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Research Article

A Cyclic Method for Solutions of a Class of Split Variational Inequality Problem in Banach Space

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In this paper, a cyclic algorithm for approximating a class of split variational inequality problem is introduced and studied in some Banach spaces. A strong convergence theorem is proved. Some applications of the theorem are presented. The results presented here improve, unify, and generalize certain recent results in the literature.

1. Introduction

Let C be a nonempty closed and convex subset of a real Banach space E, with dual E^*. Then, a mapping $T: C \longrightarrow E$ is said to be

(1) Nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in C$.

(2) Demiclosed at zero if whenever a sequence $\{v_n\}$ in C converges weakly to u and $\{v_n - Tv_n\}$ converges strongly to 0, then $u \in F(T)$.

(3) L-Lipschitz continuous on E if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \text{for all } x, y \in C. \tag{1}$$

A mapping $T: C \longrightarrow E^*$ is said to be

(1) Monotone if

$$\langle x - y, Tx - Ty \rangle \geq 0, \quad \text{for all } x, y \in C. \tag{2}$$

(2) α-inverse strongly monotone if

$$\langle x - y, Tx - Ty \rangle \geq \alpha\|Tx - Ty\|^2, \quad \text{for any } x, y \in C. \tag{3}$$

(3) Strongly monotone if

$$\langle x - y, Tx - Ty \rangle \geq \alpha\|x - y\|^2, \quad \text{for any } x, y \in C. \tag{4}$$

Problem of the type finding $u \in C$ such that

$$\langle v - u, Tu \rangle \geq 0, \quad \text{for all } v \in C, \tag{5}$$

is called a variational inequality problem, and the set of solution of such problem is denoted by $\mathrm{VI}(C, T)$.

Variational inequality problems have played a crucial role in the study of several problems arising in physics, finance, economics, network analysis, optimization, medical image and structural analysis, and so on (see, for example, [1–5]). Variational inequality problems were formulated in the late 1960's by Lions and Stampacchia [6]. Since then, various iterative algorithms for approximating solutions of such problems have been proposed by numerous researchers (see, for example, [7–12, 27]) and the references therein.

In 1976, Korpelevch [14] introduced the following extragradient method for solving the variational inequality problem when the operator T is monotone and L-Lipschitz continuous in a finite dimensional Euclidean space $\mathbb{R}^n$,

$$x_{n+1} = P_C\left(x_n - \lambda f\left(P_C\left(x_n - \lambda T x_n\right)\right)\right), \quad (6)$$

for each $n \in \mathbb{N}$, $\lambda \in (0, (1/L))$.

The split feasibility problem in the finite dimensional Hilbert space was first introduced by Censor and Elfving [15] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction.

Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be two nonempty closed convex subset of H_1 and H_2, respectively. The split feasibility problem is to find

$$u \in C \text{ such that } \mathrm{Au} \in Q. \quad (7)$$

Assuming that the split feasibility problem is consistent (i.e., (7) has a solution), it is easy to see that $x \in C$ solves (7) if and only if it solves the fixed point equation:

$$x = P_C\left(I + \gamma A^*\left(P_Q - I\right)A\right)x, \quad x \in C, \quad (8)$$

where P_C and P_Q are the orthogonal projections onto C and Q, respectively, $\gamma > 0$, and A^* is the adjoint of A. To solve (8), Byrne [16] proposed the CQ algorithm which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C\left(I + \gamma A^*\left(P_Q - I\right)A\right)x_n, \quad (9)$$

for each $n \in \mathbb{N}$, where $\gamma \in (0, (2/\lambda))$, λ being the spectral radius of the operator A^*A.

In 2010, Censor et al. [17] considered a new variational problem called split variational inequality problem (SVIP). It entails finding a solution of one variational inequality problem whose image under a bounded linear transformation is a solution of another variational inequality problem. The SVIP is formulated as

$$\text{find } u \in \mathrm{VI}(C, f) \text{ such that } Au \in \mathrm{VI}(Q, g), \quad (10)$$

where C and Q are the nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and A: $H_1 \longrightarrow H_2$ is a bounded linear operator. They constructed the following iterative algorithm to solve such problem and proved a strong convergence theorem in a Hilbert space:

$$x_{n+1} = P_C(I - \lambda f)\left(x_n + \gamma A^*\left(P_Q(I - \lambda g) - I\right)Ax_n\right), \quad n \in \mathbb{N}, \quad (11)$$

where $\gamma \in (0, (1/L))$, L is the spectral radius of the operator A^*A, and A^* is the adjoint of A.

One can easily observe that split variational inequality has the split feasibility problem as a special case.

Recently, Tian and Jiang [12], based on the work of Censor et al. [17], considered a class of SVIP which is to find

$$x \in \mathrm{VI}(C, f) \text{ such that } Ax \in F(T), \quad (12)$$

where C is a nonempty closed convex subset of a real Hilbert space H_1, $f : C \longrightarrow H_1$ is a monotone and k-Lipschitz continuous map, $A : H_1 \longrightarrow H_2$ is a bounded linear map, and $T : H_2 \longrightarrow H_2$ is a nonexpansive map. They proposed the following algorithm by combining the Korpelevich extragradient method and Byrne CQ algorithm:

$$\begin{cases} x_1 = x \in C; \\ y_n = P_C\left(x_n - \gamma_n A^*(I - T)Ax_n\right); \\ t_n = P_C\left(y_n - \lambda_n f y_n\right); \\ x_{n+1} = P_C\left(y_n - \lambda_n f t_n\right). \end{cases} \quad (13)$$

They obtained the following result.

Theorem 1 (see [12]). *Let H_1 and H_2 be real Hilbert spaces. Let C be a nonempty, closed and convex subset of H_1, $A : H_1 \longrightarrow H_2$ be a bounded linear operator such that $A \neq 0$, $f : C \longrightarrow H_1$ be a monotone and k-Lipschitz continuous map, and $T : H_2 \longrightarrow H_2$ be a nonexpansive map. Setting $\Gamma = \{z \in VI(C, f) : Az \in F(T)\}$, assume that $\Gamma \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by (13), where $\{\gamma_n\} \subset [a, b]$, $a, b \in (0, (1/\|A\|^2))$, and $\lambda_n \in (0, (1/k))$. Then, the sequence $\{x_n\}$ converges weakly to a point $z \in \Gamma$.*

Remark 1. Inspired by the results of Tian and Jiang [12], the authors raised the following motivational questions:

Q1. Can the result of Tian and Jiang hold in a more general setting of Banach space than Hilbert?

