

Research Article

Existence Results for a Class of p -Laplacian Fractional Differential Equations with Integral Boundary Conditions

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In this paper, we investigate the existence and uniqueness of solutions for a class of integral boundary value problems of nonlinear fractional differential equations with p -Laplacian operator. We obtain some existence and uniqueness results concerned with our problem by using Schaefer's fixed-point theorem and Banach contraction mapping principle. Finally, we present some examples to illustrate our main results.

1. Introduction

This paper deals with the existence and uniqueness of solutions for the following fractional integral boundary value problem with p -Laplacian operator:

$$\begin{cases} {}^c D_{0+}^\beta \varphi_p({}^c D_{0+}^\alpha x(t)) = f(t, x(t)), t \in (0, 1), \\ x(0) = \int_0^1 g(s)x(s)ds, x(1) = 0, \\ \varphi_p({}^c D_{0+}^\alpha x(0)) = \varphi_p({}^c D_{0+}^\alpha x(1)) = \int_0^1 h(s)\varphi_p({}^c D_{0+}^\alpha x(s))ds, \end{cases} \quad (1)$$

where $1 < \alpha, \beta \leq 2, 3 < \alpha + \beta < 4$, ${}^c D_{0+}^\alpha$ and ${}^c D_{0+}^\beta$ are the Caputo fractional derivatives, $f \in C([0, 1] \times \mathbf{R}, \mathbf{R})$, $g, h \in C([0, 1], \mathbf{R})$, and the p -Laplacian operator is defined as $\varphi_p(s) = |s|^{p-2}s, p > 1$.

In the past few decades, fractional differential equations have been widely applied to many fields in natural and social sciences, because they are important tools in mathematically describing many phenomena of science and engineering such as aerodynamics, control theory, signal and image processing, plasma dynamics, blood flow phenomena, and viscoelas-

tic and non-Newtonian fluid mechanics (see [1–5] and their references). In 1983, Leibenson [2] proposed the following integer-order differential equation model with p -Laplacian operator to study the turbulent flow in a porous medium in his work:

$$\left(\varphi_p(u'(t)) \right)' = f(t, u(t), u'(t)), \quad t \in (0, 1), \quad (2)$$

where φ_p is a p -Laplacian operator.

Based on Leibenson's results, many researchers have generalized his model into a variety of models with fractional order and obtained many valuable existence and uniqueness results for p -Laplacian fractional differential equations with two-point boundary conditions (see [6–14]), multipoint boundary conditions (see [15–18]), and nonlocal boundary conditions (see [19–23]). Since the 1st derivatives of unknown functions existed in Leibenson's model, it became a special study to discuss the existence results for p -Laplacian fractional differential equations where the fractional orders were in the neighborhood of 1. So, Chai [7] used the fixed-point theorem on cones to investigate the existence and multiplicity of positive solutions for fractional differential equations with p -Laplacian operator:

$$\begin{cases} D_{0+}^{\beta} \left(\varphi_p(D_{0+}^{\alpha} u(t)) \right) = -f(t, u(t)), & 0 < t < 1, \\ u(0) = 0, \\ u(1) + \sigma D_{0+}^{\gamma} u(1) = 0, \\ D_{0+}^{\alpha} u(0) = 0, \end{cases} \quad (3)$$

where $1 < \alpha \leq 2, 0 < \beta, \gamma \leq 1$, and $D_{0+}^{\alpha}, D_{0+}^{\beta}$, and D_{0+}^{γ} are the Riemann-Liouville fractional derivatives. Jong [15] established the existence and uniqueness of positive solutions for multipoint boundary value problems of nonlinear fractional differential equations with p -Laplacian operator by using the Banach contraction mapping principle

$$\begin{cases} D_{0+}^{\beta} \left(\varphi_p(D_{0+}^{\alpha} u(t)) \right) = f(t, u(t)), & 0 < t < 1, \\ u(0) = 0, \\ D_{0+}^{\gamma} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\gamma} u(\eta_i), \\ D_{0+}^{\alpha} u(0) = 0, \\ \varphi_p(D_{0+}^{\alpha} u(1)) = \sum_{i=1}^{m-2} \zeta_i \varphi_p(D_{0+}^{\alpha} u(\eta_i)), \end{cases} \quad (4)$$

where $1 < \alpha, \beta \leq 2, 0 < \gamma \leq 1$, and $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$.

Integral boundary value problems for differential equations have arisen in the study of various fields such as underground water flow, blood flow problems, and thermoelasticity. So, the study on the existence of solutions for the p -Laplacian integral boundary value problems has attracted the attention of many researchers recently (see [20–23]). Zhang et al. [23] considered the existence of symmetric positive solutions of the problem for the following nonlinear fourth-order p -Laplacian differential equations with integral boundary conditions:

$$\begin{cases} \left(\varphi_p(x''(t)) \right)'' = \omega(t)f(t, x(t)), & t \in (0, 1), \\ x(0) = x(1) = \int_0^1 g(s)x(s)ds, \\ \varphi_p(x''(0)) = \varphi_p(x''(1)) = \int_0^1 h(s)\varphi_p(x''(s))ds, \end{cases} \quad (5)$$

where $\omega, g, h \in L[0, 1]$ are the nonnegative, symmetric functions and $f \in C((0, 1] \times [0, +\infty), [0, +\infty))$. Zhang and Cui [22] investigated the existence of positive solutions for nonlinear fourth-order singular p -Laplacian differential equations with the integral boundary conditions

$$\begin{cases} \left(\varphi_p(x''(t)) \right)'' = f(t, x(t), x'(t)), & t \in (0, 1), \\ x(0) = \int_0^1 g(s)x(s)ds, x(1) = 0, \\ \varphi_p(x''(0)) = \varphi_p(x''(1)) = \int_0^1 h(s)\varphi_p(x''(s))ds, \end{cases} \quad (6)$$

where $f \in C((0, 1) \times (0, +\infty) \times (0, +\infty), [0, +\infty))$, f may be singular at $t = 0, 1, u = 0$, and $g, h \in L[0, 1]$ are nonnegative. The existence results on solutions of problem (2) are established by employing upper and lower solution methods together with maximal principle. Jiang [20] used the generalized continuous theorem to investigate the existence of solutions to the integral boundary value problem of p -Laplacian multiterm fractional differential equations at resonance

$$\begin{cases} D_{0+}^{\beta} \left(\varphi_p(D_{0+}^{\alpha} u(t)) \right) = -f(t, u(t), D_{0+}^{\alpha-1} u(t), D_{0+}^{\alpha} u(t)), & 0 < t < 1, \\ u(0) = D_{0+}^{\alpha} u(0) = 0, \\ u(1) = \int_0^1 h(t)u(t)dt, \end{cases} \quad (7)$$

where $0 < \beta \leq 1, 1 < \alpha \leq 2$, and $\int_0^1 h(t)t^{\alpha-1} dt = 1$.

