v\|_{1,\Omega} \forall (u, v) \in V_0(\Omega) \times V_0(\Omega), \\ |b_1(u, q)| &\leq \|b_1\| \|u\|_{1,\Omega} \|q\|_{0,\Omega} \forall (u, q) \in V_0(\Omega) \times M_0(\Omega), \\ |b_2(u, q)| &\leq \|b_2\| \|u\|_{1,\Omega} \|q\|_{0,\Omega} \forall (u, q) \in V_0(\Omega) \times M_0(\Omega), \\ |d(q, p)| &\leq \|d\| \|q\|_{0,\Omega} \|p\|_{0,\Omega} \forall (p, q) \in M_0(\Omega) \times M_0(\Omega), \\ |L(v)| &\leq \|L\| \|v\|_{1,\Omega} \forall v \in V_0(\Omega), \\ |G(q)| &\leq \|G\| \|q\|_{0,\Omega} \forall q \in M_0(\Omega). \end{aligned} \quad (17)$$

Furthermore, we define the following Hilbert spaces:

$$\begin{aligned} B_i &= \{u \in V_0(\Omega) \mid b_i(u, q) = 0 \forall q \in M_0(\Omega)\}, \quad \text{for } i = 1, 2, \\ D &= \{p \in M_0(\Omega) \mid d(p, q) = 0 \forall q \in M_0(\Omega)\}. \end{aligned} \quad (18)$$

In addition to assumption (17), we assume that

$$a(v, v) \geq \delta \|v\|_{1,\Omega}^2, \quad \text{for all } v \in V_0(\Omega), \quad (19)$$

b_i , for $i = 1, 2$, satisfying the inf-sup condition such that there exists a constant $\rho > 0$

$$\begin{aligned} \sup_{\|q\|=1} b_i(v, q) &\geq \rho_i \|v\|_{1,\Omega} \forall v \in B_i^{\perp}, \\ \sup_{v \in B_i^{\perp}} b_i(v, q) &> 0 \forall q \in M_0(\Omega). \end{aligned} \quad (20)$$

The bilinear form d satisfies the weak coerciveness such as there exists a constant $\varepsilon > 0$ such that

$$\sup_{q \in M_0(\Omega)} d(p, q) \geq \varepsilon \|p\|_{0,\Omega}^2 \forall p \in M_0(\Omega). \quad (21)$$

Find $(u, \psi) \in V_0(\Omega) \times M_0(\Omega)$

$$\begin{cases} a(u, v) + b_1(v, \psi) = L(v) \forall v \in V_0(\Omega), \\ b_2(u, q) - d(\psi, q) = G(q) \forall q \in M_0(\Omega), \end{cases} \quad (22)$$

where a , b_i for $i = 1, 2$ and d are bounded bilinear forms, where f and g are bounded linear functionals on $V_0(\Omega)$ and $M_0(\Omega)$, respectively. Reference [17] contains an existence and uniqueness, for the saddle point problem.

Find $(u, \psi) \in V_0(\Omega) \times M_0(\Omega)$ such that

$$\begin{cases} a(u, v) + b_1(v, \psi) = L(v) \forall v \in V_0(\Omega), \\ b_2(u, q) = G(q) \forall q \in M_0(\Omega). \end{cases} \quad (23)$$

Lemma 1. Let D be a linear operator, defined by

$$\begin{aligned} \mathbf{D} &: M_0(\Omega) \longrightarrow D^\perp, \\ \psi &\longmapsto d_\psi, \end{aligned} \tag{24}$$

such that $d(\psi, q) = (d_\psi, q)$ for all $q \in D^\perp$. Therefore, \mathbf{D} is continuous and we have $\text{Im}\mathbf{D} = D^\perp$.

Proof. $\text{Im}\mathbf{D}$ is closed in D^\perp . In fact, let $(\psi_j)_j$ be a sequence of $M_0(\Omega)$ such as $d_{\psi_j} \longrightarrow w$ on D^\perp ; then, $(d_{\psi_j})_j$ is cauchy on D^\perp .

By definition of the coercivity of the bilinear form d , we have

$$\begin{aligned} \|\psi_j - \psi_k\|_{0,\Omega} &\leq \frac{1}{\varepsilon} \sup_{q \in M_0(\Omega)} \frac{d(\psi_j - \psi_k, q)}{\|q\|_{0,\Omega}} = \frac{1}{\varepsilon} \sup_{q \in D^\perp} \frac{d(\psi_j - \psi_k, q)}{\|q\|_{0,\Omega}} \\ &= \frac{1}{\varepsilon} \|d_{\psi_j - \psi_k}\|_{0,\Omega} = \frac{1}{\varepsilon} \|d_{\psi_j} - d_{\psi_k}\|_{0,\Omega}. \end{aligned} \tag{25}$$

We deduce that ψ_j is cauchy in $M_0(\Omega)$; then, there exist $\psi \in M_0(\Omega)$ such as $\psi_j \longrightarrow \psi$ on $M_0(\Omega)$.

By the continuity of the operator \mathbf{D} , we have $d_{\psi_j} \longrightarrow d_\psi$. The uniqueness of the limit gives $w = d_\psi \in D^\perp$, so we obtain that $\text{Im}\mathbf{D}$ is a closed set of D^\perp ; it gives that $D^\perp = \text{Im}\mathbf{D} \oplus (\text{Im}\mathbf{D})^\perp$ and $\text{Im}\mathbf{D} \cap (\text{Im}\mathbf{D})^\perp = 0$.

Suppose that $(\text{Im}\mathbf{D})^\perp \neq \{0\}$; this implies that there exists $p \in (\text{Im}\mathbf{D})^\perp$ nonzero. On the other hand, we have for all $q \in M_0(\Omega)$ $d(q, p) = (d_q, p) = 0$ (because $d_q \in \text{Im}\mathbf{D}$); this implies that $p \in D$ but $p \in (\text{Im}\mathbf{D})^\perp \subset D^\perp$; this contradicts the fact that p is nonzero. Finally, $\text{Im}\mathbf{D} = D^\perp$.

Theorem 2. With assumptions (17)–(21), the generalized variational problem (22) has a unique solution $(u, \psi) \in V_0(\Omega) \times M_0(\Omega)$ for any $L \in (V_0(\Omega))'$ and $G \in (M_0(\Omega))'$ as long as

$$\delta_1 = (\delta\varepsilon)^{-1} \|b_1\| \|b_2\| < 1. \tag{26}$$

Further, the following stability estimates hold:

$$\begin{aligned} \|u\|_{1,\Omega} &\leq \frac{\delta_1 + 1}{1 - \delta_1} \|u^0\|_{1,\Omega} + \frac{\delta^{-1} \|b_1\|}{1 - \delta_1} \|\psi^0\|_{0,\Omega}, \\ \|\psi\|_{0,\Omega} &\leq \frac{\varepsilon^{-1} \|b_2\|}{1 - \delta_1} (2\|u^0\|_{1,\Omega} + \delta^{-1} \|b_1\| \|\psi^0\|_{0,\Omega}), \end{aligned} \tag{27}$$

where (u^0, ψ^0) is the solution to (23) and thus has the bounds

$$\begin{aligned} \|u^0\|_{1,\Omega} &\leq c_{11} \|L\| + c_{12} \|G\|, \\ \|\psi^0\|_{1,\Omega} &\leq c_{21} \|L\| + c_{22} \|G\|, \\ c_{11} &= \delta^{-1}, c_{12} = \rho_2^{-1} (\delta^{-1} \|a\| + 1), \\ c_{21} &= \rho_2^{-1} (c_{11} \|a\| + 1), c_{22} = \rho_2^{-1} \|a\| c_{12}. \end{aligned} \tag{28}$$

Proof. D is a closed set included in $M_0(\Omega)$, so D is a Hilbert space.

