Research Article

Continuous Dependence on the Forchheimer Coefficient of the Forchheimer Fluid Interfacing with a Darcy Fluid

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The continuous dependence of the two fluids which interface with each other in a bounded domain is derived. It is first shown how to establish the *a priori* results for the solutions for the flow in a bounded domain occupied by a viscous fluid in contact with a porous solid. It is also proved that the solution depends continuously on the Forchheimer coefficient.

1. Introduction

In this paper, we study two fluids in one bounded domain when they interface with each other. We want to know what effect they can give to each other. Let an appropriate part of the plane $z = x_3 = 0$ denotes the boundary between a porous medium which occupies a bounded region $\Omega_2 \subseteq \mathbb{R}^3$ and a nonlinear viscous fluid which occupies a bounded region $\Omega_1 \subseteq \mathbb{R}^3$. The interface is denoted by L . The remaining parts of the boundaries of Ω_1 and Ω_2 are denoted, respectively, by Γ_1 and Γ_2 . We also denote $\partial\Omega_1 = \Gamma_1 \cup L$ and $\partial\Omega_2 = \Gamma_2 \cup L$. We also note that Ω_1 is above the plane $z = x_3 = 0$ and Ω_2 is below the plane $z = x_3 = 0$.

Let (u_i, T, p) and (v_i, θ, q) denote the velocity, temperature, and pressure in Ω_1 and Ω_2 , respectively. The Forchheimer system consists of the partial differential equations (see [1])

$$b|u|u_i + (1 + \gamma T)u_i - g_i T + \frac{\partial p}{\partial x_i} = 0, \quad \text{in } \Omega_1 \times [0, \tau], \quad (1)$$

$$\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \Delta T, \quad \text{in } \Omega_1 \times [0, \tau], \quad (2)$$

$$\frac{\partial u_i}{\partial x_i} = 0, \quad \text{in } \Omega_1 \times [0, \tau], \quad (3)$$

where g_i is the gravity force function. The coefficient b is a positive constant which is named as the Forchheimer coefficient. The viscosity variation in (1) is accounted for by the term $1 + \gamma T$, i.e., we are considering a viscosity μ like $\mu = \mu_1(1 + \gamma T)$, $\gamma > 0$.

The Darcy equations are (see Nield and Bejan [2])

$$v_i - g_i \theta + \frac{\partial q}{\partial x_i} = 0, \quad \text{in } \Omega_2 \times [0, \tau], \quad (4)$$

$$\frac{\partial \theta}{\partial t} + v_i \frac{\partial \theta}{\partial x_i} = \Delta \theta, \quad \text{in } \Omega_2 \times [0, \tau], \quad (5)$$

$$\frac{\partial v_i}{\partial x_i} = 0, \quad \text{in } \Omega_2 \times [0, \tau], \quad (6)$$

where Ω_1 and Ω_2 are the bounded, simply connected, and star-shaped domains and τ is a given number satisfying $0 \leq \tau < \infty$. We impose the following boundary conditions:

$$\begin{aligned} u_i &= 0; \quad T = G(x, t), \quad \text{on } \Gamma_1 \times [0, \tau], \\ v_i n_i^{(2)} &= 0, \quad \theta = \tilde{G}(x, t), \quad \text{on } \Gamma_2 \times [0, \tau], \end{aligned} \quad (7)$$

for prescribed functions $G(x, t)$ and $\tilde{G}(x, t)$, and $n_i^{(2)}$ denotes the unit outward normal of $\partial\Omega_2$. We also let $n_i^{(1)}$ be the unit

outward normal of $\partial\Omega_1$. Obviously, $n_3^{(1)} = -n_3^{(2)}$ on L . The initial conditions are written as

$$\begin{aligned} u_i(x, 0) &= f_i(x), \quad T(x, 0) = T_0(x), \quad \text{in } \Omega_1, \\ \theta(x, 0) &= \theta_0(x), \quad \text{in } \Omega_2, \end{aligned} \quad (8)$$

for prescribed functions f_i , T_0 , and θ_0 . On the interface L , the conditions are

$$\begin{aligned} u_3 &= v_3, \\ T &= \theta, \\ T_{,3} &= \theta_{,3}, \\ q &= p. \end{aligned} \quad (9)$$

The purpose of this paper is to study the continuous dependence on the coefficients of problems (1)–(5). This type of stability is often called structural stability to distinguish it from continuous dependence on the initial data, on the boundary data, or even on the partial differential equation themselves. In continuum mechanics problems, it is necessary to be able to establish continuous dependence on the model; this is discussed in terms of differential equations by Hirsch and Smale [3]. Such stability estimates are fundamental in that one wishes to know if a small change in a coefficient in an equation or boundary data, or a small change in the equations themselves, will lead to a drastic change in the solution. When we study the continuous dependence or convergence, structural stability expresses the changes in the model itself rather than the original data. Many references to work of this nature are discussed in the monograph of Ames and Straughan [4] and the monograph of Straughan [5].

In the area of porous media, there have been many studies of continuous dependence, or structural stability, including those of Scott and Straughan [6, 7], Franchi and Straughan [8], Hoang and Ibragimov [9], Lin and Payne [10, 11], Liu [12, 13], Liu et al. [14–16], Scott [6], Scott and Straughan [7], Li [17], Cichon' et al. [18], Ma and Liu [19], Ciarletta et al. [20], and Payne et al. [18–21]. These results about structural stability mainly focus on one fluid in the domain. But in reality, there are usually two or more fluids in the domain. They can interface with each other. So the study of such type of problems may be interesting and meaningful. It is worth drawing attention to this type of study. Several quantitative results in physical problems were obtained in [22–25].