Summarizing previous results, very few papers dealt with the existence of solutions for integral boundary value problems of p -Laplacian fractional differential equations, especially, Zhang and Cui [22] who established the existence of positive solutions for fourth-order singular p -Laplacian differential equations under $p \geq 2$ and Jiang [20] who considered the existence of solutions for nonlinear multiterm fractional differential equations with $0 < \beta \leq 1$. Moreover, due to the nonlinearity of p -Laplacian operator φ_p , it is more difficult to study for the case $1 < \beta \leq 2$ rather than for the case $0 < \beta \leq 1$. Motivated by the above facts, this paper deals with the existence and uniqueness of solutions of problem (1) in which the fractional derivatives are Caputo fractional derivatives with $1 < \alpha, \beta \leq 2$. The structure of this paper is organized as follows.

In Section 2, we recall some definitions and lemmas. In Section 3, we prove the existence and uniqueness of solutions for nonlinear integral boundary value problem with p -Laplacian operator by using Schaefer’s fixed-point theorem and Banach contraction mapping principle. Finally, we give two examples to illustrate our main results in Section 4.

2. Preliminaries

The Riemann-Liouville fractional integral and the Caputo fractional derivative of order $\alpha > 0$ of a function $f : (0, \infty) \rightarrow \mathbf{R}$ is given by

$$\begin{aligned} (I_{0+}^\alpha f)(t) &:= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \\ ({}^c D_{0+}^\alpha f)(t) &:= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \end{aligned} \tag{8}$$

where $n = [\alpha] + 1$, provided that the right-hand side is point-wise defined on $(0, \infty)$ (see [4, 5]).

Lemma 1 (see [1]). *If $\alpha > 0$ and $f \in C[a, b]$, then $({}^c D_{a+}^\alpha I_{a+}^\alpha f)(t) = f(t)$.*

Lemma 2 (see [9]). *Assume that ${}^c D_{0+}^\alpha u \in C[0, 1]$, $\alpha > 0$. Then, $I_{a+}^\alpha {}^c D_{0+}^\alpha u(t) = u(t) + c_0 + c_1 t + \dots + c_{n-1} t^{n-1}$, $c_i \in \mathbf{R}$, $i = 0, 1, \dots, n - 1$, $n - 1 < \alpha \leq n$.*

For the sake of convenience, put $\sigma_1 := \int_0^1 (1-s)g(s)ds$ and $\sigma_2 := \int_0^1 h(s)ds$ and assume that $\sigma_1 \neq 1$ and $\sigma_2 \neq 1$.

Lemma 3. *Let $\sigma \in C[0, 1]$. Then, the fractional boundary value problem*

$$\begin{cases} {}^c D_{0+}^\beta z(t) = \sigma(t), & 0 < t < 1, \\ z(0) = z(1) = \int_0^1 h(s)z(s)ds \end{cases} \tag{9}$$

has a unique solution which is given by

$$z(t) = \frac{-1}{1-\sigma_2} \int_0^1 \int_0^1 h(s)G_\beta(s, \tau)\sigma(\tau)d\tau ds - \int_0^1 G_\beta(t, s)\sigma(s)ds, \tag{10}$$

where

$$G_\beta(t, s) := \frac{1}{\Gamma(\beta)} \begin{cases} t(1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \leq s \leq t \leq 1, \\ t(1-s)^{\beta-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{11}$$

Proof. In view of Lemma 2, we have that

$$z(t) = c_0 + c_1 t + I_{0+}^\beta \sigma(t). \tag{12}$$

By means of the property of the fractional integral of a continuous function, we obtain that $z(0) = c_0$.

Since $z(0) = z(1) = \int_0^1 h(s)z(s)ds$, from (12), we obtain

$$\begin{aligned} c_0 &= \int_0^1 h(s)z(s)ds, \\ c_1 &= \int_0^1 h(s)z(s)ds - I_{0+}^\beta \sigma(t) \Big|_{t=1} - \int_0^1 h(s)z(s)ds = -I_{0+}^\beta \sigma(t) \Big|_{t=1}. \end{aligned} \tag{13}$$

Hence, (12) can be written as

$$\begin{aligned} z(t) &= \int_0^1 h(s)z(s)ds + I_{0+}^\beta \sigma(t) - I_{0+}^\beta \sigma(t) \Big|_{t=1} = \int_0^1 h(s)z(s)ds \\ &\quad + \frac{1}{\Gamma(\beta)} \left[\int_0^t (t-s)^{\beta-1} \sigma(s)ds - \int_0^1 t(1-s)^{\beta-1} \sigma(s)ds \right]. \end{aligned} \tag{14}$$

Thus, we can easily get

$$z(t) = \int_0^1 h(s)z(s)ds - \int_0^1 G_\beta(t, s)\sigma(s)ds. \tag{15}$$

In the right side of (15), the term $\int_0^1 h(s)z(s)ds$ can be rewritten as

$$\begin{aligned} \int_0^1 h(s)z(s)ds &= \int_0^1 h(s) \left(\int_0^1 h(\tau)z(\tau)d\tau - \int_0^1 G_\beta(s, \tau)\sigma(\tau)d\tau \right) ds \\ &= \int_0^1 h(s)ds \int_0^1 h(s)z(s)ds - \int_0^1 \int_0^1 h(s)G_\beta(s, \tau)\sigma(\tau)d\tau ds, \end{aligned} \tag{16}$$

so we get

$$\int_0^1 h(s)z(s)ds = \frac{-1}{1-\sigma_2} \int_0^1 \int_0^1 h(s)G_\beta(s, \tau)\sigma(\tau)d\tau ds. \tag{17}$$

Therefore, the unique solution of (9) is given by

$$z(t) = \frac{-1}{1-\sigma_2} \int_0^1 \int_0^1 h(s)G_\beta(s, \tau)\sigma(\tau)d\tau ds - \int_0^1 G_\beta(t, s)\sigma(s)ds. \tag{18}$$

Conversely, let $z \in C[0, 1]$ be the function which is expressed by (10). Putting

$$A(\beta) := \frac{-1}{1-\sigma_2} \int_0^1 \int_0^1 h(s)G_\beta(s, \tau)\sigma(\tau) d\tau ds, \tag{19}$$

we get

$$\begin{aligned} z(t) &= A(\beta) - \int_0^1 G_\beta(t, s)\sigma(s)ds = A(\beta) - \int_0^t G_\beta(t, s)\sigma(s)ds \\ &\quad - \int_t^1 G_\beta(t, s)\sigma(s)ds = A(\beta) \\ &\quad - \frac{1}{\Gamma(\beta)} \left[\int_0^t (t(1-s)^{\beta-1} - (t-s)^{\beta-1}) \sigma(s) ds \right. \\ &\quad \left. + \int_t^1 t(1-s)^{\beta-1} \sigma(s)ds \right] = A(\beta) \\ &\quad - \frac{1}{\Gamma(\beta)} \int_0^1 t(1-s)^{\beta-1} \sigma(s)ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sigma(s)ds. \end{aligned} \tag{20}$$