Problem (22) became

$$\begin{cases} a(u, v) + b_1(v, \psi) = L(v) \forall v \in V_0(\Omega), \\ b_2(u, q) = G(q) \forall q \in D. \end{cases} \tag{29}$$

a , b_1 , and b_2 satisfy the conditions of theorem 3.1 of the article in [17]; then, problem (29) has a unique solution $(u, \psi_0) \in V(\Omega) \times D$ (if D is reduced to $\{0\}$, just take $\psi_0 = 0$). We are now looking for $\psi \in M_0(\Omega)$ such that

$$d(\psi, q) = -G(q) + b_2(u, q) \forall q \in M_0(\Omega). \tag{30}$$

For this u , the displacement solution of (29) is fixed; we consider the following linear form defined by $F : q \mapsto -G(q) + b_2(u, q)$. This form inherits its continuity from the continuity of b_2 and G . We define the space $D^\perp = \{p \in M_0(\Omega) \mid (p, q) = 0 \forall q \in D\}$ such that D^\perp is a Hilbert space with the norm and inner product inherited from $M_0(\Omega)$. Problem (30) is equivalent to the problem

$$d(\psi, q) = F(q) \forall q \in D^\perp. \tag{31}$$

According to the Lax Milgram theorem, problem (31) admits a unique solution $\psi \in D^\perp$. Assume that (31) is verified, let us show that (30) is verified.

D is closed in $M_0(\Omega)$, so we have $M_0(\Omega) = D \oplus D^\perp$; then, we obtain for all $q \in M_0(\Omega)$. Therefore, we have $q = q_1 + q_2$ for $q_1 \in D$ and $q_2 \in D^\perp$; it implies that $d(\psi, q) = d(\psi, q_1) + d(\psi, q_2) = 0 + F(q_2) = F(q_1 + F(q_2))$.

Conversely, let $\psi \in M_0(\Omega)$ be the solution of (30).

The linear form $q \mapsto d(\psi, q)$ is continuous on D^\perp that is a Hilbert space with the scalar product on $M_0(\Omega)$.

From the Riesz theorem, there exists a unique $d_\psi \in D^\perp$ such that

$$\begin{aligned} d(\psi, q) &= (d_\psi, q) \forall q \in D^\perp, \\ \|d_\psi\|_{0,\Omega} &= \|d(\psi, \cdot)\|_{(D^\perp)'} = \sup_{q \in D^\perp} \frac{d(\psi, q)}{\|q\|_{0,\Omega}}. \end{aligned} \tag{32}$$

This defines an operator $D : \psi \mapsto d_\psi$ which is linear and continuous on $M_0(\Omega)$ such that $\text{Im}D = \{d_\psi \mid \psi \in M_0(\Omega)\}$. From Lemma (1), we have $\text{Im}D = D^\perp$.

If $F \in (D^\perp)'$, Riesz implies that there exists $w \in D^\perp$ such that $F(q) = (w, q)_{M_0(\Omega)}$ for all $q \in D^\perp$. Since $\text{Im}D = D^\perp$, for this $w \in D^\perp$, there exists $\psi \in M_0(\Omega)$ such that $w = d_\psi$ $d(\psi, q) = d(d_\psi, q) = (w, q) = F(q)$ for all $q \in D^\perp$.

Finally, it has been shown that problems (30) and (31) are equivalent.

Lax Milgram guarantees that problem (31) admits a unique solution $\psi_1 \in D^\perp$. Just take $\psi = \psi_0 + \psi_1 \in M_0(\Omega) = D \oplus D^\perp$.

Theorem 3. *With assumptions (17)–(21), the generalized variational problem (15) has a unique solution $(u, \psi) \in V_0(\Omega) \times M_0(\Omega)$ for any $L \in (V_0(\Omega))'$ as long as*

$$\delta_1 := (\delta\varepsilon)^{-1} \|b_\Gamma\| \|b\| < 1. \tag{33}$$

Further, the following stability estimates hold:

$$\begin{aligned} \|u\|_{1,\Omega} &\leq \frac{\delta_1 + 1}{1 - \delta_1} \|u^0\|_{1,\Omega} + \frac{\delta^{-1} \|b_\Gamma\|}{1 - \delta_1} \|\psi^0\|_{0,\Omega}, \\ \|\psi\|_{0,\Omega} &\leq \frac{\varepsilon^{-1} \|b\|}{1 - \delta_1} (2\|u^0\|_{1,\Omega} + \delta^{-1} \|b_\Gamma\| \|\psi^0\|_{0,\Omega}), \end{aligned} \tag{34}$$

where (u^0, ψ^0) is the solution to (23) and thus has the bounds

$$\begin{aligned} \|u^0\|_{1,\Omega} &\leq c_{11} \|L\|, \\ \|\psi^0\|_{1,\Omega} &\leq c_{21} \|L\|, \\ c_{11} &= \delta^{-1}, \\ c_{21} &= \rho_2^{-1} (c_{11} \|a\| + 1). \end{aligned} \tag{35}$$

Proof. It is sufficient to show that the bilinear forms a, b, b_Γ , and d have checked the conditions (17)–(21), applying Theorem 2 in the case of $G = 0$.

Remark 4. We rewrite the system (15) in a single equation. For that, let us define the operators $A : V \rightarrow V^*$, $B : V \rightarrow M^*$, $B_\Gamma : V \rightarrow M^*$, and $D : M \rightarrow M^*$ by $\langle Au, v \rangle := a(u, v)$, $v \in V(\Omega)$, $\langle Bu, \psi \rangle := b(u, \psi)$, $\psi \in M(\Omega)$, $\langle B_\Gamma u, \psi \rangle := b_\Gamma(u, \psi)$, $\psi \in M(\Omega)$, and $\langle D\psi, q \rangle := d(\psi, q)$, $q \in M(\Omega)$ so that (15) becomes

$$\mathcal{L}U := \begin{pmatrix} A & B_\Gamma \\ B^T & -D \end{pmatrix} \begin{pmatrix} u \\ \psi \end{pmatrix} = \begin{pmatrix} L \\ 0 \end{pmatrix} =: F. \tag{36}$$

The well posedness [17–19] of the above equation is written in the following form. For the variational problem (15), the mapping $\mathcal{L} : V \times M \rightarrow X^* \times M^*$ defines an isomorphism if and only if a, b, b_Γ , and d satisfy assumptions (17)–(21).

4. WEB-Spline Process

Several types of mesh-less approximations were proposed for various applications. A central problem of the mesh-less Galerkin method is to incorporate boundary conditions of Dirichlet type. The weighted extended B-spline approximation takes care of not only the boundary constraints but also the issue of well conditioning of the Galerkin systems [3, 20]. This essential new feature of constructing well-conditioned

basis is of cardinal importance in this work. We are interested in applying the approximation properties of the WEB-spaces for discretizing the Navier-Lamé equations as it have been done for Stokes and Navier-Stokes problems [21, 22].

The standard uniform B-spline of degree n is defined by the recursion [5]

$$B^n(x) = \frac{x}{n} B^{n-1}(x) + \frac{n+1-x}{n} B^{n-1}(x-1), \tag{37}$$

starting from B^0 , the characteristic function of the unit interval between zero and one. Figures 1–3 show the uniform B-splines of degrees one, two, and three. These are also known as linear, quadratic, and cubic B-splines, respectively. In this paper, we use the following notational conventions [5]. For functions f and g , we write $f \leq c g$ if $f \leq c g$ with a constant c which does not depend on the grid width h , indices, or arguments of functions. The symbols \geq and \approx are defined analogously. We describe the procedure of constructing the WEB-splines and discuss the approximating properties of the web-space as a finite element space. For $k = (k_0, k_1) \in \mathbb{Z}^2$, and $h > 0$ define the scaled translates:

$$B_k(x) := \frac{1}{h} B\left(\frac{x}{h} - k_0\right) B\left(\frac{y}{h} - k_1\right). \tag{38}$$

B is the univariate B-spline of degree n of support $[0, n + 1]$; the B-splines B_k are polynomials by pieces on the h -grid with vertices $h\mathbb{Z}^2$ and scaled so that the L^2 -norm, $\|B_k\| = \|B_0\|_{0,\Omega}$, is independent of h .

Knowing that the tensor product B-spline is the extension of B-spline to higher dimensions.

In general, it can be defined as follows:

$$B_{k,h}^n = \prod_{v=1}^m B_{k_v,h}^{n_v}, \tag{39}$$

such that the B-spline $B_{k,h}^n$ is an m -variant of degree n_v in the v th variable, index $k = k_1, \dots, k_m$ and the width of the grid h .

With the convention that $n_1 = \dots = n_m$ unless otherwise indicated, for the sake of the problem, the element n can be taken as an integer, rather than an integer vector.