Q2. Can the result also be proved for a common fixed point of finite family of nonexpansive mapping?

Q3. Can strong convergence theorem be proved?

In this paper, the above questions are answered in affirmative. We study a cyclic algorithm in the setting of uniformly smooth which is also 2-uniformly convex real Banach space and 2-uniformly smooth real Banach space and prove its strong convergence to a solution of a variational inequality problem for a monotone K-Lipschitz continuous map whose image under a bounded linear operator is a common fixed point of a finite family of nonexpansive maps. Our theorems improve and extend the results of Tian and Jiang [12].

2. Preliminaries

The duality map of a Banach space E has the following properties:

(1) If E is a reflexive, strictly convex, and smooth real Banach space, then J is single-valued and bijective. In this case, the inverse $J^{-1} : E^* \longrightarrow E$ is given by $J^{-1} = J^*$ with J^* being the duality mapping of E^*.

(2) In a Hilbert space H, the duality map J and its inverse J^{-1} are the identity maps on H.

(3) If E is uniformly smooth and uniformly convex, then the dual space E^* is also uniformly smooth and uniformly convex and the normalized duality map J and its inverse, J^{-1}, are both uniformly continuous on bounded sets.

Let E be a smooth real Banach space and $\phi : E \times E \longrightarrow \mathbb{R}$ be defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \quad (14)$$

It is easy to see from the definition of ϕ that, in a real Hilbert space H, equation (14) reduces to $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in H$.

Furthermore, given $x, y, z \in E$ and $\tau \in (0, 1)$, we have the following properties (see, for example, [18]):

P1: $(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2$,

P2: $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle z - x, Jy - Jz\rangle$,

P3: $\phi(\tau x + (1 - \tau)y, z) \le \tau\phi(x, z) + (1 - \tau)\phi(y, z)$.

Definition 1. Let E be a smooth, strictly convex, and reflexive real Banach space and let C be a nonempty, closed, and convex subset of E. The map $\Pi_C : E \longrightarrow C$ defined by $\tilde{x} = \Pi_C(x) \in C$ such that $\phi(\tilde{x}, x) = \inf_{y\in C}\phi(y, x)$ is called the generalized projection of E onto C. Clearly, in a real Hilbert space H, the generalized projection Π_C coincides with the metric projection P_C from H onto C.

Definition 2. Let E_1 and E_2 be two reflexive, strictly convex, and smooth Banach spaces. The collection of mappings $A : E_1 \longrightarrow E_2$ is linear, and continuous is a normed linear space with norm defined by $\|A\| = \sup_{\|x\|\le 1}\|Ax\|$. The dual operator $A^*: E_2^* \longrightarrow E_1^*$ defined by $\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle \forall x \in E_1$, $y^* \in E_2^*$ is called the adjoint operator of A. The adjoint operator A^* has the property $\|A^*\| = \|A\|$.

Lemma 1 (see [19]). *Let C be a nonempty closed and convex subset of a smooth, strictly convex, and reflexive real Banach space E. Then,*

(1) If $x \in E$, then $\tilde{x} = \Pi_C x$ if and only if $\langle \tilde{x} - y, Jx - J\tilde{x}\rangle \ge 0$, for all $y \in C$,

(2) $\phi(y, \tilde{x}) + \phi(\tilde{x}, x) \le \phi(y, x)$, for all $x \in E$, $y \in C$.

Lemma 2 (see [20]). *Let E be q-uniformly smooth Banach space. Then, there exists a constant $d_q > 0$ such that*

$$\|x + y\|^q \le \|x\|^q + q\langle y, jx\rangle + d_q\|y\|^q. \quad (15)$$

Lemma 3 (see [21]). *Let E be a 2-uniformly convex and smooth real Banach space. Then, there exists a positive constant α such that*

$$\alpha\|x - y\|^2 \le \phi(x, y), \quad \forall x, y \in E. \quad (16)$$

Lemma 4 (see [22]). *Let C be a nonempty closed and convex subset of a reflexive space E and f, a monotone, and hemicontinuous map of C into E^*. Let $B \subset E \times E^*$ be an operator defined by*

$$Bu = \begin{cases} fu + N_C(u), & \text{if } u \in C, \\ \varnothing, & \text{if } u \notin C, \end{cases} \quad (17)$$

where $N_C(u)$ is defined as

$$N_C(u) = \{w^* \in E^* : \langle u - z, w^*\rangle \ge 0, \forall z \in C\}. \quad (18)$$

Then, B is maximal monotone and $B^{-1}0 = \mathrm{VI}(C, f)$.

Lemma 5 (see [23]). *Let E be a uniformly convex and smooth real Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\phi(x_n, y_n) = 0$, then $\lim_{n\longrightarrow\infty}\|x_n - y_n\| = 0$.*

3. Main Results

Theorem 2. *Let E_1 be a uniformly smooth and 2-uniformly convex real Banach space and E_2 a 2-uniformly smooth real Banach space with smoothness constant $d_2 \in (0, 1)$. Let C be a nonempty, closed, and convex subset of E_1. Let $f : C \longrightarrow E_1^*$ be a monotone and k-Lipschitz continuous map, and $A : E_1 \longrightarrow E_2$ be a bounded linear operator with its adjoint A^* such that $A \ne 0$. Let $T_i : E_2 \longrightarrow E_2$, $i = 1, 2, \ldots, m$ be nonexpansive mappings. Setting $\Gamma = \{z \in VI(C, f) : Az \in \cap_{i=1}^m F(T_i)\}$ and assuming $\Gamma \ne \varnothing$. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_1 = x \in C = C_1, \\ y_n = J^{-1}\left(Jx_n - \gamma A^* J_2\left(I - T_{[n]}\right)Ax_n\right), \quad \text{where } [n] = n \bmod m, \\ t_n = \Pi_C J^{-1}(Jy_n - \lambda f y_n), \\ z_n = \Pi_C J^{-1}(Jy_n - \lambda f t_n), \\ C_{n+1} = \{v \in C_n : \phi(v, z_n) \le \phi(v, y_n) \le \phi(v, x_n),\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases} \quad (19)$$

where $\gamma \in [a, b]$, $a, b \in (0, (1/d_2^*\|A\|^2))$, d_2^* being the smoothness constant of E_1^* as in Lemma 2, and $\lambda \in (0, (\alpha/k))$, α being a positive constant as in Lemma 3. Then, the sequence $\{x_n\}$ converges to a point $z \in \Gamma$.