In this paper, we derive an *a priori* convergence result which compares the solution to the Forchheimer system of partial differential equations with that of the Darcy equations. The purpose of this paper is to study the continuous dependence of a solution to the Forchheimer system to a solution to Darcy equations on the Forchheimer coefficient b and the effective viscosity coefficient γ . Different from [14, 26], there are two nonlinear terms $\gamma T u_i$ and $|u| u_i$ and there is no Laplace term in (1). So, some Sobolev inequalities do not hold for our problem. This will bring great difficulty. To get our result, we must seek a new method to overcome

the difficulty. In the next section, we derive a number of *a priori* bounds which will be used in establishing the continuous dependence result in Section 3.

2. A Priori Bounds

In this section, we want to drive bounds for the various norms of u , v , T , and θ .

2.1. Bounds for $\|u\|_{L_3(\Omega_1)}^3$, $\|u\|_{L_2(\Omega_1)}^2$, and $\|v\|_{L_2(\Omega_2)}^2$. Multiplying (1.1)₁ with u_i and integrating over Ω_1 , we obtain

$$b\|u\|_{L_3(\Omega_1)}^3 + \left\| \sqrt{1 + \gamma T} u \right\|_{L_2(\Omega_1)}^2 = \int_{\Omega_1} g_i T u_i dx - \int_{\Omega_1} \frac{\partial p}{\partial x_i} u_i dx. \quad (10)$$

Integrating by parts and using Young's inequality and the arithmetic-geometric mean inequality now lead to

$$\begin{aligned} b\|u\|_{L_3(\Omega_1)}^3 + \left\| \sqrt{1 + \gamma T} u \right\|_{L_2(\Omega_1)}^2 &= \int_{\Omega_1} g_i T u_i dx \\ &- \int_L p u_3 n_3^{(1)} dA \leq \frac{1}{2} \|u\|_{L_2(\Omega_1)}^2 + \frac{1}{2} g^2 \|T\|_{L_2(\Omega_1)}^2 \\ &+ \int_L q v_i n_i^{(2)} dA, \end{aligned} \quad (11)$$

where $g^2 = \max g_i g_i$. By using the divergence theorem and Equation (2), we get

$$\begin{aligned} 2b\|u\|_{L_3(\Omega_1)}^3 + \left\| \sqrt{1 + \gamma T} u \right\|_{L_2(\Omega_1)}^2 &\leq g^2 \|T\|_{L_2(\Omega_1)}^2 \\ &+ 2 \int_{\Omega_2} v_i (g_i \theta - v_i) dx \leq g^2 \|T\|_{L_2(\Omega_1)}^2 \\ &+ g^2 \|\theta\|_{L_2(\Omega_2)}^2 - \|v\|_{L_2(\Omega_2)}^2. \end{aligned} \quad (12)$$

So, we have

$$\begin{aligned} 2b\|u\|_{L_3(\Omega_1)}^3 + \left\| \sqrt{1 + \gamma T} u \right\|_{L_2(\Omega_1)}^2 + \|v\|_{L_2(\Omega_2)}^2 &\leq g^2 \|T\|_{L_2(\Omega_1)}^2 \\ &+ g^2 \|\theta\|_{L_2(\Omega_2)}^2. \end{aligned} \quad (13)$$

2.2. Bounds for $\|T\|_{L_2(\Omega_1)}^2$ and $\|\theta\|_{L_2(\Omega_2)}^2$. Payne et al. [27] have obtained the following result:

$$\sup_{[0, \tau]} \|T\|_{\infty} \leq T_M, \quad (14)$$

with $T_M = \max \{ \|T_0\|_{\infty}, \sup_{[0, \tau]} \|G\|_{\infty}, T_{LM} \}$ and T_{LM} is the maximum temperature on the interface L . Similarly, in the $\Omega_2 \times [0, \tau]$,

$$\sup_{[0, \tau]} \|\theta\|_{\infty} \leq \theta_M, \quad (15)$$

with $\theta_M = \max \{ \|\theta_0\|_\infty, \sup_{[0,\tau]} \|\tilde{G}\|_\infty, T_{LM} \}$. However, in the area $\Omega_1 \cup \Omega_2 \times [0, \tau]$, the maximum temperature cannot be reached on the interface L . Therefore, we observe that

$$\max \left\{ \sup_{[0,\tau]} \|T\|_\infty, \sup_{[0,\tau]} \|\theta\|_\infty, T_{LM} \right\} \leq N_M, \quad (16)$$

where $N_M = \max \{ \|T_0\|_\infty, \sup_{[0,\tau]} \|G\|_\infty, \|\theta_0\|_\infty, \sup_{[0,\tau]} \|\tilde{G}\|_\infty \}$. To derive the bounds for $\|T\|_{L_2(\Omega_1)}^2$ and $\|\theta\|_{L_2(\Omega_2)}^2$, we introduce another two functions φ and $\tilde{\varphi}$ which for each t satisfy

$$\begin{aligned} \varphi_{,\eta} - \Delta\varphi &= 0, & \text{in } \Omega_1, \\ \tilde{\varphi}_{,\eta} - \Delta\tilde{\varphi} &= 0, & \text{in } \Omega_2, \\ \varphi &= G, & \text{on } \Gamma_1, \\ \tilde{\varphi} &= \tilde{G}, & \text{on } \Gamma_2, \\ \varphi(x, 0) &= T_0, & \text{in } \Omega_1, \\ \tilde{\varphi}(x, 0) &= \theta_0, & \text{in } \Omega_2, \\ \varphi &= \tilde{\varphi}, \quad \varphi_{,3} = \tilde{\varphi}_{,3}, & \text{on } L. \end{aligned} \quad (17)$$