For $I \in \{0, \dots, n\}^m$, we partition the grid cells $Q_I = I h + [0, 1]^m h$ with $m = 2$ into interior, boundary, and exterior cells, depending on whether $Q \subset \bar{\Omega}$, the interior of Q intersects $\partial\Omega$, or $Q \cap \Omega = \emptyset$.

For $k \in K := \{I \in \mathbb{Z}^2 : \text{supp}(B_I) \cap \Omega \neq \emptyset\}$ (the relevant index set for Ω), if $\text{supp}(B_k)$ has at least one grid cell completely inside Ω , then bk is named as inner B-spline; otherwise, it is outer.

The corresponding subsets of K are I and J (c.f Figure 4): $K = I \cup J$.

More details about the construction of WEB-spline basis are given in [5].

Now, it is tempting to use $B_h := \text{span}\{B_k : k \in K\}$ as a finite element approximation space. At first sight, this does not seem feasible since B-splines do not conform to the boundary conditions. But this difficulty can be resolved easily by multiplying B_k by a smoothed distance function $\omega(x) \sim di$

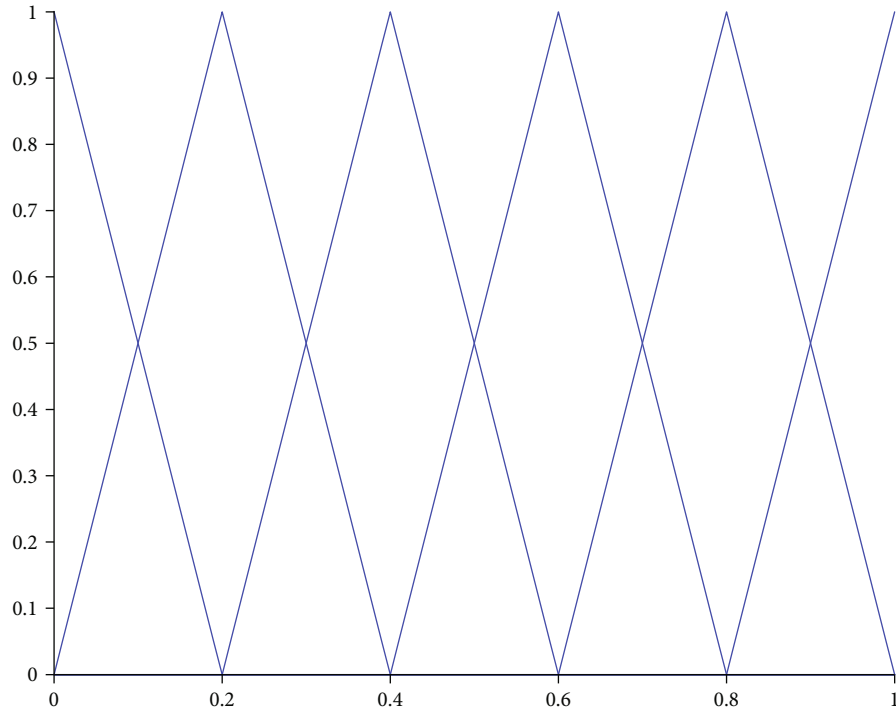


FIGURE 1: Linear B-spline.

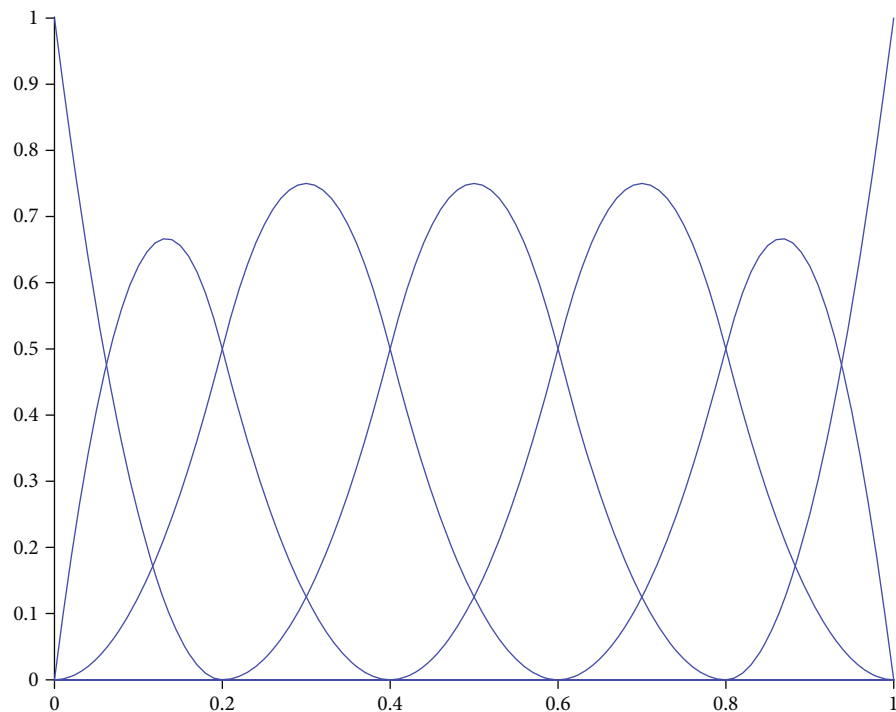


FIGURE 2: Quadratic B-spline.

$st(x, \partial\Omega)$. Then, the weighted B -spline space (WB-space) $B_\omega^h := \text{span}\{\omega B_k : k \in K\}$ spanned by weighted B -splines is a possible finite element subspace for Dirichlet boundary value problems yielding optimal order approximations. But, the

condition number of the Galerkin matrix G_h can become extremely large. This is due to the outer B -splines which have only very small support inside Ω . One might think that these basis functions do not contribute much to the approximation

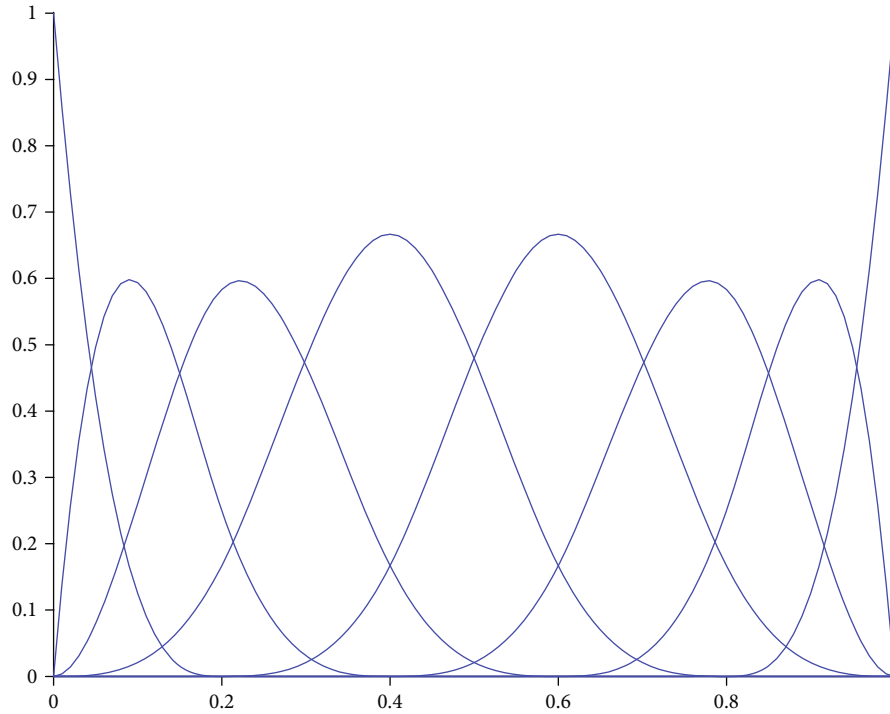


FIGURE 3: Cubic B-spline.