Then, we have that

$$z(t) = A(\beta) - tI_{0+}^\beta \sigma(t) \Big|_{t=1} + I_{0+}^\beta \sigma(t). \tag{21}$$

Since $\sigma \in C[0, 1]$, applying ${}^c D_{0+}^\beta$ to both sides of (21) and using Lemma 1, we can obtain

$${}^c D_{0+}^\beta z(t) = {}^c D_{0+}^\beta I_{0+}^\beta \sigma(t) = \sigma(t). \tag{22}$$

On the other hand, multiplying (10) by $h(t)$ and integrating on $[0, 1]$, we have

$$\begin{aligned} \int_0^1 z(t)h(t)dt &= \frac{-1}{1-\sigma_2} \int_0^1 h(t)dt \int_0^1 \int_0^1 h(s)G_\beta(s, \tau)\sigma(\tau)d\tau ds \\ &- \int_0^1 h(t) \int_0^1 G_\beta(t, s)\sigma(s)ds dt = -\frac{\sigma_2}{1-\sigma_2} \int_0^1 \int_0^1 h(s)G_\beta(s, \tau)\sigma(\tau) \\ &\cdot d\tau ds - \int_0^1 \int_0^1 h(t)G_\beta(t, s)\sigma(s) \\ &\cdot ds dt = -\frac{1}{1-\sigma_2} \int_0^1 \int_0^1 h(s)G_\beta(s, \tau)\sigma(\tau)d\tau ds. \end{aligned} \tag{23}$$

Hence, (10) can be written as

$$z(t) = -\int_0^1 G_\beta(t, s)\sigma(s)ds + \int_0^1 h(s)z(s)ds. \tag{24}$$

Since $G_\beta(0, s) = G_\beta(1, s) = 0$, we can also have that

$$z(0) = z(1) = \int_0^1 h(s)z(s)ds \tag{25}$$

Therefore, we can know that $z(t)$ is a solution of problem (9) and (9) has a unique solution which is given by (10). The proof is completed.

Lemma 4. Let $f \in C([0, 1] \times \mathbf{R}, \mathbf{R})$, then p -Laplacian integral boundary value problem (1) has a unique solution which is given by

$$\begin{aligned} x(t) &= \int_0^1 G_\alpha(t, s)\varphi_q \left(\frac{1}{1-\sigma_2} \int_0^1 \int_0^1 h(\gamma)G_\beta(\gamma, \tau)f(\tau, x(\tau)) \right. \\ &\cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma)f(\gamma, x(\gamma))d\gamma \Big) \\ &\cdot ds + \frac{1-t}{1-\sigma_1} \int_0^1 \int_0^1 G_\alpha(s, \varsigma)g(\varsigma)\varphi_q \\ &\cdot \left(\frac{1}{1-\sigma_2} \cdot \int_0^1 \int_0^1 h(\gamma)G_\beta(\gamma, \tau)f(\tau, x(\tau))d\tau d\gamma \right. \\ &\left. + \int_0^1 G_\beta(s, \gamma)f(\gamma, x(\gamma))d\gamma \right) ds d\varsigma, \end{aligned} \tag{26}$$

where q is the number that satisfies $1/p + 1/q = 1$.

Proof. The proof of this lemma is divided into two steps.

(Step 1) Let $r \in C[0, 1]$ and consider the boundary value problem

$$\begin{cases} {}^c D_{0+}^\alpha x(t) = r(t), & 0 < t < 1, \\ x(0) = \int_0^1 g(s)x(s)ds, & x(1) = 0. \end{cases} \tag{27}$$

By using Lemma 2, we have that

$$x(t) = d_0 + d_1 t + I_{0+}^\alpha r(t). \tag{28}$$

From (28) and boundary condition of (27) we can obtain

$$\begin{aligned} d_0 &= x(0) = \int_0^1 g(s)x(s)ds, \\ d_1 &= -d_0 - I_{0+}^\alpha r(t) \Big|_{t=1} = -\int_0^1 g(s)x(s)ds - I_{0+}^\alpha r(t) \Big|_{t=1}. \end{aligned} \tag{29}$$

Then, (28) can be written as

$$\begin{aligned} x(t) &= d_0 + d_1 t + I_{0+}^\alpha r(t) = \int_0^1 g(s)x(s) \\ &\cdot ds + \left(-\int_0^1 g(s)x(s)ds - I_{0+}^\alpha r(t) \Big|_{t=1} \right) t + I_{0+}^\alpha r(t) \\ &= I_{0+}^\alpha r(t) - I_{0+}^\alpha r(t) \Big|_{t=1} t + (1-t) \int_0^1 g(s)x(s) \\ &\cdot ds = \frac{1}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} r(s)ds - \int_0^1 t(1-s)^{\alpha-1} r(s)ds \right] \\ &+ (1-t) \int_0^1 g(s)x(s)ds = -\int_0^1 G_\alpha(t, s)r(s) \\ &\cdot ds + (1-t) \int_0^1 g(s)x(s)ds. \end{aligned} \tag{30}$$

In a similar way to the proof of Lemma 3, it can be easily seen that

$$\int_0^1 g(s)x(s)ds = \frac{-1}{1-\sigma_1} \int_0^1 \int_0^1 g(s)G_\alpha(s, \tau)r(\tau)d\tau ds. \tag{31}$$

Hence, we have that the unique solution of (27) is given by

$$x(t) = -\int_0^1 G_\alpha(t, s)r(s)ds - \frac{1-t}{1-\sigma_1} \int_0^1 \int_0^1 g(s)G_\alpha(s, \tau)r(\tau)d\tau ds. \tag{32}$$

Conversely, let $x(t)$ be the function which is expressed by (32). Then, from the definition of $G_\alpha(t, s)$, (32) can be written as

$$x(t) = -tI_{0+}^\alpha r(t)|_{t=1} + I_{0+}^\alpha r(t) - \frac{1-t}{1-\sigma_1} \int_0^1 \int_0^1 g(s)G_\alpha(s, \tau)r(\tau)d\tau ds. \tag{33}$$

Since $r \in C[0, 1]$ and $1 < \alpha \leq 2$, applying ${}^cD_{0+}^\alpha$ to both sides of (33), we can obtain

$${}^cD_{0+}^\alpha x(t) = {}^cD_{0+}^\alpha I_{0+}^\alpha r(t) = r(t). \tag{34}$$