Then, from the identities

$$\begin{aligned} \int_0^t \int_{\Omega_1} (T - \varphi) \left[(T - \varphi)_{,\eta} + u_i T_{,i} - \Delta(T - \varphi) \right] dx d\eta &= 0, \\ \int_0^t \int_{\Omega_2} (\theta - \tilde{\varphi}) \left[(\theta - \tilde{\varphi})_{,\eta} + v_i \theta_{,i} - \Delta(\theta - \tilde{\varphi}) \right] dx d\eta &= 0, \end{aligned} \quad (18)$$

we have

$$\begin{aligned} &\|T - \varphi\|_{L_2(\Omega_1)}^2 + \|\theta - \tilde{\varphi}\|_{L_2(\Omega_2)}^2 + 2 \int_0^t \|\nabla(T - \varphi)\|_{L_2(\Omega_1)}^2 d\eta \\ &+ 2 \int_0^t \|\nabla(\theta - \tilde{\varphi})\|_{L_2(\Omega_2)}^2 d\eta = 2 \int_0^t \int_{\Omega_1} (T - \varphi)_{,i} u_i T dx d\eta \\ &+ 2 \int_0^t \int_{\Omega_2} (\theta - \tilde{\varphi})_{,i} v_i \theta dx d\eta. \end{aligned} \quad (19)$$

So, (20) leads to that

$$\begin{aligned} &\|T - \varphi\|_{L_2(\Omega_1)}^2 + \|\theta - \tilde{\varphi}\|_{L_2(\Omega_2)}^2 \\ &+ 2 \int_0^t \|\nabla(T - \varphi)\|_{L_2(\Omega_1)}^2 d\eta + 2 \int_0^t \|\nabla(\theta - \tilde{\varphi})\|_{L_2(\Omega_2)}^2 d\eta \\ &\leq 2N_M \left(\int_0^t \|\nabla(T - \varphi)\|_{L_2(\Omega_1)}^2 d\eta \int_0^t \|u\|_{L_2(\Omega_1)}^2 d\eta \right)^{1/2} \\ &+ 2N_M \left(\int_0^t \|\nabla(\theta - \tilde{\varphi})\|_{L_2(\Omega_2)}^2 d\eta \int_0^t \|v\|_{L_2(\Omega_2)}^2 d\eta \right)^{1/2}. \end{aligned} \quad (20)$$

It follows by Lemma 1 that

$$\|T - \varphi\|_{\Omega_1}^2 + \|\theta - \tilde{\varphi}\|_{\Omega_2}^2 \leq N_M^2 \left(\int_0^t \|u\|_{L_2(\Omega_1)}^2 d\eta + \int_0^t \|v\|_{L_2(\Omega_2)}^2 d\eta \right), \quad (21)$$

where

$$\begin{aligned} \|T - \varphi\|_{\Omega_1}^2 &= \|T - \varphi\|_{L_2(\Omega_1)}^2 + \int_0^t \|\nabla(T - \varphi)\|_{L_2(\Omega_1)}^2 d\eta, \\ \|\theta - \tilde{\varphi}\|_{\Omega_2}^2 &= \|\theta - \tilde{\varphi}\|_{L_2(\Omega_2)}^2 + \int_0^t \|\nabla(\theta - \tilde{\varphi})\|_{L_2(\Omega_2)}^2 d\eta. \end{aligned} \quad (22)$$

Since

$$\begin{aligned} \|T\|_{\Omega_1}^2 &\leq \|T - \varphi\|_{\Omega_1}^2 + \|\varphi\|_{\Omega_1}^2, \\ \|\theta\|_{\Omega_2}^2 &\leq \|\theta - \tilde{\varphi}\|_{\Omega_2}^2 + \|\tilde{\varphi}\|_{\Omega_2}^2, \end{aligned} \quad (23)$$

we have

$$\begin{aligned} \|T\|_{\Omega_1}^2 + \|\theta\|_{\Omega_2}^2 &\leq N_M^2 \left(\int_0^t \|u\|_{L_2(\Omega_1)}^2 d\eta + \int_0^t \|v\|_{L_2(\Omega_2)}^2 d\eta \right) \\ &+ \|\varphi\|_{\Omega_1}^2 + \|\tilde{\varphi}\|_{\Omega_2}^2. \end{aligned} \quad (24)$$

In computing the bounds for $\|\varphi\|_{\Omega_1}^2$ and $\|\tilde{\varphi}\|_{\Omega_2}^2$, we introduce another two functions h and $\tilde{h}$ which for each t satisfy

$$\begin{aligned} \Delta h &= 0, & \text{in } \Omega_1, \\ \Delta \tilde{h} &= 0, & \text{in } \Omega_2, \\ h &= G, & \text{on } \Gamma_1, \\ \tilde{h} &= \tilde{G}, & \text{on } \Gamma_2, \\ h &= \tilde{h}, \quad h_{,3} = \tilde{h}_{,3}, & \text{on } L. \end{aligned} \quad (25)$$