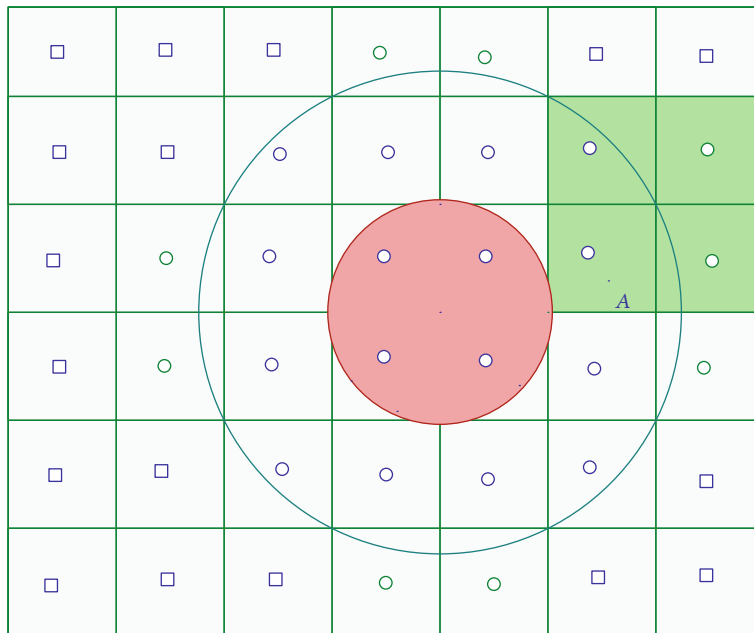


FIGURE 4: Relevant quadratic B-splines for domain A , marked at the center of their supports—inner B-splines with circles and outer ones with squares.

power and can simply be omitted. Unfortunately, this is not the case. A suitable solution to the problem of controlling the unstable outer B -splines is provided by adjoining them appropriately with the inner B -splines. It has been done in such a way that the approximation power of the finite element subspace is retained.

Definition 5 (see [20]). For $i \in I$, WEB-spline B_i is defined as

$$B_i = \frac{\omega}{\omega(x_i)} \left[b_i + \sum_{j \in J} e_{ij} b_j \right], \tag{40}$$

where x_i denotes the center of a grid cell, which is completely inside the domain Ω . The coefficients $e_{i,j}$ satisfy $|e_{i,j}| \leq 1$, $e_{i,j} = 0$ for $\|i - j\| \geq 1$ and are chosen so that all weighted polynomials (ωp) of order n are contained in the WEB-space $B_h := \text{span}\{B_i : i \in I\}$.

Theorem 6 (see [20]). *For an outer index $j \in J$, let $I(j) = l + \{0, \dots, n\}^m \subset I$ be an m -dimensional array of inner indices closest to j assuming that h is small enough so that such an array exists. Then, the coefficients*

$$e_{i,j} = \prod_{v=1}^m \prod_{\mu=0, l_v+\mu \neq i_v}^n \frac{j_v - l_v - \mu}{i_v - l_v - \mu} \tag{41}$$

are admissible for constructing WEB-splines according to Definition 5.

5. $C_{A,B}$ Boundary Conditions with Resolution Adapted

The imposition of inhomogeneous Dirichlet boundary conditions is essential in numerical analysis of a structure. It is especially difficult and no longer straightforward whenever nonconformal mesh is used to discretize a structure. One of the contributions of this paper is to develop a weighted extended spline basis with high computing accuracy and appropriate with the mixed formulation of boundary conditions $C_{A,B}$ that generalize all the cases that can be encountered on the edge of Ω .

$$Au + B \left(\mu \frac{\partial u}{\partial n} + \lambda \nabla \cdot un \right) = g \text{ on } \Gamma. \tag{42}$$

With A and B are two reversible square (in case 2D) matrices, $A, B \in L^\infty(\Gamma)^{2 \times 2}$ and $g \in H^{1/2}(\Gamma)$ where the function g is a priori known function of spatial coordinates. The inhomogeneous Dirichlet condition is expressed in the case of $B = 0$; then, we obtain $u = A^{-1}g$, on Γ . In general, a boundary condition function g may not be known explicitly or is not prescribed globally. More typically, the boundary conditions are prescribed in a piecewise manner, i.e., the boundary conditions are specified by individual functions g_i on each portion of boundary, $u = A^{-1}g_i$, on Γ_i , $i = 1, \dots, N$, and $\Gamma = \cup_{i=1}^N \Gamma_i$, $\cap_i \Gamma_i = \emptyset$. If each Dirichlet boundary Γ_i is geometrically represented by implicit function ω_i so that $\omega_i = 0$ on Γ_i [23],

$$u = A^{-1}g_i, \forall \omega_i = 0. \tag{43}$$

In fact, as the weighting function and boundary value function are expressed in the form of implicit functions, no unique expressions exist but typical forms are given below.

Weighting function $\omega(x)$ acts as a multiplier to modify the original interpolation functions and has to vanish on any Dirichlet boundary ($\omega_i(x) = 0$). As implicit function is not unique for the definition, weighting function of different forms can be used, as discussed in [24]. Here, two construc-

tion techniques are presented. First, a straightforward method is to construct the weighting function by means of the product $\omega(x) = \prod_{i=1}^m \omega_i(x)$. Clearly, $\omega(x) = 0$ if any $\omega_i(x) = 0$ on Γ_i . Nevertheless, a product of all the implicit functions might lead to a surge of function values which will cause a numerical overflow and then result in poor computing accuracy and robustness. Therefore, each implicit function ω_i is preferred to be normalized in advance according to the size of the physical domain. For instance, the implicit function of a circle can be normalized in terms of its radius.

$$\omega(x) = 1 - \left(\frac{x - x_0}{R} \right)^2 - \left(\frac{y - y_0}{R} \right)^2. \tag{44}$$

Now, we will choose a basis function web-spline that will be called the web-spline basis function adapted. This basis will respond to any type of nonhomogeneous boundary conditions especially $C_{A,B}$ in the case $B = 0$. Let u be the displacement solution of problem (15), so we can write $u = u_0 + u_\Gamma$ with u_0 being the solution of (15) in case of homogeneous Dirichlet boundary and $u_\Gamma = A^{-1}g$ is the nonhomogeneous Dirichlet boundary. u is written as a linear combination of the B_i web-spline family cited in Definition (5)

$$u = \sum_{i=1}^N u_i B_i + A^{-1}g. \tag{45}$$

It can be rewritten again in the form

$$u = \sum_{i=1}^N u_i B_i + \sum_{i=1}^N b_i A^{-1}g_i, \tag{46}$$

where $b_i(x)$ denotes the i th weighting coefficient associated with $A^{-1}g_i$ in which basic properties of $b_i(x)$ are as follows [23].

$b_i = \delta_{ij}$, $i, j = 1, 2 \dots N$, with δ_{ij} denoting the Kronecker delta function. With this condition, different forms are proposed below for the definition of b_i .

Transfinite interpolation form: the weighting coefficients can be defined by extending the transfinite interpolation that was studied by Rvachev et al. [25] in the CAD community.

$$b_i = \frac{\prod_{j=1, i \neq j}^N \omega_j^\theta}{\sum_{k=1}^N \prod_{j=1, i \neq j}^N \omega_j^\theta}. \tag{47}$$

The above equation shares a property of symmetry and similarity. $\omega_i(x)$ is generally supposed to be positive in the physical domain to ensure the nonzero value of the denominator. Evidently, $b_i = 1$ only if $\omega_i = 0$ and the partition of unity holds with $\sum_{i=1}^N b_i = 1$. The value of θ is used to interpolate normal derivatives prescribed on Γ , which must be 1 greater than the order of the prescribed derivatives.

Finally, the basis that we can propose for the nonhomogeneous boundary condition is as follows:

$$\{\phi_i\}_{i=1,\dots,N} = \{B_i + b_i\}_{i=1,\dots,N}. \tag{48}$$

6. Discretization of the Navier-Lamé Problem Using WEB-Spline Basis

In the Navier-Lamé equation, Δu and $\nabla\psi$ are the terms with derivatives of the highest order for the displacement and div-displacement (ψ), respectively. Thus, the orders of the differential operators differ by 1. This suggests the rule of thumb. The degree of the basis functions used to approximate the displacement should be one larger than the approximation of the div-displacement. Also, to satisfy the Dirichlet boundary conditions, the displacement basis functions are multiplied by a suitable weight function ω . Hence, in this article, we choose ϕ_j -linear weighted extended B-spline for displacement approximation along with φ_i -Haar wavelet basis function for div-displacement approximation as our linear-constant element and for quadratic-linear element we denote ϕ_j as quadratic WEB-spline and φ_i the mean zero linear function (see Figures 1 and 2) as defined above. In the following, the inf-sup condition and, therefore, the well posedness of the discrete Navier-Lamé problem are settled first for the linear-constant element and then for the quadratic-linear element.