On the other hand, multiplying (32) by $g(t)$ and integrating on $[0, 1]$, we have that

$$\begin{aligned} \int_0^1 g(t)x(t)dt &= -\int_0^1 g(t) \int_0^1 G_\alpha(t, s)r(s) \cdot dsdt - \frac{1}{1-\sigma_1} \int_0^1 (1-t)g(t) \cdot dt \int_0^1 \int_0^1 g(s)G_\alpha(s, \tau)r(\tau) \cdot d\tau ds = -\int_0^1 \int_0^1 G_\alpha(t, s)g(t)r(s) \cdot dsdt - \frac{\sigma_1}{1-\sigma_1} \int_0^1 \int_0^1 g(s)G_\alpha(s, \tau)r(\tau) \cdot d\tau ds = -\left(1 + \frac{\sigma_1}{1-\sigma_1}\right) \int_0^1 \int_0^1 G_\alpha(t, s)g(t)r(s) \cdot dsdt = \frac{-1}{1-\sigma_1} \int_0^1 \int_0^1 G_\alpha(t, s)g(t)r(s)dsdt. \end{aligned} \tag{35}$$

So, we can rewrite (32) as

$$x(t) = -\int_0^1 G_\alpha(t, s)r(s)ds + (1-t) \int_0^1 g(t)x(t)dt. \tag{36}$$

Since $G_\alpha(0, s) = G_\alpha(1, s) = 0$, we can get

$$x(0) = \int_0^1 g(t)x(t)dt, \quad x(1) = 0. \tag{37}$$

Therefore, we can know that $x(t)$ is the solution of (27).

(Step 2) Now let $x(t)$ be the solution of (1). Putting $y(t) := {}^cD_{0+}^\alpha x(t)$, then we have that

$$x(t) = -\int_0^1 G_\alpha(t, s)y(s)ds - \frac{1-t}{1-\sigma_1} \int_0^1 \int_0^1 g(s)G_\alpha(s, \tau)y(\tau)d\tau ds. \tag{38}$$

Also denoting $\varphi_p(y(t))$ by $z(t)$, then by Lemma 3, we can see that that

$$z(t) = \frac{-1}{1-\sigma_2} \int_0^1 \int_0^1 h(s)G_\beta(s, \tau)f(\tau, x(\tau)) \cdot d\tau ds - \int_0^1 G_\beta(t, s)f(s, x(s))ds. \tag{39}$$

Since it is well-known that $\varphi_p^{-1} = \varphi_q$, combining (38) and (39) yields

$$\begin{aligned} x(t) &= \int_0^1 G_\alpha(t, s)\varphi_p^{-1} \left(\frac{1}{1-\sigma_2} \int_0^1 \int_0^1 h(\gamma)G_\beta(\gamma, \tau)f(\tau, x(\tau)) \cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma)f(\gamma, x(\gamma))d\gamma \right) \cdot ds + \frac{1-t}{1-\sigma_1} \int_0^1 \int_0^1 G_\alpha(\varsigma, s)g(\varsigma)\varphi_p^{-1} \cdot \left(\frac{1}{1-\sigma_2} \cdot \int_0^1 \int_0^1 h(\gamma)G_\beta(\gamma, \tau)f(\tau, x(\tau))d\tau d\gamma + \int_0^1 G_\beta(s, \gamma)f(\gamma, x(\gamma))d\gamma \right) dsd\varsigma. \end{aligned} \tag{40}$$

The proof is completed.

Remark 5. From the definition of $G_\alpha(t, s)$ and $G_\beta(t, s)$, it is easy to know that those functions are continuous in $[0, 1] \times [0, 1]$.

Lemma 6 (see [24]). *Schaefer's fixed-point theorem.* Let X be the Banach space and $T : X \rightarrow X$ be completely continuous operator. If the set $E := \{u \in X \mid u = \rho Tu, 0 < \rho < 1\}$ is bounded, then T has at least one fixed point in X .

The basic properties of the p -Laplacian operator which will be used in the following studies are listed below (see [10]).

(i) If $1 < p \leq 2$, $xy > 0$, and $|x|, |y| \geq m > 0$, then

$$|\varphi_p(x) - \varphi_p(y)| \leq (p-1)m^{p-2}|x-y| \tag{41}$$

(ii) If $p > 2$ and $|x|, |y| \leq M$, then

$$|\varphi_p(x) - \varphi_p(y)| \leq (p-1)M^{p-2}|x-y| \tag{42}$$

3. Main Results

In this section, we establish the existence and uniqueness of solutions of problem (1) by using Schaefer's fixed-point theorem and the Banach contraction mapping principle.

Let us consider the Banach space $X = C[0, 1]$ endowed with the norm $\|u\| := \max_{0 \leq t \leq 1} |u(t)|$.

Define an operator $T : X \rightarrow X$ by

$$\begin{aligned}
 Tx(t) = & \int_0^1 G_\alpha(t, s) \varphi_q \left(\frac{1}{1 - \sigma_2} \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) \right. \\
 & \cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \Big) \\
 & \cdot ds + \frac{1-t}{1-\sigma_1} \int_0^1 \int_0^1 G_\alpha(\varsigma, s) g(\varsigma) \\
 & \cdot \varphi_q \left(\frac{1}{1-\sigma_2} \cdot \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) \right. \\
 & \cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \Big) ds + \frac{1-t}{1-\sigma_1}.
 \end{aligned} \tag{43}$$

Then, Equation (26) is equivalent to the operator equation

$$x = Tx, x \in C[0, 1]. \tag{44}$$

From Lemma 4, the existence of solutions for the problem (1) refers to the existence of fixed points of Equation (44). Therefore, it is sufficient to prove the existence of fixed points of (44).

Lemma 7. *The operator T is completely continuous.*

Proof. Since $f \in C([0, 1] \times \mathbf{R}, \mathbf{R})$ and φ_q is continuous, we can know that $T : X \rightarrow X$ is continuous.

Let $\Omega \subset X$ be a bounded subset, then for any $u \in \Omega$, there exists $M_0 > 0$ such that $\|u\| \leq M_0$.