Proof. We divide the proof into five steps. □

Step 1. We show that C_n is closed and convex for any $n \ge 1$.

Since $C = C_1$, C_1 is closed and convex.

Assume C_n is closed and convex for some $n \ge 1$. Since for any $v \in C_n$,

$$\phi(v, y_n) \le \phi(v, x_n) \Longleftrightarrow 2\langle v, Jx_n - Jy_n\rangle \le \|x_n\|^2 - \|y_n\|^2,$$
$$\phi(v, z_n) \le \phi(v, y_n) \Longleftrightarrow 2\langle v, Jy_n - Jz_n\rangle \le \|y_n\|^2 - \|z_n\|^2, \quad (20)$$

we have that C_{n+1} is closed and convex. Therefore, C_n is closed and convex for any $n \ge 1$.

Step 2. We prove $\Gamma \subset C_n$ for any $n \ge 1$.

For $n = 1$, $\Gamma \subset C = C_1$.

Assume $\Gamma \subset C_n$ for some $n \ge 1$. Let $u \in \Gamma$, then

$$\begin{aligned}\phi(u,y_n) &= \phi\left(u,J^{-1}\left(Jx_n-\gamma A^*J_2\left(I-T_{[n]}\right)Ax_n\right)\right)\\ &=\|u\|^2+\left\|J^{-1}\left(Jx_n+\gamma A^*J_2\left(T_{[n]}-I\right)Ax_n\right)\right\|^2-2\langle u,Jx_n+\gamma A^*J_2\left(T_{[n]}-I\right)Ax_n\rangle\\ &=\|u\|^2+\left\|Jx_n+\gamma A^*J_2\left(T_{[n]}-I\right)Ax_n\right\|^2-2\langle u,Jx_n\rangle-2\gamma\langle Au,J_2\left(T_{[n]}-I\right)Ax_n\rangle.\end{aligned} \tag{21}$$

From the fact that E_1^* is 2-uniformly smooth, we have

$$\begin{aligned}\phi(u,y_n) &\le \|u\|^2+\left\|Jx_n\right\|^2+2\gamma\langle x_n,A^*J_2\left(T_{[n]}-I\right)Ax_n\rangle\\ &+d_2^*\gamma^2\|A\|^2\left\|\left(T_{[n]}-I\right)Ax_n\right\|^2-2\langle u,Jx_n\rangle-2\gamma\langle Au,J_2\left(T_{[n]}-I\right)Ax_n\rangle\\ &=\phi(u,x_n)+d_2^*\gamma^2\|A\|^2\left\|\left(T_{[n]}-I\right)Ax_n\right\|^2+2\gamma\langle Ax_n-Au,J_2\left(T_{[n]}-I\right)Ax_n\rangle\\ &=\phi(u,x_n)+d_2^*\gamma^2\|A\|^2\left\|\left(T_{[n]}-I\right)Ax_n\right\|^2\\ &+2\gamma\langle Ax_n-T_{[n]}Ax_n+T_{[n]}Ax_n-Au,J_2\left(T_{[n]}-I\right)Ax_n\rangle\\ &=\phi(u,x_n)+d_2^*\gamma^2\|A\|^2\left\|\left(T_{[n]}-I\right)Ax_n\right\|^2-2\gamma\left\|\left(T_{[n]}-I\right)Ax_n\right\|^2\\ &+2\gamma\langle T_{[n]}Ax_n-Au,J_2\left(T_{[n]}-I\right)Ax_n\rangle.\end{aligned} \tag{22}$$

Using the fact that E_2 is 2-uniformly smooth and $T_{[n]}$ being nonexpansive, we have

$$\begin{aligned}&2\langle T_{[n]}Ax_n-Au,J_2\left(T_{[n]}-I\right)Ax_n\rangle\\ &\le d_2\left\|T_{[n]}Ax_n-Au\right\|^2+\left\|T_{[n]}Ax_n-Ax_n\right\|^2\\ &-\left\|\left(T_{[n]}Ax_n-Au\right)-\left(T_{[n]}Ax_n-Ax_n\right)\right\|^2\\ &\le (d_2-1)\left\|Ax_n-Au\right\|^2+\left\|\left(T_{[n]}-I\right)Ax_n\right\|^2.\end{aligned} \tag{23}$$

From (22) and (23), we get

$$\begin{aligned}\phi(u,y_n) &\le \phi(u,x_n)+\gamma^2d_2^*\|A\|^2\left\|\left(T_{[n]}-I\right)Ax_n\right\|^2\\ &-2\gamma\left\|\left(T_{[n]}-I\right)Ax_n\right\|^2+\gamma(d_2-1)\left\|Ax_n-Au\right\|^2\\ &+\gamma\left\|\left(T_{[n]}-I\right)Ax_n\right\|^2\\ &\le\phi(u,x_n)-\gamma\left(1-d_2^*\gamma\|A\|^2\right)\left\|\left(T_{[n]}-I\right)Ax_n\right\|^2\\ &-\gamma(1-d_2)\left\|Ax_n-Au\right\|^2\\ &\le\phi(u,x_n)-\gamma\left(1-d_2^*\gamma\|A\|^2\right)\left\|\left(T_{[n]}-I\right)Ax_n\right\|^2\\ &\le\phi(u,x_n).\end{aligned} \tag{24}$$

Also by Lemma 1, we have

$$\begin{aligned}\phi(u,z_n) &\le \phi\left(u,J^{-1}[Jy_n-\lambda ft_n]\right)-\phi\left(z_n,J^{-1}[Jy_n-\lambda ft_n]\right)\\ &=\|u\|^2-2\langle u,Jy_n-\lambda ft_n\rangle-\left\|z_n\right\|^2+2\langle z_n,Jy_n-\lambda ft_n\rangle\\ &=\phi(u,y_n)-\phi(z_n,y_n)+2\langle u-z_n,\lambda ft_n\rangle\\ &=\phi(u,y_n)-\phi(z_n,y_n)+2\lambda\langle u-t_n,ft_n\rangle+2\lambda\langle t_n-z_n,ft_n\rangle.\end{aligned} \tag{25}$$