6.1. Linear Constant Element. More precisely, we give V_h and M_h , the displacement and div-displacement finite element spaces, respectively, as follows:

$$V_h = \left\{ (u_1, u_2)/u_k = \sum_{j \in I^u} \alpha_j^k \phi_j, \alpha_j^k \in \mathbb{R}, k = 1, 2 \right\}, \tag{49}$$

$$M_h = \left\{ \psi/\psi = \sum_{i \in I^\psi} \beta_i^k \varphi_i, \beta_i \in \mathbb{R} \right\}. \tag{50}$$

A sufficient condition for (V_h, M_h) to satisfy the inf-sup condition is given in [21, 22]. A similar result has been proved in [26] for a different pair (V_h, M_h) .

6.2. Quadratic-Linear Element. For $j \in I^\psi$,

$$\varphi_j(x, y) = \frac{1}{h} \varphi\left(\frac{x}{h} - j_0\right) \varphi\left(\frac{y}{h} - j_1\right), \tag{51}$$

where φ is defined as

$$\varphi(x) := \begin{cases} x & \text{on } [0, 1), \\ 2 - x & \text{on } [1, 2), \\ 2 - x & \text{on } [2, 3), \\ x - 4 & \text{on } [3, 4). \end{cases} \tag{52}$$

For constructing the displacement approximation space, we do the following. For $i \in I^u$, let ϕ_i be the WEB-spline of order 3, which is given by (40) and (48) where ϕ_i is defined as in (51), the tensor product of the scaled translate of the function ϕ

$$\phi(x) := \begin{cases} \frac{x^2}{2} & \text{on } [0, 1), \\ \frac{1}{2} + (x-1) - (x-1)^2 & \text{on } [1, 2), \\ \frac{(3-x)^2}{2} & \text{on } [2, 3), \end{cases} \tag{53}$$

$$V_b := \oplus_{K \in T_h} V_K,$$

where the quadrangulation T_h is the collection of all cells $K \subset \Omega$ such that $K \cap \partial\Omega = \emptyset$ (those which are fully inside the domain Ω) and $K_{\partial\Omega} := K \cap \partial\Omega \neq \emptyset$ such that $K \cap \partial\Omega \neq \emptyset$ (those portions which intersect the boundary $\partial\Omega$) and V_K is the one dimensional subspace spanned by the function b^K given by

$$b^K := \omega(x, y) b\left(\frac{x}{h} - k_0\right) b\left(\frac{y}{h} - k_1\right). \tag{54}$$

The function $b(x) := x(x-1)$ is chosen so that the bubble function b^K vanishes on the edges of the cell K . (k_0, k_1) is the node corresponding to the cell K . The weight function ω is multiplied to ensure that the bubble function also vanishes on the boundary $\partial\Omega$. The distance function is used near the boundary, where it is free of singularities and blended smoothly with a plateau inside the domain. More precisely, ω is defined as $\omega = 1 - (\max(\delta - d(x), 0)/\delta)^l$, $d(x) = \text{dist}(x, \partial\Omega)$, where δ controls the height of the plateau and l the smoothness of the weight function. The plateau facilitates the use of precomputed values while assembling the Galerkin matrix and also avoids the use of high-order quadratures for the integration of the bubble functions supported on the cells which are fully inside the domain.

The displacement approximation space is taken to be $V_h^2 := V_h \oplus V_b$.

This pair (V_h^2, M_h) of discretization spaces satisfies the discrete inf-sup condition.

Lemma 7 (see [3]). *A web-basis $\{B_i\}_{i \in I}$ is a stable basis with respect to the L^2 -norm, $C_1 \|\{a_i\}_{i \in I}\| \leq \|\sum_{i \in I} B_i\|_0 \leq C_2 \|\{a_i\}_{i \in I}\|$, where C_2, C_1 are constants.*

We can also bound higher order Sobolev norms in terms of the 2-norm of the coefficients.

Lemma 8 (see [3]). *A web-basis $\{B_i\}_{i \in I}$ satisfies $\|\sum_{i \in I} a_i B_i\|_1 \leq C_3 h^{-1} \|\{a_i\}_{i \in I}\|$, and $C_3 > 0$ is a constant.*

Lemma 9 (inverse estimate). *Let V_h be the finite element space considered as in (49). Then, there exists a constant C such that $\|\sum_{i \in I} a_i B_i\|_1 \leq C_6 h^{-1} \|\sum_{i \in I} a_i B_i\|_0$.*

The proof follows from Lemmas 7 and 8.

Lemma 10 (see [21, 22]). *Suppose T_h is a family of uniform quadrilateral mesh on the domain Ω and \prod_h is the L^2 -projection operator onto $V_h \subset H_0^1(\Omega)$. Then*

$$\left\| \prod_h v \right\|_1 \leq C \|v\|_1, \forall v \in H_0^1(\Omega), \quad (55)$$

where C is a constant which is h -independent.

Lemma 11. *There exists an operator $P_h : H^1(\Omega) \longrightarrow V_h^2$ which satisfies the following properties:*

$$b(P_h v, q_h) = b(v, q_h), \forall q_h \in M_h, \quad (56)$$

$$b_\Gamma(P_h v, q_h) = b_\Gamma(v, q_h), \forall q_h \in M_h, \quad (57)$$

$$\|P_h v\|_1 \leq C \|v\|_1, C > 0 \text{ independent of } h. \quad (58)$$

Proof. Let $P_h^0 : H^1(\Omega) \longrightarrow V_h$ be the usual L^2 -projection operator with

$$\|P_h^0 v\|_1 \leq c_1 \|v\|_1, \quad (59)$$

$$\|v - P_h^0 v\|_0 \leq c_2 h \|v\|_1. \quad (60)$$

The first inequality (59) follows from Lemma (10) and the density of $H_0^1(\Omega)$ in $H^1(\Omega)$ ($H_0^1(\bar{\Omega}) \cap V^{\text{H}^1, \Omega} \subset H^1(\Omega)$), and the second (62) is an L^2 -error estimate. Now we fix a linear mapping $P_h^1 : L^2(\Omega) \longrightarrow V_b$ such that

$$\int_k P_h^1 v = \int_k v, \text{ for each } K \in T_h. \quad (61)$$

We may interpret the map P_h^1 as a process with two steps. First, we apply the L^2 -projection onto the space of piecewise constant functions. Afterwards, in each cell K the constant is replaced by a bubble function with the same integral. In this way, we get

$$\|P_h^1 v\|_0 \leq c_3 \|v\|_0. \quad (62)$$

More precisely, we define P_h^1 as follows:

$$P_h^1 v|_K(x, y) = \beta(K) b^K(x, y), \text{ for each } K \in T_h. \quad (63)$$

In view of condition (61), the constant $\beta(K)$ is taken to be $\int_k v / \int_k b^K$. Observe that for $\int_k b^K = c_4 h^2$ and $\int_k (b^K)^2 = c_5 h^2$, where $c_4, c_5 > 0$ are constants,

$$\begin{aligned} \|P_h^1 v\|_0^2 &= \sum_{K \in T_h} \int_k |P_h^1 v|^2 = \sum_{K \in T_h} \int_k \beta(K)^2 (b^K)^2 = \sum_{K \in T_h} \frac{\int_k v}{\int_k b^K} \int_k (b^K)^2 \\ &\leq \frac{1}{(c_4)^2 h^4} \sum_{K \in T_h} \left(\int_k v^2 \right) |K| c_5 h^2 = \left(\frac{c_5}{(c_4)^2} \right) \|v\|_0^2. \end{aligned} \quad (64)$$

The above inequality is due to Cauchy-Schwarz and $|K| = h^2$ the area of the cell K . Let us set

$$P_h v = P_h^0 v + P_h^1 (v - P_h^0 v). \quad (65)$$

Verify the two properties in the statement of the lemma. By the virtue of the construction of P_h and we impose that $v - P_h^0 v = 0$ on $\partial\Omega$

$$\int_K (v - P_h v) = \int_K (I - P_h^1)(v - P_h^0 v) = 0 \forall K \in T_h. \quad (66)$$

Because q_h is piecewise linear, we apply Green's theorem

$$\begin{aligned} b(v - P_h v, q_h) &= \int_\Omega \operatorname{div} (v - P_h v) q_h = \int_{\partial\Omega} (v - P_h v) n q_h \\ &\quad - \int_\Omega (v - P_h v) \nabla q_h = - \sum_{K \in T_h} C_K \int_\Omega (v - P_h v) = 0. \end{aligned} \quad (67)$$

It is easy to prove (57) by applying Green's theorem and using (56)

$$\begin{aligned} \|P_h v\|_1 &\leq \|P_h^0 v\|_1 + \|P_h^1 (v - P_h^0 v)\|_1 \leq c_1 \|v\|_1 + c_6 h^{-1} \|P_h^1 (v - P_h^0 v)\|_0 \\ &\leq c_1 \|v\|_1 + c_6 h^{-1} c_3 \|v - P_h^0 v\|_0 \leq c_1 \|v\|_1 + c_6 h^{-1} c_3 c_2 h \|v\|_1 \\ &\leq c \|v\|_1. \end{aligned} \quad (68)$$

This proves the assertion (58) with $c = c_1 + c_6 c_3 c_2$, such as c_6 of Lemma (9).