We will show that $T(\Omega)$ is relatively compact in X . Since f is a continuous function, there exists $M_f > 0$ such that $|f(t, u(t))| \leq M_f, t \in [0, 1], u \in \Omega$. Then, we have

$$\begin{aligned}
 |Tx(t)| \leq & \int_0^1 |G_\alpha(t, s)| \cdot \left| \varphi_q \left(\frac{1}{1 - \sigma_2} \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) \right. \right. \\
 & \cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \Big) \\
 & \cdot ds + \frac{1-t}{1-\sigma_1} \int_0^1 \int_0^1 |G_\alpha(\varsigma, s) g(\varsigma)| \\
 & \cdot \left| \varphi_q \left(\frac{1}{1-\sigma_2} \cdot \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) \right. \right. \\
 & \cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \Big) \Big| \\
 & \cdot dsd\varsigma \leq \int_0^1 |G_\alpha(t, s)| \\
 & \cdot \left| \varphi_q \left(\frac{1}{1-\sigma_2} \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) \right. \right. \\
 & \cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \Big) \Big| \\
 & \cdot ds + \frac{1-t}{1-\sigma_1} \int_0^1 \int_0^1 |G_\alpha(\varsigma, s) g(\varsigma)| \\
 & \cdot \varphi_q \left(\frac{1}{1-\sigma_2} \cdot \int_0^1 \int_0^1 |h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau))| \right. \\
 & \cdot |d\tau d\gamma + \int_0^1 |G_\beta(s, \gamma) f(\gamma, x(\gamma))| d\gamma \Big) dsd\varsigma.
 \end{aligned} \tag{45}$$

And since $(t, s) \in [0, 1] \times [0, 1]$, evaluating the upper bound of $|G_\beta(t, s)|$ gives

$$|G_\beta(t, s)| \leq \frac{1}{\Gamma(\beta)}. \tag{46}$$

So, we obtain

$$\begin{aligned}
 & \frac{1}{1-\sigma_2} \int_0^1 \int_0^1 |h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau))| \\
 & \cdot d\tau d\gamma + \int_0^1 |G_\beta(s, \gamma) f(\gamma, x(\gamma))| \\
 & \cdot d\gamma \leq \frac{M_f}{\Gamma(\beta)} \left(1 + \frac{\|h\|}{|1-\sigma_2|} \right) =: M_1.
 \end{aligned} \tag{47}$$

Then, we can get easily that

$$\begin{aligned}
 |Tx(t)| \leq & \int_0^1 |G_\alpha(t, s)| \cdot \varphi_q(M_1) \\
 & \cdot ds + \frac{1-t}{1-\sigma_1} \int_0^1 \int_0^1 |G_\alpha(\varsigma, s) g(\varsigma)| \cdot \varphi_q(M_1) \\
 & \cdot dsd\varsigma \leq \frac{\varphi_q(M_1)}{\Gamma(\alpha)} \left(1 + \frac{\|g\|}{|1-\sigma_1|} \right).
 \end{aligned} \tag{48}$$

In view of the definition of the p -Laplacian operator, we have that

$$\|Tx\| \leq \frac{M_1^{q-1}}{\Gamma(\alpha)} \left(1 + \frac{\|g\|}{|1-\sigma_1|} \right). \tag{49}$$

This shows that $T(\Omega)$ is uniformly bounded in X . For any $x \in \Omega, 0 \leq t_1 < t_2 \leq 1$, we have that

$$\begin{aligned}
 |Tx(t_2) - Tx(t_1)| = & \left| \int_0^1 G_\alpha(t_2, s) \right. \\
 & \cdot \varphi_q \left(\frac{1}{1-\sigma_2} \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) \right. \\
 & \cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \Big) \\
 & \cdot ds + \frac{1-t_2}{1-\sigma_1} \int_0^1 \int_0^1 G_\alpha(\varsigma, s) g(\varsigma) \\
 & \cdot \varphi_q \left(\frac{1}{1-\sigma_2} \cdot \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) \right. \\
 & \cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \Big) \\
 & \cdot dsd\varsigma - \int_0^1 G_\alpha(t_1, s) \varphi_q \left(\frac{1}{1-\sigma_2} \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) \right. \\
 & \cdot f(\tau, x(\tau)) d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \Big)
 \end{aligned}$$

$$\begin{aligned}
 & \cdot ds - \frac{1-t_1}{1-\sigma_1} \int_0^1 \int_0^1 G_\alpha(\varsigma, s) g(\varsigma) \\
 & \cdot \varphi_q \left(\frac{1}{1-\sigma_2} \cdot \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) \right. \\
 & \cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \Big) \\
 & \cdot ds d\varsigma \Big| = \left| \int_0^1 (G_\alpha(t_2, s) - G_\alpha(t_1, s)) \right. \\
 & \cdot \varphi_q \left(\frac{1}{1-\sigma_2} \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) \right. \\
 & \cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \Big) \quad (50) \\
 & \cdot ds + \frac{t_1-t_2}{1-\sigma_1} \int_0^1 \int_0^1 G_\alpha(\varsigma, s) g(\varsigma) \\
 & \cdot \varphi_q \left(\frac{1}{1-\sigma_2} \cdot \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) \right. \\
 & \cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \Big) \\
 & \cdot ds d\varsigma \leq M_1^{q-1} \cdot \int_0^1 |G_\alpha(t_2, s) - G_\alpha(t_1, s)| \\
 & \cdot ds + M_1^{q-1} \cdot \frac{t_2-t_1}{|1-\sigma_1|} \frac{\|g\|}{\Gamma(\alpha)}.
 \end{aligned}$$

And since

$$G_\alpha(t_2, s) - G_\alpha(t_1, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t_2(1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} - t_1(1-s)^{\alpha-1} + (t_1-s)^{\alpha-1}, & s \leq t_1, \\ t_2(1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} - t_1(1-s)^{\alpha-1}, & t_1 \leq s \leq t_2, \\ t_2(1-s)^{\alpha-1} - t_1(1-s)^{\alpha-1}, & s \geq t_2, \end{cases} \quad (51)$$

the integral term $\int_0^1 |G_\alpha(t_2, s) - G_\alpha(t_1, s)| ds$ can be divided into the following three parts:

$$\begin{aligned}
 & \int_0^{t_1} |G_\alpha(t_2, s) - G_\alpha(t_1, s)| \\
 & \cdot ds = \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |t_2(1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} \\
 & - (t_1(1-s)^{\alpha-1} - (t_1-s)^{\alpha-1})| \\
 & \cdot ds \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} ((t_2-t_1)(1-s)^{\alpha-1} + ((t_2-s)^{\alpha-1} \\
 & - (t_1-s)^{\alpha-1}) ds \leq \frac{1}{\Gamma(\alpha)} (t_2-t_1) \int_0^{t_1} (1-s)^{\alpha-1} \\
 & \cdot ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}) \\
 & \cdot ds \leq \frac{t_2-t_1}{\Gamma(\alpha+1)} + \frac{(t_2^\alpha - t_1^\alpha) - (t_2-t_1)^\alpha}{\Gamma(\alpha+1)}. \quad (52)
 \end{aligned}$$

In a similar way to this, we can obtain

$$\begin{aligned}
 & \int_{t_1}^{t_2} |G_\alpha(t_2, s) - G_\alpha(t_1, s)| ds \leq \frac{t_2-t_1}{\Gamma(\alpha+1)} + \frac{(t_2-t_1)^\alpha}{\Gamma(\alpha+1)}, \\
 & \int_{t_2}^1 |G_\alpha(t_2, s) - G_\alpha(t_1, s)| ds \leq \frac{t_2-t_1}{\Gamma(\alpha+1)}. \quad (53)
 \end{aligned}$$