Then, from the identities

$$\int_0^t \int_{\Omega_1} (\varphi - h) \left[(\varphi - h)_{,\eta} - \Delta(\varphi - h) + h_{,\eta} \right] dx d\eta = 0, \quad (26)$$

$$\int_0^t \int_{\Omega_2} (\tilde{\varphi} - \tilde{h}) \left[(\tilde{\varphi} - \tilde{h})_{,\eta} - \Delta(\tilde{\varphi} - \tilde{h}) + \tilde{h}_{,\eta} \right] dx d\eta = 0, \quad (27)$$

it follows from (26) that

$$\begin{aligned} & \|\varphi - h\|_{L_2(\Omega_1)}^2 + \|\tilde{\varphi} - \tilde{h}\|_{L_2(\Omega_2)}^2 + 2 \int_0^t \|\nabla(\varphi - h)\|_{L_2(\Omega_1)}^2 d\eta \\ & + 2 \int_0^t \|\nabla(\tilde{\varphi} - \tilde{h})\|_{L_2(\Omega_2)}^2 d\eta = \|T_0 - h(x, 0)\|_{L_2(\Omega_1)}^2 \\ & + \|\theta_0 - \tilde{h}(x, 0)\|_{L_2(\Omega_2)}^2 \\ & + 2 \left(\int_0^t \|\varphi - h\|_{L_2(\Omega_1)}^2 d\eta \int_0^t \|h_{,\eta}\|_{L_2(\Omega_1)}^2 d\eta \right)^{1/2} \\ & + 2 \left(\int_0^t \|\tilde{\varphi} - \tilde{h}\|_{L_2(\Omega_2)}^2 d\eta \int_0^t \|\tilde{h}_{,\eta}\|_{L_2(\Omega_2)}^2 d\eta \right)^{1/2}. \end{aligned} \quad (28)$$

From (28), we have

$$\begin{aligned} & \|\varphi - h\|_{L_2(\Omega_1)}^2 + \|\tilde{\varphi} - \tilde{h}\|_{L_2(\Omega_2)}^2 \\ & \leq \int_0^t \left(\|\varphi - h\|_{L_2(\Omega_1)}^2 + \|\tilde{\varphi} - \tilde{h}\|_{L_2(\Omega_2)}^2 \right) d\eta + m_1(t), \end{aligned} \quad (29)$$

where

$$\begin{aligned} m_1(t) &= \|T_0 - h(x, 0)\|_{L_2(\Omega_1)}^2 + \|\theta_0 - \tilde{h}(x, 0)\|_{L_2(\Omega_2)}^2 \\ &+ \int_0^t \|h_{,\eta}\|_{L_2(\Omega_1)}^2 d\eta + \int_0^t \|\tilde{h}_{,\eta}\|_{L_2(\Omega_2)}^2 d\eta. \end{aligned} \quad (30)$$

Upon integration of (29), we have

$$\int_0^t \left(\|\varphi - h\|_{L_2(\Omega_1)}^2 + \|\tilde{\varphi} - \tilde{h}\|_{L_2(\Omega_2)}^2 \right) d\eta \leq \int_0^t m_1(\eta) e^{t-\eta} d\eta. \quad (31)$$

Inserting (31) into (28), we get

$$\begin{aligned} & \|\varphi - h\|_{\Omega_1}^2 + \|\tilde{\varphi} - \tilde{h}\|_{\Omega_2}^2 \leq \|T_0 - h(x, 0)\|_{L_2(\Omega_1)}^2 \\ & + \|\theta_0 - \tilde{h}(x, 0)\|_{L_2(\Omega_2)}^2 + \int_0^t m_1(\eta) e^{t-\eta} d\eta \\ & + \int_0^t \|h_{,\eta}\|_{L_2(\Omega_1)}^2 d\eta + \int_0^t \|\tilde{h}_{,\eta}\|_{L_2(\Omega_2)}^2 d\eta. \end{aligned} \quad (32)$$

So

$$\begin{aligned} & \|\varphi\|_{\Omega_1}^2 + \|\tilde{\varphi}\|_{\Omega_2}^2 \leq \|\varphi - h\|_{\Omega_1}^2 + \|\tilde{\varphi} - \tilde{h}\|_{\Omega_2}^2 + \|h\|_{\Omega_1}^2 \\ & + \|\tilde{h}\|_{\Omega_2}^2 \leq \|h\|_{\Omega_1}^2 + \|\tilde{h}\|_{\Omega_2}^2 + \|T_0 - h(x, 0)\|_{L_2(\Omega_1)}^2 \\ & + \|\theta_0 - \tilde{h}(x, 0)\|_{L_2(\Omega_2)}^2 + \int_0^t m_1(\eta) e^{t-\eta} d\eta \\ & + \int_0^t \|h_{,\eta}\|_{L_2(\Omega_1)}^2 d\eta + \int_0^t \|\tilde{h}_{,\eta}\|_{L_2(\Omega_2)}^2 d\eta. \end{aligned} \quad (33)$$