By the fact that $u\in \mathrm{VI}(C,f)$ and using property P2, we have

$$\begin{aligned}\phi(u,z_n) &\le \phi(u,y_n)-\phi(z_n,y_n)+2\lambda\langle t_n-z_n,ft_n\rangle\\ &=\phi(u,y_n)-\phi(z_n,t_n)-\phi(t_n,y_n)\\ &+2\langle z_n-t_n,Jy_n-\lambda ft_n-Jt_n\rangle.\end{aligned} \tag{26}$$

Also from the fact that $t_n=\Pi_CJ^{-1}(Jy_n-\lambda fy_n)$, $z_n\in C$, the Lipschitz continuity of f, Lemma 1, and Lemma 3, we obtain that

$$\begin{aligned}\langle z_n-t_n,Jy_n-\lambda ft_n-Jt_n\rangle &=\langle z_n-t_n,Jy_n-\lambda fy_n-Jt_n\rangle\\ &+\lambda\langle z_n-t_n,fy_n-ft_n\rangle\\ &\le\lambda\langle z_n-t_n,fy_n-ft_n\rangle\\ &\le k\lambda\left\|z_n-t_n\right\|\left\|y_n-t_n\right\|\\ &\le\frac{k\lambda}{2}\left(\left\|z_n-t_n\right\|^2+\left\|y_n-t_n\right\|^2\right)\\ &\le\frac{k\lambda}{2\alpha}\left(\phi(z_n,t_n)+\phi(t_n,y_n)\right).\end{aligned} \tag{27}$$

Thus,

$$\begin{aligned}\phi(u,z_n) &\le \phi(u,y_n)-\phi(t_n,y_n)-\phi(z_n,t_n)\\ &+\frac{k\lambda}{\alpha}\left(\phi(z_n,t_n)+\phi(t_n,y_n)\right)\\ &=\phi(u,y_n)-\left(1-\frac{k\lambda}{\alpha}\right)\left(\phi(t_n,y_n)+\phi(z_n,t_n)\right)\end{aligned} \tag{28}$$

$$\leq \phi(u, y_n). \tag{29}$$

Hence, $\Gamma \subset C_n$ for any $n \geq 1$.

Step 3. We shall show that $\{x_n\}$ is a Cauchy sequence.

Since $\Gamma \subset C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \subset C_n$, then by Lemma 1, we have that $\phi(x_{n+1}, x_1) \leq \phi(u, x_1)$ and also $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. Hence, $\phi(x_n, x_1)$ is nondecreasing. So, $\lim_{n\longrightarrow\infty}\phi(x_n, x_1)$ exists. By property $P1$, $\{x_n\}$ is bounded. Also, it follows from (24), (28), and the fact that A is a bounded linear operator that $\{y_n\}$, $\{z_n\}$, and $\{Ax_n\}$ are bounded.

From Lemma 1, we have that

$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_1) \leq \phi(x_m, x_1) - \phi(x_n, x_1) \longrightarrow 0$ as $n, m \longrightarrow \infty$. Hence, $\{x_n\}$ is a Cauchy sequence.

Step 4. We show that

$$\lim_{n\longrightarrow\infty}\|x_n - y_n\| = \lim_{n\longrightarrow\infty}\left\|\left(T_{[n]} - I\right)Ax_n\right\| = \lim_{n\longrightarrow\infty}\|z_n - t_n\| = \lim_{n\longrightarrow\infty}\|z_n - y_n\| = 0. \tag{30}$$

Since $x_{n+1} \in C_{n+1} \subset C_n$,

$$\begin{aligned}\phi(x_n, y_n) &= \phi(x_n, x_{n+1}) + \phi(x_{n+1}, y_n) + 2\langle x_{n+1} - x_n, Jy_n - Jx_{n+1}\rangle \\ &\leq \phi(x_n, x_{n+1}) + \phi(x_{n+1}, x_n) + 2\langle x_{n+1} - x_n, Jy_n - Jx_{n+1}\rangle \\ &= 2\langle x_n - x_{n+1}, Jx_n - Jx_{n+1}\rangle - 2\langle x_n - x_{n+1}, Jy_n - Jx_{n+1}\rangle.\end{aligned} \tag{31}$$

Taking limit as $n \longrightarrow \infty$, we have $\lim_{n\longrightarrow\infty}\phi(x_n, y_n) = 0$.

Similarly,

$$\phi(x_n, z_n) \leq 2\langle x_n - x_{n+1}, Jx_n - Jx_{n+1}\rangle - 2\langle x_n - x_{n+1}, Jz_n - Jx_{n+1}\rangle. \tag{32}$$

Taking limit as $n \longrightarrow \infty$, we have $\lim_{n\longrightarrow\infty}\phi(x_n, z_n) = 0$.

Since $\{x_n\}$ is bounded, it follows from Lemma 5 that $\lim_{n\longrightarrow\infty}\|x_n - y_n\| = 0$ and $\lim_{n\longrightarrow\infty}\|x_n - z_n\| = 0$.

Now,

$$\begin{aligned}\phi(u, x_n) - \phi(u, y_n) &= \|x_n\|^2 - \|y_n\|^2 - 2\langle u, Jx_n - Jy_n\rangle \\ &= \left(\|x_n\| - \|y_n\|\right)\left(\|x_n\| + \|y_n\|\right) + 2\langle u, Jx_n - Jy_n\rangle \\ &\leq \|x_n - y_n\|\left(\|x_n\| + \|y_n\|\right) + 2\langle u, Jx_n - Jy_n\rangle.\end{aligned} \tag{33}$$

Taking limit as $n \longrightarrow \infty$, we have $\lim_{n\longrightarrow\infty}(\phi(u, x_n) - \phi(u, y_n)) = 0$.

In a similar way, we also have $\lim_{n\longrightarrow\infty}(\phi(u, x_n) - \phi(u, z_n)) = 0$.