These inequalities yield

$$\begin{aligned}
 & \int_0^1 |G_\alpha(t_2, s) - G_\alpha(t_1, s)| \\
 & \cdot ds = \int_0^{t_1} |G_\alpha(t_2, s) - G_\alpha(t_1, s)| \\
 & \cdot ds + \int_{t_1}^{t_2} |G_\alpha(t_2, s) - G_\alpha(t_1, s)| \\
 & \cdot ds + \int_{t_2}^1 |G_\alpha(t_2, s) - G_\alpha(t_1, s)| \\
 & \cdot ds \leq \frac{3(t_2-t_1) + (t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)}. \quad (54)
 \end{aligned}$$

Therefore, we get

$$\begin{aligned}
 |Tx(t_2) - Tx(t_1)| & \leq M_1^{q-1} \cdot \frac{3(t_2-t_1) + (t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)} \\
 & + M_1^{q-1} \cdot \frac{t_2-t_1}{|1-\sigma_1|} \frac{\|g\|}{\Gamma(\alpha)} = \frac{M_1^{q-1}}{\Gamma(\alpha)} \left(\frac{3}{\alpha} + \frac{\|g\|}{|1-\sigma_1|} \right) \\
 & \cdot (t_2-t_1) + \frac{M_1^{q-1}}{\Gamma(\alpha+1)} \cdot (t_2^\alpha - t_1^\alpha). \quad (55)
 \end{aligned}$$

This shows that $T(\Omega)$ is equicontinuous in X . By using the Arzela-Ascoli theorem, we can see that $T(\Omega)$ is relatively compact in X . As a consequence of the above discussion, the operator $T : X \rightarrow X$ is completely continuous. The proof is completed.

Denote as follows:

$$A := \left[\frac{1}{\Gamma(\alpha)} \left(1 + \frac{\|g\|}{|1-\sigma_1|} \right) \right]^{p-1} \frac{1}{\Gamma(\beta)} \left(1 + \frac{\|h\|}{|1-\sigma_2|} \right). \quad (56)$$

In this article, the following hypotheses will be used.

(H1). There exist nonnegative functions $a, b \in C[0, 1]$ such that

- (i) $|f(t, x)| \leq a(t) + b(t)|x|^{p-1}, t \in [0, 1], x \in \mathbf{R}$
- (ii) $A\|b\| < 1$

Theorem 8. Assume that the hypothesis (H1) holds, then problem (1) has at least one solution.

Proof. Consider the following set:

$$E := \{x \in X \mid x = \rho Tx, 0 < \rho < 1\}. \tag{57}$$

For any $x \in E$, it can be easily seen that

$$\|x\| = \rho \|Tx\| < \|Tx\|. \tag{58}$$

On the other hand, from the condition (i) of the hypothesis (H1) and (49), we have that

$$\begin{aligned} \|Tx\| &\leq \frac{1}{\Gamma(\alpha)} \left(1 + \frac{\|g\|}{|1 - \sigma_1|}\right) \\ &\cdot \left(\frac{1}{\Gamma(\beta)} \left(1 + \frac{\|h\|}{|1 - \sigma_2|}\right) (\|a\| + \|b\| \|x\|^{p-1})\right)^{q-1}. \end{aligned} \tag{59}$$

From (58), we get

$$\begin{aligned} \|x\| &< \frac{1}{\Gamma(\alpha)} \left(1 + \frac{\|g\|}{|1 - \sigma_1|}\right) \\ &\cdot \left(\frac{1}{\Gamma(\beta)} \left(1 + \frac{\|h\|}{|1 - \sigma_2|}\right) (\|a\| + \|b\| \|x\|^{p-1})\right)^{q-1}. \end{aligned} \tag{60}$$

Since $1/p + 1/q = 1$ implies $(p - 1)(q - 1) = 1$, we obtain

$$\|x\|^{p-1} < A (\|a\| + \|b\| \|x\|^{p-1}). \tag{61}$$

Therefore, we have that

$$\|x\|^{p-1} (1 - A \|b\|) < A \|a\|, \tag{62}$$

and by using the condition (ii) of the hypothesis (H1), we can see that

$$\|x\| < \left(\frac{A \|a\|}{1 - A \|b\|}\right)^{q-1}. \tag{63}$$

So, we can know that E is bounded. In view of Schaefer's fixed-point theorem (Lemma 6), the operator $T : X \rightarrow X$ has at least one fixed point which is the solution of the problem (1). The proof is completed.

Here, put $r := (A \|a\| / (1 - A \|b\|))^{q-1}$ and list more hypotheses to obtain the uniqueness results for our problem.

(H2). There exists a real number $L > 0$ such that

$$|f(t, x) - f(t, y)| < L |x - y|, \quad t \in [0, 1], x, y \in [-r, r]. \tag{64}$$

For the readers' convenience, denote as follows:

$$\begin{aligned} M_0 &:= \left(1 + \frac{\|h\|}{|1 - \sigma_2|}\right) (\|a\| + \|b\| r^{p-1}), \\ \theta &:= \frac{L(q-1)M_0^{q-2}}{\Gamma(\alpha)\Gamma(\beta)} \left(1 + \frac{\|g\|}{|1 - \sigma_1|}\right) \left(1 + \frac{\|h\|}{|1 - \sigma_2|}\right). \end{aligned} \tag{65}$$

Theorem 9. *Let $1 < p < 2$. Assume that the hypotheses (H1) and (H2) are satisfied and $0 < \theta < 1$, then problem (1) has a unique solution.*

Proof. Put $B = \{x \in X \mid \|x\| \leq r\}$.

Firstly, we will show that $T(B) \subset B$. In fact, from the hypothesis (H1) and (49), for any $x \in B$, we have that

$$\begin{aligned} \|Tx\| &\leq \frac{1}{\Gamma(\alpha)} \left(1 + \frac{\|g\|}{|1 - \sigma_1|}\right) \left(\frac{1}{\Gamma(\beta)} \left(1 + \frac{\|h\|}{|1 - \sigma_2|}\right)\right. \\ &\cdot (\|a\| + \|b\| \|x\|^{p-1})\left.\right)^{q-1} \\ &= \left[\left(\frac{1}{\Gamma(\alpha)} \left(1 + \frac{\|g\|}{|1 - \sigma_1|}\right)\right)^{p-1} \frac{1}{\Gamma(\beta)}\right. \\ &\cdot \left.\left(1 + \frac{\|h\|}{|1 - \sigma_2|}\right)\right]^{q-1} (\|a\| + \|b\| \|x\|^{p-1})^{q-1} \\ &\leq A^{q-1} (\|a\| + \|b\| r^{p-1})^{q-1} \\ &= A^{q-1} \left(\|a\| + \frac{A \|a\| \|b\|}{1 - A \|b\|}\right)^{q-1} \\ &= \left(\frac{A \|a\|}{1 - A \|b\|}\right)^{q-1} = r. \end{aligned} \tag{66}$$

Next, we will prove the uniqueness of solutions for problem (1).