The terms $\|h\|_{L_2(\Omega_1)}^2$, $\int_0^t \|h_{,\eta}\|_{L_2(\Omega_1)}^2 d\eta$ as well as $\|\tilde{h}\|_{L_2(\Omega_2)}^2$, $\int_0^t \|\tilde{h}_{,\eta}\|_{L_2(\Omega_2)}^2 d\eta$ may be bounded by boundary data by using a Rellich identity, cf. [6, 7]. Combining (13), (22), (24), and (33), we may have

$$\begin{aligned} & \|T\|_{L_2(\Omega_1)}^2 + \|\theta\|_{L_2(\Omega_2)}^2 + \int_0^t \|\nabla T\|_{L_2(\Omega_1)}^2 d\eta \\ & + \int_0^t \|\nabla \theta\|_{L_2(\Omega_2)}^2 d\eta \leq N_M^2 g^2 \int_0^t \left(\|T\|_{L_2(\Omega_1)}^2 + \|\theta\|_{L_2(\Omega_2)}^2 \right) \\ & \cdot d\eta + m_2(t), \end{aligned} \quad (34)$$

where

$$\begin{aligned} m_2(t) &= \|h\|_{\Omega_1}^2 + \|\tilde{h}\|_{\Omega_2}^2 + \|T_0 - h(x, 0)\|_{L_2(\Omega_1)}^2 \\ &+ \|\theta_0 - \tilde{h}(x, 0)\|_{L_2(\Omega_2)}^2 + \int_0^t m_1(\eta) e^{t-\eta} d\eta \\ &+ \int_0^t \|h_{,\eta}\|_{L_2(\Omega_1)}^2 d\eta + \int_0^t \|\tilde{h}_{,\eta}\|_{L_2(\Omega_2)}^2 d\eta. \end{aligned} \quad (35)$$

After integrating (34), we have

$$\int_0^t \left(\|T\|_{L_2(\Omega_1)}^2 + \|\theta\|_{L_2(\Omega_2)}^2 \right) d\eta \leq \int_0^t m_2(\eta) e^{N_M^2 g^2 (t-\eta)} d\eta. \quad (36)$$

Inserting (36) back into (34), we have the following lemma.

Lemma 1. *Let (u_i, T, p) and (v_i, θ, q) be the solutions of (1)-(5) with $T_0, \theta_0, G, \tilde{G} \in L_\infty$. Then,*

$$\begin{aligned} & \|T\|_{L_2(\Omega_1)}^2 + \|\theta\|_{L_2(\Omega_2)}^2 + \int_0^t \|\nabla T\|_{L_2(\Omega_1)}^2 d\eta \\ & + \int_0^t \|\nabla \theta\|_{L_2(\Omega_2)}^2 d\eta \leq P_1(t), \end{aligned} \quad (37)$$

where

$$P_1(t) = N_M^2 g^2 \int_0^t m_2(\eta) e^{N_M^2 g^2 (t-\eta)} d\eta + m_2(t). \quad (38)$$

Combining (13) and (34), we can obtain the following lemma.

Lemma 2. *Let (u_i, T, p) and (v_i, θ, q) be the solutions of (1)-(5) with $T_0, \theta_0, G, \tilde{G} \in L_\infty$. Then,*

$$2b\|u\|_{L_3(\Omega_1)}^3 + \left\| \sqrt{I + \gamma T u} \right\|_{L_2(\Omega_1)}^2 + \|v\|_{L_2(\Omega_2)}^2 \leq g^2 P_1(t) + P_2(t). \quad (39)$$

3. Continuous Dependence on the Forchheimer Coefficient

In this section, we want to derive an a priori estimate showing how (u_i, T, p) and (v_i, θ, q) depend continuously on the Forchheimer coefficient b . Let (u_i, T, p) and (v_i, θ, q) be solutions of (1)-(5) with $b = b_1$, and (u_i^*, T^*, p^*) and (v_i^*, θ^*, q^*) be solutions of (1)-(5) with $b = b_2$, respectively. We define

$$\begin{aligned} w_i &= u_i - u_i^*, \\ \Sigma &= T - T^*, \\ \pi &= p - p^*, \\ \sigma &= b_1 - b_2, \end{aligned} \quad (40)$$

$$\begin{aligned} w_i^m &= v_i - v_i^*, \\ \Sigma^m &= \theta - \theta^*, \\ \pi^m &= q - q^* \end{aligned} \quad (41)$$

Then, (w_i, Σ, π) satisfy the following equations

$$\begin{aligned} \sigma |u|u_i + b_2[u_i|u| - u_i^*|u^*|] + (1 + \gamma T^*)w_i \\ + \gamma \Sigma u_i - g_i \Sigma + \pi_{,i} = 0, \quad \text{in } \Omega_1 \times [0, \tau], \\ \frac{\partial \Sigma}{\partial t} + w_i T_{,i} + u_i^* (\Sigma)_{,i} = \Delta \Sigma, \quad \text{in } \Omega_1 \times [0, \tau], \\ \frac{\partial w_i}{\partial x_i} = 0, \quad \text{in } \Omega_1 \times [0, \tau], \end{aligned} \quad (42)$$

and (w_i^m, Σ^m, π^m) satisfy equations

$$\begin{aligned} w_i^m - g_i \Sigma^m + \pi_{,i}^m = 0, \quad \text{in } \Omega_2 \times [0, \tau], \\ \frac{\partial \Sigma^m}{\partial t} + w_i^m \theta_{,i} + v_i^* (\Sigma^m)_{,i} = \Delta \Sigma^m, \quad \text{in } \Omega_2 \times [0, \tau], \\ \frac{\partial w_i^m}{\partial x_i} = 0, \quad \text{in } \Omega_2 \times [0, \tau]. \end{aligned} \quad (43)$$