From (24), we obtain

$$0 < \left(\gamma - d_2^*\gamma^2\|A\|^2\right)\left\|\left(T_{[n]} - I\right)Ax_n\right\|^2 \leq \phi(u, x_n) - \phi(u, y_n). \tag{34}$$

Thus,

$$\lim_{n\longrightarrow\infty}\left\|\left(T_{[n]} - I\right)Ax_n\right\|^2 = 0. \tag{35}$$

From (24) and (28),

$$\phi(t_n, y_n) \leq \frac{1}{(1 - (k\lambda/\alpha))}\left(\phi(u, x_n) - \phi(u, z_n)\right), \tag{36}$$

$$\phi(z_n, t_n) \leq \frac{1}{(1 - (k\lambda/\alpha))}\left(\phi(u, x_n) - \phi(u, z_n)\right), \tag{37}$$

(35) and (36), respectively, implies that

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\phi(t_n, y_n) = 0, \\ &\lim_{n\longrightarrow\infty}\phi(z_n, t_n) = 0.\end{aligned} \tag{38}$$

Since $\{z_n\}$ and $\{y_n\}$ are bounded, by Lemma 5, we get

$$\lim_{n\longrightarrow\infty}\|t_n - y_n\| = 0, \tag{39}$$

$$\lim_{n\longrightarrow\infty}\|z_n - t_n\| = 0. \tag{40}$$

Thus, (39) and (40) imply

$$\lim_{n\longrightarrow\infty}\|z_n - y_n\| = 0. \tag{41}$$

Step 5. We show that $\{x_n\}$ converges to an element of Γ

Since $\{x_n\}$ is a Cauchy sequence, we may assume that $x_n \longrightarrow x^*$.

From the fact that $\lim_{n\longrightarrow\infty}\|x_n - y_n\| = \lim_{n\longrightarrow\infty}\|x_n - z_n\| = \lim_{n\longrightarrow\infty}\|t_n - y_n\| = 0$, we obtain that $y_n \longrightarrow x^*$, $z_n \longrightarrow x^*$, and $t_n \longrightarrow x^*$. Since A is a bounded linear operator, we have that $Ax_n \longrightarrow Ax^*$.

From (35), we have

$$\lim_{n\longrightarrow\infty}\left\|\left(T_{[n]} - I\right)Ax_n\right\| = 0. \tag{42}$$

Thus, for $i \in \{1, 2, 3, \ldots, m\}$, $\lim_{n\longrightarrow\infty}\|(T_i - I)Ax_n\| = 0$.

Since T_i is nonexpansive for each $i \in \{1, 2, 3, \ldots, m\}$, we have that $I - T_i$ is demiclosed at 0 for $i \in \{1, 2, 3, \ldots, m\}$. And therefore, $Ax^* \in F(T_i)$for each $i \in \{1, 2, 3, \ldots, m\}$.

Thus, $Ax^* \in \cap_{i=1}^{m} F(T_i)$. Next, we show that $x^* \in \mathrm{VI}(C, f)$

Define

$$Bv = \begin{cases} fv + N_C(v), & \text{if } v \in C, \\ \varnothing, & \text{if } v \notin C. \end{cases} \tag{43}$$

By Lemma 4, B is a maximal monotone and $0 \in Bv$ if and only if $v \in \mathrm{VI}(C, f)$.

Let $(v, w) \in G(B)$. $(v, w) \in G(B) \Longrightarrow w \in Bv = fv + N_C(v) \Longrightarrow w - fv \in N_C(v)$.

Thus, $\langle v - p, w - fv\rangle \geq 0, \quad \forall p \in C.$ Since $z_n = \Pi_C J^{-1}(Jy_n - \lambda f t_n)$ and $v \in C$, we have by Lemma 4 that $\langle z_n - v, Jy_n - \lambda f t_n - Jz_n\rangle \geq 0$.

Thus,

$$\left\langle v - z_n, \frac{Jz_n - Jy_n}{\lambda} + ft_n\right\rangle \geq 0, \quad n \geq 0. \tag{44}$$

Using the fact that $z_n \in C$ and $w - fv \in N_C(v)$, we have

$$\begin{aligned}\langle v - z_n, w\rangle &\geq \langle v - z_n, fv\rangle \\ &\geq \langle v - z_n, fv\rangle - \langle v - z_n, \frac{Jz_n - Jy_n}{\lambda} + ft_n\rangle \\ &= \langle v - z_n, fv - fz_n\rangle + \langle v - z_n, fz_n - ft_n\rangle \\ &- \langle v - z_n, \frac{Jz_n - Jy_n}{\lambda}\rangle \\ &\geq \langle v - z_n, fz_n - ft_n\rangle - \langle v - z_n, \frac{Jz_n - Jy_n}{\lambda}\rangle.\end{aligned} \tag{45}$$

Using the fact that J is uniformly continuous on bounded sets and f is Lipschitz continuous, as $n \longrightarrow \infty$, we have

$$\langle v - x^*, w\rangle \geq 0. \tag{46}$$

Since B is a maximal monotone, $0 \in Bx^*$, and hence $x^* \in VI(C, f)$, therefore $x^* \in \Gamma$.

Corollary 1. *Let E_1 be a uniformly smooth and 2-uniformly convex real Banach space and E_2 a 2-uniformly smooth real Banach space with smoothness constant $d_2 \in (0,1)$. Let C be a nonempty, closed, and convex subset of E_1. Let $f : C \longrightarrow E_1^*$ be a monotone and k-Lipschitz continuous map and $A : E_1 \longrightarrow E_2$ be a bounded linear operator with its adjoint A^* such that $A \neq 0$. Let $T : E_2 \longrightarrow E_2$ be a nonexpansive map. Let $\Gamma = \{z \in VI(C, f) : Az \in F(T)\} \neq \emptyset$. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_1 = x \in C = C_1, \\ y_n = J^{-1}(Jx_n - \gamma A^* J_2 (I - T) A x_n), \\ t_n = \Pi_C J^{-1}(Jy_n - \lambda f y_n), \\ z_n = \Pi_C J^{-1}(Jy_n - \lambda f t_n), \\ C_{n+1} = \{v \in C_n \colon \phi(v, y_n) \leq \phi(v, x_n), \phi(v, z_n) \leq \phi(v, y_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases} \tag{47}$$

where $\gamma \in [a,b]$, $a,b \in (0, (1/d_2^* \|A\|^2))$, d_2^* being the smoothness constant of E_1^* as in Lemma 2, and $\lambda \in (0, (\alpha/k))$, α being a positive constant as in Lemma 3. Then, the sequence $\{x_n\}$ converges to a point $z \in \Gamma$.