Theorem 12. *Let the space be $B_h = \{u \in V_h^2 \mid b(P_h u, q_h) = 0, \forall q_h \in M_h(\Omega)\}$ such that $V_h^2 = B_h \oplus B_h^\perp$. The pair (V_h^2, M_h) satisfies the discrete inf-sup condition:*

$$\exists \lambda_0 > 0, \sup_{\|q_h\|_0=1} \frac{b(v_h, q_h)}{\|v_h\|_{V_h^2}} \geq \lambda_0, \forall v_h \in B_h^\perp, \quad (69)$$

$$\sup_{v_h \in B_h^\perp} b(v_h, q_h) > 0, \forall q_h \in M_h(\Omega).$$

Proof. Let $P_h v \in B_h^\perp$ be given. By the continuous form of the inf-sup condition (20) and (56) and (57) of Lemma 11, there exist $c > 0$ and $\rho > 0$ such that $(1/c) \|P_h v\|_{V_h^2} \leq \|v\|_{1, \Omega} \leq (1/\rho) \sup_{\|q_h\|=1} b(v, q_h) = (1/\rho) \sup_{\|q_h\|=1} b(P_h v, q_h)$; then, we find $\sup_{\|q_h\|=1} b(P_h v, q_h) > (c/\rho) \|P_h v\|_{V_h^2}$, $\forall P_h v \in B_h^\perp$. According to (56) and the second condition of (20), we have $\sup_{v_h \in B_h^\perp} b(v_h, q_h) = \sup_{v \in B^\perp} b(v, q_h) > 0, \forall q_h \in M_h(\Omega)$.

Remark 13. With a similar demonstration as above, the bilinear form b_Γ (16) on $V_h^2 \times M_h$ verifies the inf-sup condition cited in Theorem 12.

7. Matrix Form of the Navier-Lamé Problem

After establishing the discrete inf-sup condition, we are ready to prove the existence and uniqueness of the discrete solution. (This part of the section is common to both the linear-constant and quadratic-linear elements, so we call the finite element spaces as V_h and M_h in general.) To find the discrete solution pair $(u_h, \psi_h) \in V_h \times M_h$, it is enough to

find the coefficient vectors of displacement $\bar{u} := (\bar{u}^1, \bar{u}^2)^T$, where $\bar{u}^\alpha = ((\bar{u}_i^\alpha)_{i \in I^u})^T$, $\alpha = 1, 2$, and div-displacement $\bar{\psi} := ((\bar{\psi}_j)_{j \in I^\psi})^T$.

Assuming that $(B^{-1}A)_{\alpha\beta} = \xi_{\alpha\beta}$, and $(B^{-1})_{\alpha\beta} = \zeta_{\alpha\beta}$ for $\alpha, \beta = 1, 2$. Let us define

$$\begin{aligned} A_{i,k}^\alpha &= \int_\Omega \mu \nabla \phi_i \cdot \nabla \phi_k d\Omega + \int_\Gamma \xi_{\alpha\alpha} \phi_i \phi_k d\Gamma, \\ B_{i,j}^\alpha &= \int_\Omega (\lambda + \mu) \varphi_j \frac{\partial \phi_i}{\partial x_\alpha} d\Omega, \\ B_{\Gamma,i,j}^\alpha &= \int_\Omega (\lambda + \mu) \varphi_j \frac{\partial \phi_i}{\partial x_\alpha} d\Omega + \int_\Gamma \mu \varphi_j n \phi_i d\Gamma, \\ D_{m,n} &= \int_\Omega (\lambda + \mu) \varphi_m \varphi_n d\Omega, \\ L_s^1 &= \int_\Omega f_1 \phi_s d\Omega + \int_\Gamma (\zeta_{11} + \zeta_{21}) g_1 \phi_s d\Gamma, \\ L_s^2 &= \int_\Omega f_2 \phi_s d\Omega + \int_\Gamma (\zeta_{22} + \zeta_{12}) g_2 \phi_s d\Gamma. \end{aligned} \tag{70}$$

The matrix form of the discrete Navier-Lamé equations is written as

$$\begin{pmatrix} A^1 & 0 & B_\Gamma^1 \\ 0 & A^2 & B_\Gamma^2 \\ B^{1,T} & B^{2,T} & -D \end{pmatrix} \begin{pmatrix} \bar{u}^1 \\ \bar{u}^2 \\ \bar{\psi} \end{pmatrix} = \begin{pmatrix} L^1 \\ L^2 \\ 0 \end{pmatrix}. \tag{71}$$

By defining the assembled matrices $A = \text{Diag}(A^1, A^2)$, $B_\Gamma = (B_\Gamma^1, B_\Gamma^2)^T$, $B = (B^1, B^2)$, $F = (L^1, L^2)^T$, we rewrite the above matrix form as

$$\begin{pmatrix} A & B_\Gamma \\ B^T & -D \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{\psi} \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}. \tag{72}$$

Decoupling of div-displacement and displacement: from the above matrix form, we write

$$\begin{cases} A\bar{u} + B_\Gamma \bar{\psi} = F, \\ B^T \bar{u} - D\bar{\psi} = 0. \end{cases} \tag{73}$$

Premultiplying the first equation above by A^{-1} , followed by B^T , and using the second equation, we get

$$(D + B^T A^{-1} B_\Gamma) \bar{\psi} = B^T A^{-1} F. \tag{74}$$

In the following, we show that $D + B^T A^{-1} B_\Gamma$ is invertible, and hence, the above equation can be solved for $\bar{\psi}$.

The positive definiteness follows from the positive $D + B^T A^{-1} B_\Gamma$.

Let us define $B_\Gamma = B + b_\Gamma$, with $(b_\Gamma)_{i,j} = \int_\Gamma \mu \varphi_j n \phi_i d\Gamma$,

$$\begin{aligned} \langle (D + B^T A^{-1} B_\Gamma) \psi, \psi \rangle_{\mathbb{R}^M} &= \langle D\psi, \psi \rangle_{\mathbb{R}^M} + \langle B^T A^{-1} B_\Gamma \psi, \psi \rangle_{\mathbb{R}^M} \\ &= \langle D\psi, \psi \rangle_{\mathbb{R}^M} + \langle A^{-1} (B + b_\Gamma) \psi, B\psi \rangle_{\mathbb{R}^M} \\ &= \langle D\psi, \psi \rangle_{\mathbb{R}^M} + \langle A^{-1} B\psi, B\psi \rangle_{\mathbb{R}^M} \\ &\quad + \langle A^{-1} b_\Gamma \psi, B\psi \rangle_{\mathbb{R}^M}, \end{aligned} \tag{75}$$

where $M = \text{cardinality of } I^\psi$.