The boundary conditions are

$$\begin{aligned} w_i = 0, \quad \Sigma = 0, \quad \text{on } \Gamma_1 \times [0, \tau], \\ w_i^m n_i^{(2)} p = 0, \quad \Sigma^m = 0, \quad \text{on } \Gamma_2 \times [0, \tau]. \end{aligned} \quad (44)$$

The initial conditions can be written as

$$\begin{aligned} w_i(x, 0) = 0, \quad \Sigma(x, 0) = 0, \quad \text{in } \Omega_1, \\ \Sigma^m(x, 0) = 0, \quad \text{in } \Omega_2. \end{aligned} \quad (45)$$

The interface L conditions are

$$\begin{aligned} w_3 &= w_3^m, \\ \Sigma &= \Sigma^m, \\ \Sigma_{,3} &= \Sigma_{,3}^m, \\ \pi^m &= \pi. \end{aligned} \quad (46)$$

We observe for later convenience that $(4.3)_1$ may be rearranged as

$$\sigma |u^*|u_i^* + b_1[u_i|u| - u_i^*|u^*|] + (1 + \gamma T)w_i + \gamma \Sigma u_i^* - g_i \Sigma + \pi_{,i} = 0. \quad (47)$$

Our main result is the following theorem.

Theorem 3. Let (u_i, T, p) and (v_i, θ, q) be the the classical solutions to the initial-boundary value problem (1)-(5) corresponding to b_1 , and (u_i^*, T^*, p^*) and (v_i^*, θ^*, q^*) also be the the classical solutions to the initial-boundary value problem (1)-(5) but corresponding to b_2 . Then, for any $t > 0$, we have

$$(u_i, T, p) \longrightarrow (u_i^*, T^*, p^*), (v_i, \theta, q) \longrightarrow (v_i^*, \theta^*, q^*), \quad (48)$$

as $b_1 \longrightarrow b_2$. The differences of velocities satisfy

$$\int_0^t \left(\|w\|_{L_2(\Omega_1)}^2 + \|w^m\|_{L_2(\Omega_2)}^2 \right) d\eta \leq \frac{\sigma^2}{b_1 b_2} \int_0^t P_2(s) e^{N_M^2 \int_s^t a_1(\eta) d\eta} ds. \quad (49)$$

Furthermore, there is a positive function $a_1(t)$, given specifically in (58), such that

$$\|\Sigma\|_{L_2(\Omega_1)}^2 + \|\Sigma^m\|_{L_2(\Omega_2)}^2 \leq \frac{\sigma^2}{b_1 b_2} N_M^2 \int_0^t P_2(s) e^{N_M^2 \int_s^t a_1(\eta) d\eta} ds, \quad (50)$$

where w, w^m, Σ, Σ^m , and σ have been defined in (40) and (41).

Proof. We begin with the identity

$$\begin{aligned} \int_{\Omega_1} [\sigma |u|u_i + b_2[u_i|u| - u_i^*|u^*|] \\ + (1 + \gamma T^*)w_i + \gamma \Sigma u_i - g_i \Sigma + \pi_{,i}] w_i dx = 0. \end{aligned} \quad (51)$$

From (51), it follows that

$$\begin{aligned} \frac{1}{2} b_2 \left\| \sqrt{|u| + |u^*|} w \right\|_{L_2(\Omega_1)}^2 + \left\| \sqrt{(1 + \gamma T^*)} w \right\|_{L_2(\Omega_1)}^2 \\ = - \int_{\Omega_1} \pi_{,i} w_i dx - \sigma \int_{\Omega_1} |u| u_i w_i dx - \gamma \int_{\Omega_1} \Sigma u_i w_i dx \\ + \int_{\Omega_1} g_i w_i \Sigma dx. \end{aligned} \quad (52)$$

Using the divergence theorem, we have

$$\begin{aligned}
& \frac{1}{2} b_2 \left\| \sqrt{|u| + |u^*|} w \right\|_{L_2(\Omega_1)}^2 + \left\| \sqrt{(1 + \gamma T^*)} w \right\|_{L_2(\Omega_1)}^2 \\
&= -\sigma \int_{\Omega_1} |u| u_i w_i dx - \int_L \pi w_3 n_3^{(1)} dA - \gamma \int_{\Omega_1} \Sigma u_i w_i dx \\
&+ \int_{\Omega_1} g_i w_i \Sigma dx = -\sigma \int_{\Omega_1} |u| u_i w_i dx + \int_L \pi^m w_3^m n_3^{(2)} dA \\
&- \gamma \int_{\Omega_1} \Sigma u_i w_i dx + \int_{\Omega_1} g_i w_i \Sigma dx - \sigma \int_{\Omega_1} |u| u_i w_i dx \\
&+ \int_{\Omega_2} (-w_i^m + g_i \Sigma^m) w_i^m dx - \gamma \int_{\Omega_1} \Sigma u_i w_i dx \\
&+ \int_{\Omega_1} g_i w_i \Sigma dx.
\end{aligned} \tag{53}$$