Proof. The result followed from Theorem 2 by setting $T_{[n]} = T$, $\forall n \in \mathbb{N}$. □

Corollary 2. *Let H_1 and H_2 be real Hilbert spaces. Let C be a nonempty, closed, and convex subset of H_1. $A : H_1 \longrightarrow H_2$ be a bounded linear operator such that $A \neq 0$, $f : C \longrightarrow H_1$ be a monotone and k-Lipschitz continuous map, and $T_i \colon H_2 \longrightarrow H_2$, $i = 1, 2, \ldots, m$, be a nonexpansive map. Let $\Gamma = \{z \in VI(C, f) \colon Az \in F(T)\} \neq \emptyset$. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_1 = x \in C = C_1, \\ y_n = x_n - \gamma A^* \left(I - T_{[n]}\right) A x_n, \quad \text{where } [n] = n \bmod m, \\ t_n = P_C (y_n - \lambda f y_n), \\ z_n = P_C (y_n - \lambda f t_n), \\ C_{n+1} = \left\{v \in C_n : \|y_n - v\| \leq \|x_n - v\|, \|z_n - v\| \leq \|y_n - v\|\right\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases} \tag{48}$$

where $\gamma \subset [a,b]$, $a,b \in (0, (1/\|A\|^2))$, and $\lambda \in (0, (1/k))$. Then, the sequence $\{x_n\}$ converges strongly to a point $z \in \Gamma$.

Proof. Setting $E_1 = H_1$ and $E_2 = H_2$ in Theorem 2, the result is as follows. □

Remark 2. Corollary 2 complements Theorem 1 (the result of Tian and Jiang [12]) in the sense that strong convergence of the sequence generated is obtained to a solution of the split variational inequality problem involving common fixed points of finite family of nonexpansive mappings, while weak convergence of the scheme is obtained in Theorem 1 involving only a single nonexpansive map. Though, additional condition of projecting the iterates on the half spaces C_n at each step is imposed.

Lemma 6. *Let H be a real Hilbert space and Q be a nonempty closed convex subset of H. Let $g : Q \longrightarrow H$ be an α- inverse strongly monotone mapping, that is, $\langle x - y, gx - gy\rangle \geq \alpha \|gx - gy\|^2$ for any $x, y \in Q$. Let $\mu \in (0, 2\alpha)$. Then, the mapping $P_Q(I - \mu g)$ is nonexpansive.*

Proof. Let$x, y \in H$:

$$\begin{aligned}&\left\|P_Q(x - \mu gx) - P_Q(y - \mu gy)\right\|^2 \leq \|(x - \mu gx) - (y - \mu gy)\|^2 \\ &= \|x - y\|^2 - 2\mu\langle x - y, gx - gy\rangle + \mu^2 \|gx - gy\|^2 \leq \|x - y\|^2 \\ &\quad - \mu\left(2 - \frac{\mu}{\alpha}\right)\langle x - y, gx - gy\rangle \\ &\leq \|x - y\|^2.\end{aligned} \tag{49}$$

Hence, $P_Q(I - \mu g)$ is nonexpansive. □

Corollary 3. *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be two nonempty, closed, and convex subset of H_1 and H_2, respectively. Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator such that $A \neq 0$, $f : C \longrightarrow H_1$ be a monotone and k-Lipschitz continuous map, and $g \colon H_2 \longrightarrow H_2$ be an inverse strongly monotone mapping. Setting $\Gamma = \{z \in VI(C, f) : Az \in VI(Q, g)\} \neq \emptyset$. Let a sequence $\{x_n\}$ be defined by*

$$\begin{cases} y_n = x_n - \gamma A^*\big(I - P_Q(I-\mu g)\big)Ax_n\big), \\ t_n = P_C(y_n - \lambda f y_n), \\ z_n = P_C(y_n - \lambda f t_n), \\ C_{n+1} = \big\{v \in C_n\colon \|y_n - v\| \le \|x_n - v\|, \|z_n - v\| \le \|y_n - v\|\big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases} \tag{50}$$

where $\gamma \subset [a,b]$, $a, b \in (0, (1/\|A\|^2))$, $\lambda \in (0(1/k))$, *and* $\mu \in (0, 2\alpha)$. *Then, the sequence* $\{x_n\}$ *converges to a point* $z \in \Gamma$.

Proof. Since $P_Q(I-\mu g)$ is nonexpansive for $\mu \in (0, 2\alpha)$ and $z \in VI(Q, g)$ if and only if $z = P_Q(I-\mu g)z$ for $\mu > 0$, putting $T = P_Q(I-\mu g)$ in Corollary 1, we get the desired result. □

4. Application to Equilibrium Problem

Let C be a nonempty closed convex subset of a real Banach space E and let $F\colon C \times C \longrightarrow \mathbb{R}$ be a bifunction.

The equilibrium problem with respect to F and C is to find $z \in C$ such that

$$F(z, y) \ge 0, \quad \forall y \in C. \tag{51}$$

The set of solutions of the equilibrium problem mentioned above is denoted by $\mathrm{EP}(F)$. For solving the equilibrium problem, we assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \le 0$ $\forall x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \le F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinous.

Lemma 7 (see [24]). *Let E be a reflexive, strictly convex, and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4), then for any $x \in E$ and $r > 0$, there exists a unique point $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, jz - jx\rangle \ge 0, \quad \forall y \in C. \tag{52}$$

Lemma 8 (see [24]). *Let E be a reflexive, strictly convex, and smooth Banach space and C be a nonempty closed convex subset of E. Let $F\colon C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4), then for any $x \in E$ and $r > 0$, define a mapping $T_r\colon E \longrightarrow C$ by*

$$T_r x = \left\{x \in C\colon F(z, y) + \frac{1}{r}\langle y - z, jz - jx\rangle \ge 0 \,\forall y \in C\right\}. \tag{53}$$

Then, the following holds:

(1) T_r is single valued;

(2) T_r is a firmly nonexpansive type, i.e.,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \le \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in E; \tag{54}$$

(3) $F(T_r) = \mathrm{EP}(F)$;

(4) $\mathrm{EP}(F)$ is closed and convex.