For any $x \in B$ and any $t \in [0, 1]$, by the hypothesis (H1), we get

$$\begin{aligned} &\frac{1}{|1 - \sigma_2|} \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) \\ &\cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) \\ &\cdot d\gamma \leq \frac{\|h\|}{|1 - \sigma_2|} \int_0^1 \int_0^1 G_\beta(\gamma, \tau) f(\tau, x(\tau)) \\ &\cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) \\ &\cdot d\gamma \leq \frac{\|h\|}{|1 - \sigma_2|} \int_0^1 \int_0^1 G_\beta(\gamma, \tau) (a(\tau) + b(\tau) r^{p-1}) \\ &\cdot d\tau d\gamma + \int_0^1 G_\beta(s, \gamma) (a(\gamma) + b(\gamma) r^{p-1}) \\ &\cdot d\gamma \leq \left(\frac{\|h\|}{|1 - \sigma_2|} + 1\right) (\|a\| + \|b\| r^{p-1}) = M_0. \end{aligned} \tag{67}$$

Since $1 < p < 2$, we can see $q > 2$. Hence, from (H2) and one basic property of p -Laplacian operator (42), for any $x, y \in B$ and any $t \in [0, 1]$, we have that

$$\begin{aligned}
 |Tx(t) - Ty(t)| &= \left| \int_0^1 G_\alpha(t, s) \right. \\
 &\cdot \varphi_q \left(\frac{1}{1 - \sigma_2} \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) d\tau d\gamma \right. \\
 &+ \left. \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \right) ds + \frac{1-t}{1-\sigma_1} \int_0^1 \int_0^1 G_\alpha(\varsigma, s) g(\varsigma) \\
 &\cdot \varphi_q \left(\frac{1}{1 - \sigma_2} \cdot \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) \right. \\
 &\cdot d\tau d\gamma + \left. \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \right) \\
 &\cdot ds d\varsigma - \int_0^1 G_\alpha(t, s) \varphi_q \left(\frac{1}{1 - \sigma_2} \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, y(\tau)) \right. \\
 &\cdot d\tau d\gamma + \left. \int_0^1 G_\beta(s, \gamma) f(\gamma, y(\gamma)) d\gamma \right) \\
 &\cdot ds - \frac{1-t}{1-\sigma_1} \int_0^1 \int_0^1 G_\alpha(\varsigma, s) g(\varsigma) \varphi_q \left(\frac{1}{1 - \sigma_2} \cdot \int_0^1 \int_0^1 h(\gamma) \right. \\
 &\cdot G_\beta(\gamma, \tau) f(\tau, y(\tau)) d\tau d\gamma + \left. \int_0^1 G_\beta(s, \gamma) f(\gamma, y(\gamma)) d\gamma \right) \\
 &\cdot ds d\varsigma \Big| \leq \left| \int_0^1 G_\alpha(t, s) \left(\varphi_q \left(\frac{1}{1 - \sigma_2} \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) \right. \right. \right. \\
 &\cdot f(\tau, x(\tau)) d\tau d\gamma + \left. \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \right) \\
 &\cdot ds - \varphi_q \left(\frac{1}{1 - \sigma_2} \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, y(\tau)) \right. \\
 &\cdot d\tau d\gamma + \left. \int_0^1 G_\beta(s, \gamma) f(\gamma, y(\gamma)) d\gamma \right) \Big| \\
 &\cdot ds + \frac{1-t}{1-\sigma_1} \int_0^1 \int_0^1 G_\alpha(\varsigma, s) g(\varsigma) \\
 &\cdot \left(\varphi_q \left(\frac{1}{1 - \sigma_2} \cdot \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, x(\tau)) \right. \right. \\
 &\cdot d\tau d\gamma + \left. \int_0^1 G_\beta(s, \gamma) f(\gamma, x(\gamma)) d\gamma \right) \\
 &\cdot - \varphi_q \left(\frac{1}{1 - \sigma_2} \cdot \int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) f(\tau, y(\tau)) \right. \\
 &\cdot d\tau d\gamma + \left. \int_0^1 G_\beta(s, \gamma) f(\gamma, y(\gamma)) d\gamma \right) \\
 &\cdot ds d\varsigma \Big| \leq \frac{(q-1)M_0^{q-2}}{|1-\sigma_2|} \int_0^1 \left| G_\alpha(t, s) \left(\int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) \right. \right. \\
 &\cdot (f(\tau, x(\tau)) - f(\tau, y(\tau))) d\tau d\gamma \Big| \\
 &\cdot ds + (q-1)M_0^{q-2} \int_0^1 \left| G_\alpha(t, s) \int_0^1 G_\beta(s, \gamma) \right. \\
 &\cdot (f(\gamma, x(\gamma)) - f(\gamma, y(\gamma))) d\gamma \Big| \\
 &\cdot ds + \frac{(1-t)(q-1)M_0^{q-2}}{|1-\sigma_1| \cdot |1-\sigma_2|} \cdot \int_0^1 \int_0^1 \left| G_\alpha(\varsigma, s) g(\varsigma) \left(\int_0^1 \int_0^1 h(\gamma) \right. \right. \\
 &\cdot G_\beta(\gamma, \tau) (f(\tau, x(\tau)) - f(\tau, y(\tau))) d\tau d\gamma \Big| \\
 &\cdot ds d\varsigma + \frac{(1-t)(q-1)M_0^{q-2}}{|1-\sigma_1|} \int_0^1 \int_0^1 \left| G_\alpha(\varsigma, s) g(\varsigma) \right. \\
 &\cdot \left. \left(\int_0^1 G_\beta(s, \gamma) (f(\gamma, x(\gamma)) - f(\gamma, y(\gamma))) d\gamma \right) \right| \\
 &\cdot ds d\varsigma \leq \frac{L(q-1)M_0^{q-2}}{|1-\sigma_2|} \int_0^1 G_\alpha(t, s) \\
 &\cdot \left(\int_0^1 \int_0^1 h(\gamma) G_\beta(\gamma, \tau) |x(\tau) - y(\tau)| d\tau d\gamma \right) \\
 &\cdot ds + L(q-1)M_0^{q-2} \int_0^1 G_\alpha(t, s) \int_0^1 G_\beta(s, \gamma) |x(\gamma) - y(\gamma)| \\
 &\cdot d\gamma ds + \frac{L(1-t)(q-1)M_0^{q-2}}{|1-\sigma_1| \cdot |1-\sigma_2|} \int_0^1 \int_0^1 G_\alpha(\varsigma, s) g(\varsigma) \int_0^1 \int_0^1 h(\gamma) \\
 &\cdot G_\beta(\gamma, \tau) |x(\tau) - y(\tau)| d\tau d\gamma ds d\varsigma + \frac{L(1-t)(q-1)M_0^{q-2}}{|1-\sigma_1|} \\
 &\cdot \int_0^1 \int_0^1 G_\alpha(\varsigma, s) g(\varsigma) \int_0^1 G_\beta(s, \gamma) |x(\gamma) - y(\gamma)| \\
 &\cdot d\gamma ds d\varsigma \leq \frac{L(q-1)M_0^{q-2}}{\Gamma(\alpha)\Gamma(\beta)} \\
 &\cdot \left(\frac{\|h\|}{|1-\sigma_2|} + 1 + \frac{\|g\|}{|1-\sigma_1|} \frac{\|h\|}{|1-\sigma_2|} + \frac{\|g\|}{|1-\sigma_1|} \right) \\
 &\cdot \|x - y\| = \theta \|x - y\|. \tag{68}
 \end{aligned}$$