We use the Cauchy-Schwarz inequality to have

$$\begin{aligned}
-\sigma \int_{\Omega_1} |u| u_i w_i dx &\leq \sigma \left\| \sqrt{|u|} w \right\|_{L_2(\Omega_1)} \|u\|_{L_3(\Omega_1)}^{3/2} \\
&\leq \frac{b_1 + b_2}{4} \left\| \sqrt{|u|} w \right\|_{L_2(\Omega_1)}^2 + \frac{\sigma^2}{2b_1(b_1 + b_2)} P_2(t).
\end{aligned} \tag{54}$$

Using Hölder's inequality and the A-G mean inequality, we have

$$\int_{\Omega_2} (-w_i^m + g_i \Sigma^m) w_i^m dx \leq -\frac{1}{2} \|w^m\|_{L_2(\Omega_2)}^2 + \frac{1}{2} g^2 \|\Sigma^m\|_{L_2(\Omega_2)}^2. \tag{55}$$

We find that the result given in Appendix B of Liu [13] for $\|u\|_{L_4(\Omega_1)}^2$

$$\|u\|_{L_4(\Omega_1)}^2 \leq k \left[\|u\|_{L_2(\Omega_1)}^2 + \|u\|_{L_2(\Omega_1)}^{1/2} \|\nabla u\|_{L_2(\Omega_1)}^{3/2} \right], \quad k > 0, \tag{56}$$

or

$$\|u\|_{L_4(\Omega_1)}^2 \leq k \left[\left(1 + \frac{\delta}{4} \right) \|u\|_{L_2(\Omega_1)}^2 + \frac{3}{4} \delta^{-1/3} \|\nabla u\|_{L_2(\Omega_1)}^2 \right], \quad k > 0, \tag{57}$$

for arbitrary positive constant $\delta > 0$. We use this inequality for $\|\Sigma\|_{L_4(\Omega_1)}^2$ with $\delta = 1$ and (2.30), (2.32) in (55) to get

$$\begin{aligned}
-\gamma \int_{\Omega_1} \Sigma u_i w_i dx &\leq \gamma \left\| \sqrt{|u|} w \right\|_{L_2(\Omega_1)} \|\Sigma\|_{L_4(\Omega_1)} \|u\|_{L_2(\Omega_1)}^{1/2} \\
&\leq \frac{b_1 + b_2}{4} \left\| \sqrt{|u|} w \right\|_{L_2(\Omega_1)}^2 + \frac{k\gamma^2}{b_1 + b_2} P_2^{1/2}(t) \left[\|\Sigma\|_{L_2(\Omega_1)}^2 \right. \\
&+ \left. \|\Sigma\|_{L_2(\Omega_1)}^{1/2} \|\nabla \Sigma\|_{L_2(\Omega_1)}^{3/2} \right] \leq \frac{b_1 + b_2}{4} \left\| \sqrt{|u|} w \right\|_{L_2(\Omega_1)}^2 \\
&+ \frac{3}{4} \delta_1 \|\nabla \Sigma\|_{L_2(\Omega_1)}^2 + \left(\frac{k\gamma^2}{b_1 + b_2} P_2^{1/2}(t) + \frac{k^4 P_2^2(t) \gamma^8}{64(b_1 + b_2)^4 \delta_1^3} \right) \\
&\times \|\Sigma\|_{L_2(\Omega_1)}^2,
\end{aligned} \tag{58}$$

for arbitrary $\delta_1 > 0$. We also have

$$\int_{\Omega_1} g_i w_i \Sigma dx \leq \frac{1}{2} \|w\|_{L_2(\Omega_1)}^2 + \frac{1}{2} g^2 \|\Sigma\|_{L_2(\Omega_1)}^2. \tag{59}$$

Combining (53), (54), (55), (58), and (59), we obtain

$$\begin{aligned}
& \frac{1}{2} b_2 \left\| \sqrt{|u| + |u^*|} w \right\|_{L_2(\Omega_1)}^2 + \left\| \sqrt{(1 + \gamma T^*)} w \right\|_{L_2(\Omega_1)}^2 \\
&+ \frac{1}{2} \|w^m\|_{L_2(\Omega_2)}^2 \leq \frac{b_1 + b_2}{2} \left\| \sqrt{|u|} w \right\|_{L_2(\Omega_1)}^2 \\
&+ \frac{\sigma^2}{2b_1(b_1 + b_2)} P_2(t) + \frac{1}{2} g^2 \|\Sigma^m\|_{L_2(\Omega_2)}^2 \\
&+ \left(\frac{k\gamma^2}{b_1 + b_2} P_2^{1/2}(t) + \frac{k^4 \gamma^8}{64(b_1 + b_2)^4 \delta_1^3} P_2^2(t) + \frac{1}{2} g^2 \right) \\
&\cdot \|\Sigma\|_{L_2(\Omega_1)}^2 + \frac{3}{4} \delta_1 \|\nabla \Sigma\|_{L_2(\Omega_1)}^2 + \frac{1}{2} \|w\|_{L_2(\Omega_1)}^2.
\end{aligned} \tag{60}$$