Theorem 3. *Let E_1 be a uniformly smooth and 2-uniformly convex real Banach space and E_2 a 2-uniformly smooth real Banach space with smoothness constant $d_2 \in (0, 1)$. Let C and Q be two nonempty, closed, and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \longrightarrow E_2$ be a bounded linear operator with its adjoint A^* such that $A \ne 0$, $f : C \longrightarrow E_1^*$ be a monotone and k-Lipschitz continuous map, and $F : Q \times Q \longrightarrow$ be a bifunction satisfying conditions (A1)–(A4). Let $\Gamma = \{z \in VI(C, f) : Az \in EP(F)\} \ne \varnothing$. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_1 = x \in C = C_1, \\ y_n = J^{-1}(Jx_n - \gamma A^* J_2(I - T_r)Ax_n), \\ t_n = \Pi_C J^{-1}(Jy_n - \lambda f y_n), \\ z_n = \Pi_C J^{-1}(Jy_n - \lambda f t_n), \\ C_{n+1} = \{v \in C_n\colon \phi(v, y_n) \le \phi(v, x_n), \phi(v, z_n) \le \phi(v, y_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases} \tag{55}$$

where $\gamma \in [a, b]$, $a, b \in (0, (1/d_2^*\|A\|^2))$, d_2^* is the smoothness constant of E_1^* as in Lemma 2, $\lambda \in (0, (\alpha/k))$, α being a positive constant as in Lemma 3, and T_r is the resolvent of F for $r > 0$. Then, the sequence $\{x_n\}$ converges to a point $z \in \Gamma$.

Proof. Putting $T = T_r$ in Corollary 1, we get the desired result. □

5. Application to Maximal Monotone Operator

A set valued mapping $B \subset E \times E^*$ with domain $D(B) = \{x \in E : Bx \ne \varnothing\}$ and range $R(B) = \cup\{Bx\colon x \in D(B)\}$ is said to be monotone if $\langle x - y, x^* - y^*\rangle \ge 0$ whenever $(x, x^*), (y, y^*) \in B$. A monotone operator $B \subset E \times E^*$ is said to be maximal monotone if its graph $G(B) = \{(x, y) : y \in Bx\}$ is not properly contained in the graph of any other monotone mapping. We know that if B is a maximal monotone, then the zero of B, $B^{-1}(0) = \{x \in E : 0 \in Bx\}$ is closed and convex. If E is a smooth, strictly convex, and reflexive Banach space, then a monotone operator $B : E \longrightarrow E^*$ is maximal if and only if $R(J + rB) = E^*$ for each $r > 0$. Let E be a smooth, strictly convex, and reflexive Banach space, C be a nonempty closed convex subset of E, and $B \subset E \times E^*$ be a monotone operator satisfying

$$D(B) \subset C \subset J^{-1}(J + rB), \tag{56}$$

for all $r > 0$. If B is a maximal monotone, then (56) holds for $C = \overline{D(B)}$ and we can define the resolvent S_r: $C \longrightarrow D(B)$ by

$$S_r x = \{x \in E: Jx \in Jz + rBz\}, \quad (57)$$

for all $x \in C$, i.e., $S_r = (J + rB)^{-1} J$. We know the following (see [23, 25–27]):

(1) S_r is single valued,

(2) $F(S_r) = B^{-1}0$, where $F(S_r)$ is the set of fixed points of S_r,

(3) S_r is a firmly nonexpansive type, i.e.,

$$\langle S_r x - S_r y, JS_r x - JS_r y\rangle \le \langle S_r x - S_r y, x - y\rangle, \quad \forall x, y \in E. \quad (58)$$

Theorem 4. *Let E_1 be a uniformly smooth and 2-uniformly convex real Banach space and E_2 a 2-uniformly smooth real Banach space with smoothness constant $d_2 \in (0,1)$. Let C and Q be two nonempty, closed, and convex subsets of E_1 and E_2, respectively; $A : E_1 \longrightarrow E_2$ be a bounded linear operator with its adjoint A^* such that $A \ne 0$; $f : C \longrightarrow E_1^*$ be a monotone and k-Lipschitz continuous map; and $B : E \longrightarrow E^*$ be a maximal monotone operator. Let $\Gamma = \{z \in VI(C, f)$: $Az \in B^{-1}(0)\} \ne \emptyset$. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_1 = x \in C = C_1, \\ y_n = J^{-1}(Jx_n - \gamma A^* J_2 (I - S_r) A x_n), \\ t_n = \Pi_C J^{-1}(Jy_n - \lambda f y_n), \\ z_n = \Pi_C J^{-1}(Jy_n - \lambda f t_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n), \phi(v, z_n) \le \phi(v, y_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases} \quad (59)$$

where $\gamma \in [a,b]$, $a, b \in (0, (1/d_2^* \|A\|^2))$, d_2^* being the smoothness constant of E_1^* as in Lemma 2, $\lambda \in (0, (\alpha/k))$, α being a positive constant as in Lemma 3, and S_r is the resolvent of B for $r > 0$. Then, the sequence $\{x_n\}$ converges to a point $z \in \Gamma$.

Proof. Putting $T = S_r$ in Corollary 1, we get the desired result. □

6. Application to Constrained Convex Minimization Problem

The problem of finding $y \in C$ such that

$$\phi(y) = \min_{x \in C} \phi(x), \quad (60)$$

where C is a nonempty closed convex subset of H and ϕ is a real-valued convex function is called constraint convex minimization problem. We denote the set of solution of the constraint convex minimization problem by $\operatorname{argmin}_{x \in C} \phi(x)$.

Lemma 9 (see [12]). *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let ϕ be a convex function of H into $\mathbb{R}$. If ϕ is differentiable, then z is a solution of the constraint convex minimization problem if and only if $z \in VI(C, \nabla\phi)$.*

Theorem 5. *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be two nonempty, closed, and convex subset of H_1 and H_2, respectively. Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator such that $A \ne 0$, and $f : C \longrightarrow H_1$ be a monotone and k-Lipschitz continuous. Let ϕ: $H_2 \longrightarrow \mathbb{R}$ be a differentiable convex function and suppose $\nabla\phi$ is α-inverse strongly monotone mapping. Setting $\Gamma = \{z \in VI(C, f) : Az \in \operatorname{argmin}_{y \in Q} \phi(y)\}$, assume that $\Gamma \ne \emptyset$. Let a sequence*

$$\begin{cases} y_n = x_n - \gamma A^* \big(I - P_Q (I - \mu \nabla\phi) A x_n\big), \\ t_n = P_C (y_n - \lambda f y_n), \\ z_n = P_C (y_n - \lambda f t_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|, \|z_n - v\| \le \|y_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases} \quad (61)$$

where $\gamma \subset [a,b]$, $a, b \in (0, (1/\|A\|^2))$, $\lambda \in (0, (1/k))$, and $\mu \in (0, 2\alpha)$. Then, the sequence $\{x_n\}$ converges to a point $z \in \Gamma$.