The positive definiteness of $B^T A^{-1} B$ follows from the positive definiteness of matrix A (which is guaranteed by the coercivity condition) and the following inequality which is an equivalent condition to the inf-sup condition (20).

$$\|B\psi\| \geq \rho \|\psi\|. \tag{76}$$

Indeed, one has

$$\langle A^{-1} B\psi, B\psi \rangle_{\mathbb{R}^M} \geq \beta \|B\psi\|^2 \geq \beta \rho^2 \|\psi\|^2. \tag{77}$$

Condition (21) gives

$$\langle D\psi, \psi \rangle_{\mathbb{R}^M} \geq \varepsilon \|\psi\|^2. \tag{78}$$

The inequality of Cauchy-Schwarz gives

$$\langle A^{-1} b_\Gamma \psi, B\psi \rangle_{\mathbb{R}^M} \geq -\|A^{-1} b_\Gamma \psi\| \|B\psi\|. \tag{79}$$

On the other hand,

$$\begin{aligned} \|A^{-1} b_\Gamma \psi\| &= \sum_{i \in I^u} (A^{-1} b_\Gamma \psi)_i^{1/2} = \left(\sum_{i \in I^u} \left(\sum_{m \in I^\psi} \sum_{k \in I^u} A_{ik}^{-1} b_{km} \psi_{m1} \right)^2 \right)^{1/2} \\ &\leq \left(\sum_{i \in I^u} \left(\text{card } I^u \text{ card } I^\psi \max_{k \in I^u} \max_{m \in I^\psi} |A_{ik}^{-1} b_{km}| \sum_{m \in I^\psi} \psi_{m1} \right)^2 \right)^{1/2} \\ &\leq \left((\text{card } (I^u))^2 \text{ card } (I^\psi) \max_{k,i \in I^u} \max_{m \in I^\psi} |A_{ik}^{-1} b_{km}| \right)^{1/2} \|\psi\|. \end{aligned} \tag{80}$$

By taking $\text{card } (I^\psi) = M$, $\text{card } (I^u) = N$, and $\max_{k,i \in I^u} \max_{m \in I^\psi} |A_{ik}^{-1} b_{km}| = \alpha$. Then, we find that

$$\langle A^{-1} b_\Gamma \psi, B\psi \rangle_{\mathbb{R}^M} \geq -N(M\alpha)^{1/2} \rho \|\psi\|^2. \tag{81}$$

By combining (77), (78), and (81), we obtain

$$\langle (D + B^T A^{-1} B_\Gamma) \psi, \psi \rangle_{\mathbb{R}^M} \geq (\beta \rho^2 + \varepsilon - N\rho(M\alpha)^{1/2}) \|\psi\|^2, \tag{82}$$

as soon as $\beta \rho^2 + \varepsilon - N\rho(M\alpha)^{1/2} > 0$.

This proves the existence of the discrete solution (u_h, ψ_h) .

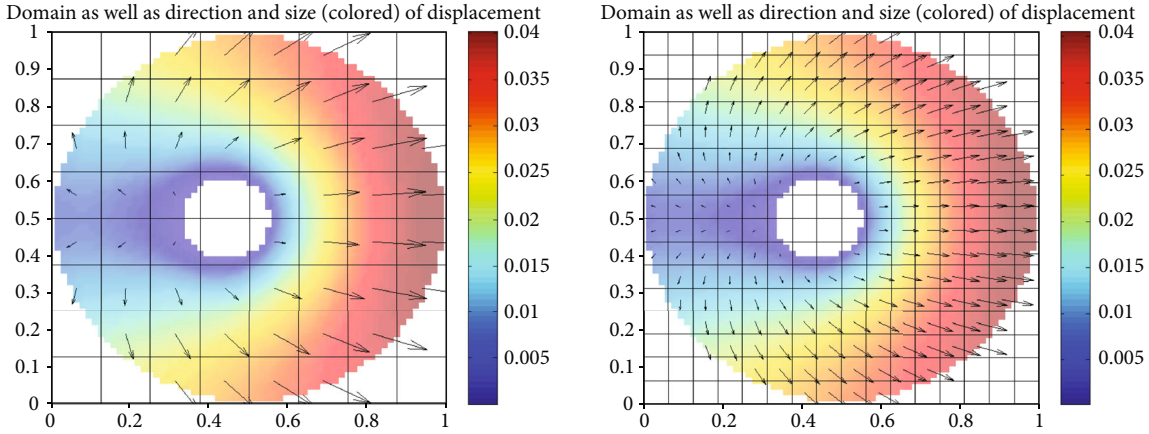


FIGURE 5: Numerical solution for $H = 8, 16$.

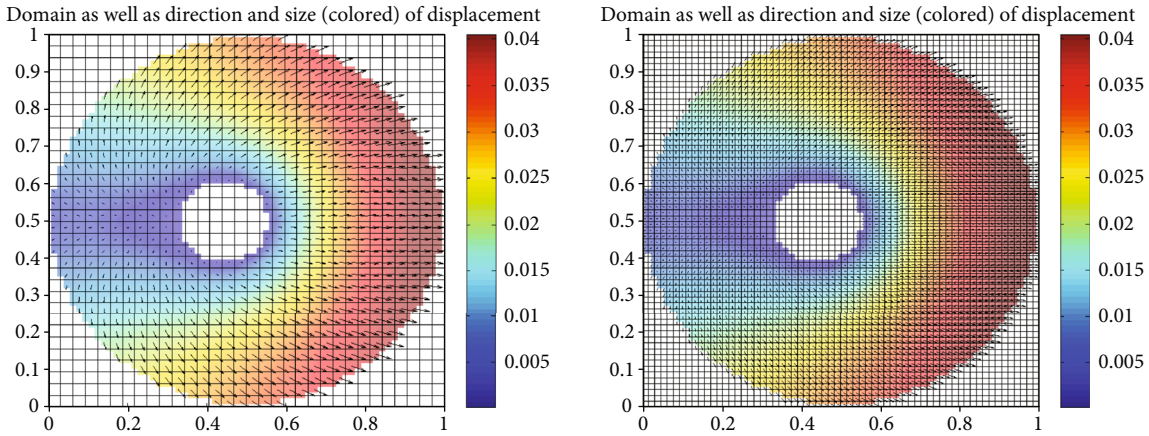


FIGURE 6: Numerical solution for $H = 32, 64$.

8. Numerical Results

We tested the performance of the WEB-spline method for the Navier-Lamé problem (6). We take the centrifugal force $\|f\| = r$ and the domain Ω is a rotating steel disc defined by $\Omega = \{(x, y) / (x - (1/2))^2 + (y - (1/2))^2 \leq (1/2)^2\} \setminus \{(x, y) / (x - (1/2) + d)^2 + (y - (1/2))^2 \leq r^2\}$, with $r = 0.1$, $d = 0.06$, and the edge of Ω consists of two disjoint parts in the form of two circles $\partial\Omega = \Gamma = \Gamma_D \cup \Gamma_N$ such as $\Gamma_D = \{(x, y) / (x - (1/2) + d)^2 + (y - (1/2))^2 - r^2 = 0\}$, $\Gamma_N = \{(x - (1/2))^2 + (y - (1/2))^2 - (1/2)^2 = 0\}$.

We give in (42) the Dirichlet matrix $A = I_2$ on Γ_D , 0_2 on Γ_N , and we give the Neumann matrix $B = I_2$ on Γ_N , 0_2 on Γ_D , the traction force $g = (0, 0)$ on $\partial\Omega$.

Figures 5 and 6 show the numerical solution for rotating steel disc with $(E = 1, \nu = 0.28)$ that is fixed at a slightly eccentric axis with radius r . We show the domain as well as direction and size (colored) of displacement for each $H = (8, 16, 32, 64)$ number of cells per coordinate direction.

As mentioned in Table 1, we check for this example the accuracy of the numerical approximations. Since in view of the smoothness of the weighted B-spline basis functions that

TABLE 1: Different errors.

H	Error of pde	Rate
8	$1.118e + 000$	NaN
16	$3.644e - 001$	$1.618e + 000$
32	$1.831e - 002$	$4.314e + 000$
64	$7.003e - 003$	$1.387e + 000$
128	$1.775e - 003$	$1.980e + 000$
256	$4.462e - 004$	$1.992e + 000$

are continuously differentiable for $n \geq 2$ we can compute the pointwise residuals [27] for the Navier-Lamé boundary value problem (6), error pde = $\|-\mu\Delta u_h - (\lambda + \mu)\nabla\psi_h - f\|_0$ and rate $r = [\text{NaN} - \text{diff}(\log(e))/\log(2)]$. We can substitute the finite element approximation into the partial differential equation Navier-Lamé where $g = 0$ for the rotating disc. This would not be possible for standard finite element approximation which usually merely belong to $H^1(\Omega)$.