This means that

$$\|Tx - Ty\| \leq \theta \|x - y\|. \tag{69}$$

Since $0 < \theta < 1$, we can see that $T : B \rightarrow B$ is a contraction mapping. By means of the Banach contraction mapping principle, we can prove that T has a unique fixed point in B . That is, problem (1) has a unique solution. The proof is completed.

4. Examples

The following examples are concerned with the illustration of Theorem 8 and Theorem 9.

Example 1. Consider the following integral boundary value problem:

$$\begin{cases}
 {}^c D_{0+}^{1.8} \varphi_{1.2} \left({}^c D_{0+}^{1.5} x(t) \right) = 1 + \frac{1}{3} \sin t + \frac{e^{-t}}{\sqrt{10+t}} x^{0.2}(t), & t \in (0, 1), \\
 x(0) = \int_0^1 (1 + e^{-s}) x(s) ds, & x(1) = 0, \\
 \varphi_{1.2} \left({}^c D_{0+}^{1.5} x(0) \right) = \varphi_{1.2} \left({}^c D_{0+}^{1.5} x(1) \right) = \int_0^1 \frac{1}{\sqrt{100+s^2}} \varphi_{1.2} \left({}^c D_{0+}^{1.5} x(s) \right) ds.
 \end{cases} \tag{70}$$

The problem (70) can be rated as the boundary value problem where $\alpha = 1.5, \beta = 1.8, p = 1.2$,

$$\begin{aligned}
 f(t, x) &= 1 + (1/3) \sin t + (e^{-t}/\sqrt{100+t})x^{0.2}, \quad a(t) = 1 + (1/3) \\
 &\sin t, \quad b(t) = (e^{-t}/\sqrt{10+t}), \quad g(t) = 1 + e^{-t}, \quad \text{and} \quad h(t) = (1/\sqrt{100+t^2}).
 \end{aligned}$$

By simple calculation, we have $\|a\| = 4/3, \|b\| = 1/\sqrt{10}, \|g\| = 2, \|h\| = 1/10,$

$\sigma_1 = \int_0^1 (1-s)g(s)ds \approx 0.87 \neq 1,$ and $\sigma_2 = \int_0^1 h(s)ds \approx 0.1 \neq 1.$

So, testing whether the hypothesis (H1) holds or not, we get

$$A = \left[\frac{1}{\Gamma(\alpha)} \left(1 + \frac{\|g\|}{|1-\sigma_1|} \right) \right]^{p-1} \frac{1}{\Gamma(\beta)} \left(1 + \frac{\|h\|}{|1-\sigma_2|} \right) \quad (71)$$

$$\approx 2.13, A\|b\| \approx 0.67 < 1.$$

Therefore, by Theorem 8, the problem (70) has at least one solution.

Example 2. Consider the boundary value problem

$$\begin{cases} {}^c D_{0+}^{1.7} \varphi_{1.5} \left({}^c D_{0+}^{1.3} x(t) \right) = 0.4t + 0.1 \sin(x(t)), & t \in (0, 1), \\ x(0) = \int_0^1 s x(s) ds, & x(1) = 0, \\ \varphi_{1.5} \left({}^c D_{0+}^{1.3} x(0) \right) = \varphi_{1.5} \left({}^c D_{0+}^{1.3} x(1) \right) = \int_0^1 \frac{1}{10+s^2} \varphi_{1.5} \left({}^c D_{0+}^{1.3} x(s) \right) ds. \end{cases} \quad (72)$$

Comparing with the problem (1), it can be easily seen that $\alpha = 1.3, \beta = 1.7, p = 1.5, f(t, x) = 0.4t + 0.1 \sin x, a(t) = 0.4t, b(t) = 0.1, g(t) = t$, and $h(t) = 1/(10 + t^2)$.

Since $\|a\| = 0.4, \|b\| = 0.1, \|g\| = 1, \|h\| = 0.1, \sigma_1 = \int_0^1 s(1-s) ds \approx 0.17 \neq 1$, and $\sigma_2 = \int_0^1 (1/10 + s^2) ds \approx 0.1 \neq 1$, we obtain

$$|f(t, x)| \leq 0.4t + 0.1|\sin x| \leq 0.4t + 0.1 \cdot \min\{1, |x|\} \leq 0.4t + 0.1 \cdot |x|^{0.5},$$

$$A = \left[\frac{1}{\Gamma(\alpha)} \left(1 + \frac{\|g\|}{|1-\sigma_1|} \right) \right]^{p-1} \frac{1}{\Gamma(\beta)} \left(1 + \frac{\|h\|}{|1-\sigma_2|} \right) \approx 1.91,$$

$$A\|b\| \approx 0.19 < 1. \quad (73)$$

So, we can see that the hypothesis (H1) is satisfied. Calculating the radius r , we have

$$r = \left(\frac{A\|a\|}{1 - A\|b\|} \right)^{q-1} \approx 0.9. \quad (74)$$

Since $r < 1$, for any $t \in [0, 1]$ and any $x, y \in [-r, r]$, we get

$$|f(t, x) - f(t, y)| = 0.1|\sin x - \sin y| \leq 0.1|x - y|,$$

$$M_0 = \left(1 + \frac{\|h\|}{|1-\sigma_2|} \right) (\|a\| + \|b\| r^{p-1}) \approx 0.55,$$

$$\theta = \frac{L(q-1)M_0^{q-2}}{\Gamma(\alpha)\Gamma(\beta)} \left(1 + \frac{\|g\|}{|1-\sigma_1|} \right) \left(1 + \frac{\|h\|}{|1-\sigma_2|} \right) \approx 0.33 < 1. \quad (75)$$

From the above discussion, we can see that the hypothesis (H2) holds. Therefore, it follows by Theorem 9 that the problem (72) has a unique solution.

Data Availability

No datasets are generated or analyzed during this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Authors' Contributions

All authors carried out the proof and conceived of the study. All authors read and approved the final manuscript.

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