Proof. Putting $T = \nabla\phi$ in Corollary 3, by Lemma 9 we get the desired result. □

7. Application to Split Minimization Problem

Let H_1 and H_2 be real Hilbert spaces. Let C and Q be two nonempty, closed, and convex subset of H_1 and H_2, respectively, and ϕ_1: $H_1 \longrightarrow \mathbb{R}$, ϕ_2: $H_2 \longrightarrow \mathbb{R}$ be two convex functions. Let $AH_1 \longrightarrow H_2$ be a bounded linear operator.

The problem of finding z satisfying the conditions

$$z \in \operatorname{argmin}_{x \in C} \phi_1(x) \text{ such that } Az \in \operatorname{argmin}_{y \in Q} \phi_2 s(y), \quad (62)$$

is called the split minimization problem.

Theorem 6. *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be two nonempty, closed, and convex subset of H_1 and H_2, respectively. Let A: $H_1 \longrightarrow H_2$ be a bounded linear operator such that $A \ne 0$, and ϕ_1: $H_1 \longrightarrow \mathbb{R}$ and ϕ_2: $H_2 \longrightarrow \mathbb{R}$ be two differentiable convex functions. Suppose that $\nabla\phi_1$ is k-Lipschitz continuous and $\nabla\phi_2$ is α-inverse strongly monotone mapping, let $\Gamma = \{z \in \operatorname{argmin}_{x \in C} \phi_1(x)$: $Az \in \operatorname{argmin}_{y \in Q} \phi_2 (y)\} \ne \emptyset$. Let a sequence*

n	$X(n)$
1	0.0059
2	0.0029
3	0.0015
4	0.0007
5	0.0004
6	0.0002
7	0.0001
8	0.0000
9	0.0000
10	0.0000
11	0.0000
12	0.0000
13	0.0000
14	0.0000
15	0.0000

Values of the sequence $x(n)$ generated by (67)

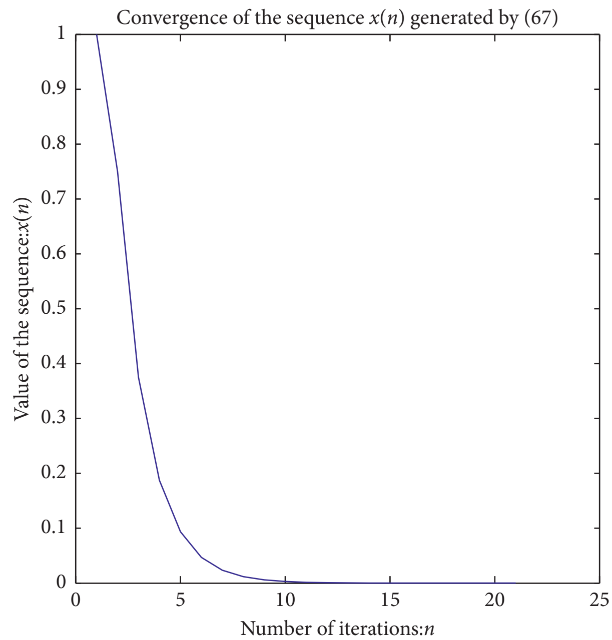

Figure 1: The table and graph of sequence $\{x_n\}$ generated by (67) versus number of iterations $n := \{0, 1, \ldots, 15\}$ with an initial choice of $x_0 = 1.0000$.

$$\begin{cases} y_n = x_n - \gamma A^*\left(I - P_Q\left(I - \mu\nabla\phi_1\right)Ax_n\right), \\ t_n = P_C\left(y_n - \lambda\nabla\phi_2 y_n\right), \\ z_n = P_C\left(y_n - \lambda\nabla\phi_2 t_n\right), \\ C_{n+1} = \left\{v \in C_n : \left\|y_n - v\right\| \le \left\|x_n - v\right\|, \left\|z_n - v\right\| \le \left\|y_n - v\right\|\right\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases} \tag{63}$$

where $\gamma \subset [a, b]$, $a, b \in (0, (1/\|A\|^2))$, $\lambda \in (0, (1/k))$, *and* $\mu \in (0, 2\alpha)$. *Then, the sequence* $\{x_n\}$ *converges to a point* $z \in \Gamma$.

Proof. Since ϕ_1 is convex, we have $\nabla\phi_1$ is monotone. Putting $f = \nabla\phi_1$ and $g = \nabla\phi_2$ in Corollary 3 and by Lemma 9, we get the desired result. □

8. Numerical Example

In this section, we present a numerical example to show the convergence of a sequence generated by our algorithm. Let $E_1 = E_2 = \mathbb{R}$, $C = [0, \infty)$.

Set $m = 2$ and Let $T_0, T_1 : C \longrightarrow C$ be defined by

$$T_0 x = \frac{2}{3}x, \quad \forall x \in C,$$
$$T_1 x = \frac{1}{2}x, \quad \forall x \in C. \tag{64}$$

Then, T_0 and T_1 are nonexpansive.

Let $f : C \longrightarrow C$ be defined by

$$fx = \frac{1}{3}x, \quad \forall x \in C. \tag{65}$$

Then, f is monotone and $\mathrm{VI}(C, f) = 0$.

Let $A : C \longrightarrow C$ be defined by

$$Ax = \frac{1}{2}x, \quad \forall x \in C. \tag{66}$$

Then, A is a bounded linear operator and $\|A\|^2 = (1/4)$, $A^* y = (1/2)y$.

When $z \in \mathrm{VI}(C, f)$, $Az = 0 \in F(T_0) \cap F(T_1)$.

So, $\Gamma = \{z \in \mathrm{VI}(C, f) : Az \in F(T_0) \cap F(T_1)\} \neq \varnothing$.

Clearly, $\Gamma = \{0\}$. Taking $\lambda = (3/n^2)$ and $\gamma = (4/n^2)$. It follows from Theorem 2 that a sequence $\{x_n\}$ is generated by the following algorithm:

$$\begin{cases} y_n = x_n - \gamma A^*\left(\left(I - T_{[n]}\right)Ax_n\right), \\ t_n = P_C\left(y_n - \lambda f y_n\right), \\ z_n = P_C\left(y_n - \lambda f t_n\right), \\ C_{n+1} = \left\{v \in C_n : v \le \dfrac{x_n^2 - z_n^2}{2x_n - 2z_n}\right\}, \\ x_{n+1} = P_{C_{n+1}} x_1 = \dfrac{x_n^2 - z_n^2}{2x_n - 2z_n}, \end{cases} \tag{67}$$

converges strongly to $0 \in \Gamma$ (see Figure 1).

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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