We show the relative residual for grid widths $1/H = 1/8, \dots, 1/256$. While the residuals decay, estimates of the rates exhibit a regular increasing behavior when H is

changed from 64 to 256 (e.g., the error pde norm [27] has dropped from $1.118e+000$ to $4.462e-004$ when H is changed from 8 to 256) which we attribute to numerical stabilities.

Note that in the case when μ is quite small, we will refer the readers to see [28–31].

9. Conclusion

Using the equations of linear elasticity as a model problem with the boundary condition $C_{A,B}$, we have described the implementation of finite element methods with weighted B-splines using mixed finite element. In this article, we have shown that our WEB-spline-based quadratic-linear finite elements satisfy the inf-sup condition, which is necessary for the existence and uniqueness of the solution, and we proved the existence of the discrete solution. We have shown the convergence of the numerical solution for the quadratic case. Because of limited regularity, the Navier-Lamé problem will not change by increasing the degree of the WEB-spline.

We have computed the relative errors and rates and have shown that it is of order $1/H$, which is theoretically correct; these numerical results are attributed to the numerical stabilities of the solution. Moreover, the advantage of this problem with $C_{A,B}$ boundary condition is the program level MATLAB; it is enough to make a single program MATLAB and can be reduced to ordinary problems as Dirichlet and Neumann.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] O. C. Zienkiewicz and R. I. Taylor, *Finite Element Method, Vols. I–III*, Butterworth and Heinemann, 2000.
- [2] O. Koubaiti, J. El-Mekkaoui, and A. Elkhalfi, “Elasticity with mixed finite element,” *Communications in Applied Analysis*, vol. 22, no. 4, 2018.
- [3] K. Höllig, U. Reif, and J. Wipper, “Weighted extended B-spline approximation of Dirichlet problems,” *SIAM Journal on Numerical Analysis*, vol. 39, no. 2, pp. 442–462, 2001.
- [4] J. A. Cottrell, T. J. R. Hughes, and Y. Bazilevs, *Isogeometric Analysis: Toward Integration of CAD and FEA*, Wiley, 2009.
- [5] K. Höllig, *Finite Element Methods with B-Splines*, SIAM, Philadelphia, 2003.
- [6] K. Höllig, J. Hörner, and A. Hoffacker, “Finite element analysis with B-splines: weighted and isogeometric methods,” in *Curves and Surfaces 2011*, J. D. Boissonnat, Ed., vol. 6920 of LNCS, , pp. 330–350, Springer, 2012.
- [7] O. Koubaiti, J. El-mekkaoui, and A. Elkhalfi, “Complete study for solving Navier-Lamé equation with new boundary condition using mini element method,” *International Journal of Mechanics*, vol. 12, pp. 46–58, 2018.
- [8] P. Das, “An a posteriori based convergence analysis for a nonlinear singularly perturbed system of delay differential equations on an adaptive mesh,” *Numerical Algorithms*, vol. 81, no. 2, pp. 465–487, 2019.
- [9] P. Das, S. Rana, and H. Ramos, “A perturbation-based approach for solving fractional-order Volterra–Fredholm integro differential equations and its convergence analysis,” *International Journal of Computer Mathematics*, pp. 1–21, 2019.
- [10] P. Das and J. Vigo-Aguiar, “Parameter uniform optimal order numerical approximation of a class of singularly perturbed system of reaction diffusion problems involving a small perturbation parameter,” *Journal of Computational and Applied Mathematics*, vol. 354, pp. 533–544, 2019.
- [11] P. Das, “A higher order difference method for singularly perturbed parabolic partial differential equations,” *Journal of Difference Equations and Applications*, vol. 24, no. 3, pp. 452–477, 2018.
- [12] P. Das, “Comparison of a priori and a posteriori meshes for singularly perturbed nonlinear parameterized problems,” *Journal of Computational and Applied Mathematics*, vol. 290, pp. 16–25, 2015.
- [13] M. Chandru, P. Das, and H. Ramos, “Numerical treatment of two-parameter singularly perturbed parabolic convection diffusion problems with non-smooth data,” *Mathematical Methods in the Applied Sciences*, vol. 41, no. 14, pp. 5359–5387, 2018.
- [14] M. Chandru, T. Prabha, P. Das, and V. Shanthi, “A numerical method for solving boundary and interior layers dominated parabolic problems with discontinuous convection coefficient and source terms,” *Differential Equations and Dynamical Systems*, vol. 27, no. 1-3, pp. 91–112, 2019.
- [15] M. H. Sadd, *Elasticity: Theory, Application and Numerics*, Elsevier Butterworth Heinemann, Amsterdam, 2005.
- [16] S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York, 1985.
- [17] P. Ciarlet Jr., J. Huang, and J. Zou, “Some observations on generalized saddle-point problems,” *SIAM Journal on Matrix Analysis and Applications*, vol. 25, no. 1, pp. 224–236, 2003.
- [18] K. Atkinson and W. Han, *Theoretical Numerical Analysis—A Functional Analysis Framework*, TAM 39, Springer, Berlin, 2000.
- [19] S. Kesavan, *Topics in Functional Analysis and Applications*, Wiley Eastern Limited, New Delhi, India, 1989.
- [20] K. Höllig, U. Reif, and J. Wipper, “Multigrid methods with WEB-splines,” *Numerische Mathematik*, vol. 91, no. 2, pp. 237–255, 2002.
- [21] V. V. K. Srinivas Kumar, B. V. Rathish Kumar, and P. C. Das, “WEB-spline based mesh-free finite element analysis for the approximation of the stationary Navier–Stokes problem,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 11, pp. 3266–3282, 2008.
- [22] V. V. K. Srinivas Kumar, B. V. Rathish Kumar, and P. C. Das, “A new class of stabilized mesh-free finite elements for the approximation of the Stokes problem,” *Numerical Methods for Partial Differential Equations*, vol. 20, no. 5, pp. 703–722, 2004.
- [23] W. Zhang and L. Zhao, “Exact imposition of inhomogeneous Dirichlet boundary conditions based on weighted finite cell method and level-set function,” *Computer Methods in Applied Mechanics and Engineering*, vol. 307, pp. 316–338, 2016.

- [24] W. Zhang, L. Zhao, and S. Cai, "Shape optimization of Dirichlet boundaries based on weighted B-spline finite cell method and level-set function," *Computer Methods in Applied Mechanics and Engineering*, vol. 294, pp. 359–383, 2015.
- [25] V. L. Rvachev, T. I. Sheiko, V. Shapiro, and I. Tsukanov, "Transfinite interpolation over implicitly defined sets," *Computer Aided Geometric Design*, vol. 18, no. 3, pp. 195–220, 2001.
- [26] H. J. Choe, H. H. Kim, D. W. Kim, and Y. Kim, "Mesh-less method for the stationary incompressible Navier-Stokes equations," *Discrete and Continuous Dynamical Systems—Series B*, vol. 1, no. 4, pp. 495–526, 2001.
- [27] K. Höllig and J. Hörner, "Programming finite element methods with weighted B-splines," *Computers and Mathematics with Applications*, vol. 70, no. 7, pp. 1441–1456, 2015.
- [28] P. Das and S. Natesan, "Higher-order parameter uniform convergent schemes for Robin type reaction-diffusion problems using adaptively generated grid," *International Journal of Computational Methods*, vol. 9, no. 4, article 1250052, 2013.
- [29] P. Das and S. Natesan, "A uniformly convergent hybrid scheme for singularly perturbed system of reaction-diffusion Robin type boundary-value problems," *Journal of Applied Mathematics and Computing*, vol. 41, no. 1-2, pp. 447–471, 2013.
- [30] P. Das, S. Rana, and J. Vigo-Aguiar, "Higher order accurate approximations on equidistributed meshes for boundary layer originated mixed type reaction diffusion systems with multiple scale nature," *Applied Numerical Mathematics*, vol. 148, pp. 79–97, 2020.
- [31] P. Das and V. Mehrmann, "Numerical solution of singularly perturbed convection-diffusion-reaction problems with two small parameters," *BIT Numerical Mathematics*, vol. 56, no. 1, pp. 51–76, 2016.