We follow a similar procedure starting from (47) to obtain

$$\begin{aligned}
& \frac{1}{2} b_1 \left\| \sqrt{|u| + |u^*|} w \right\|_{L_2(\Omega_1)}^2 + \left\| \sqrt{(1 + \gamma T)} w \right\|_{L_2(\Omega_1)}^2 \\
&+ \frac{1}{2} \|w^m\|_{L_2(\Omega_2)}^2 \leq \frac{b_1 + b_2}{2} \left\| \sqrt{|u^*|} w \right\|_{L_2(\Omega_1)}^2 \\
&+ \frac{\sigma^2}{2b_2(b_1 + b_2)} P_2(t) + \frac{1}{2} g^2 \|\Sigma^m\|_{L_2(\Omega_2)}^2 \\
&+ \left(\frac{k\gamma^2}{b_1 + b_2} P_2^{1/2}(t) + \frac{k^4 \gamma^8}{64(b_1 + b_2)^4 \delta_1^3} P_2^2(t) + \frac{1}{2} g^2 \right) \\
&\cdot \|\Sigma\|_{L_2(\Omega_1)}^2 + \frac{3}{4} \delta_1 \|\nabla \Sigma\|_{L_2(\Omega_1)}^2 + \frac{1}{2} \|w\|_{L_2(\Omega_1)}^2.
\end{aligned} \tag{61}$$

We add (60) and (61) to find

$$\begin{aligned}
& \left\| \sqrt{(1 + \gamma T + \gamma T^*)} w \right\|_{L_2(\Omega_1)}^2 + \|w^m\|_{L_2(\Omega_2)}^2 \\
&\leq \frac{\sigma^2}{2b_1 b_2} P_2(t) + a_1(t) \left(\|\Sigma\|_{L_2(\Omega_1)}^2 + \|\Sigma^m\|_{L_2(\Omega_2)}^2 \right) \\
&+ \frac{3}{2} \delta_1 \|\nabla \Sigma\|_{L_2(\Omega_1)}^2,
\end{aligned} \tag{62}$$

where

$$a_1(t) = \frac{2k\gamma^2}{b_1 + b_2} P_2^{1/2}(t) + \frac{k^4\gamma^8}{32(b_1 + b_2)^4\delta_1^3} P_2^2(t) + g^2. \quad (63)$$

To bound $\|\Sigma\|_{L_2(\Omega_1)}^2$ and $\|\Sigma^m\|_{L_2(\Omega_2)}^2$, we multiply (3.3)₂ and (3.4)₂ by Σ and Σ^m , respectively, and integrate by parts to find

$$\begin{aligned} & \left[\|\Sigma\|_{L_2(\Omega_1)}^2 + \|\Sigma^m\|_{L_2(\Omega_2)}^2 \right] \\ &= -2 \int_0^t \|\nabla \Sigma\|_{L_2(\Omega_1)}^2 d\eta - 2 \int_0^t \|\nabla \Sigma^m\|_{L_2(\Omega_2)}^2 d\eta \\ & \quad + 2 \int_0^t \int_{\Omega_1} w_i T \Sigma_{,i} dx d\eta + 2 \int_0^t \int_{\Omega_2} w_i^m \theta \Sigma_{,i}^m dx d\eta \quad (64) \\ &\leq - \int_0^t \|\nabla \Sigma\|_{L_2(\Omega_1)}^2 d\eta - \int_0^t \|\nabla \Sigma^m\|_{L_2(\Omega_2)}^2 d\eta \\ & \quad + N_M^2 \int_0^t \|w\|_{L_2(\Omega_1)}^2 d\eta + N_M^2 \int_0^t \|w^m\|_{L_2(\Omega_2)}^2 d\eta, \end{aligned}$$

or

$$\begin{aligned} & \|\Sigma\|_{L_2(\Omega_1)}^2 + \|\Sigma^m\|_{L_2(\Omega_2)}^2 + \int_0^t \|\nabla \Sigma\|_{L_2(\Omega_1)}^2 d\eta \\ & \quad + \int_0^t \|\nabla \Sigma^m\|_{L_2(\Omega_2)}^2 d\eta \leq N_M^2 \int_0^t \|w\|_{L_2(\Omega_1)}^2 d\eta \quad (65) \\ & \quad + N_M^2 \int_0^t \|w^m\|_{L_2(\Omega_2)}^2 d\eta. \end{aligned}$$

Combining (62) and (4.26), we find that

$$\begin{aligned} & \|w\|_{L_2(\Omega_1)}^2 + \|w^m\|_{L_2(\Omega_2)}^2 \leq \frac{\sigma^2}{2b_1b_2} P_2(t) + \frac{3}{2} \delta_1 \|\nabla \Sigma\|_{L_2(\Omega_1)}^2 \\ & \quad + a_1(t) N_M^2 \left(\int_0^t \|w\|_{L_2(\Omega_1)}^2 d\eta + \int_0^t \|w^m\|_{L_2(\Omega_2)}^2 d\eta \right), \quad (66) \end{aligned}$$

or

$$\Xi'(t) \leq \frac{\sigma^2}{2b_1b_2} P_2(t) + \frac{3}{2} \delta_1 \|\nabla \Sigma\|_{L_2(\Omega_1)}^2 + a_1(t) N_M^2 \Xi(t), \quad (67)$$

where

$$\Xi(t) = \int_0^t \|w\|_{L_2(\Omega_1)}^2 d\eta + \int_0^t \|w^m\|_{L_2(\Omega_2)}^2 d\eta. \quad (68)$$

Thus, after integration, we may derive from (67) the estimate:

$$\begin{aligned} \Xi(t) &\leq \frac{\sigma^2}{2b_1b_2} \int_0^t P_2(s) e^{N_M^2 \int_s^t a_1(\eta) d\eta} ds \\ & \quad + \frac{3}{2} \delta_1 e^{N_M^2 \int_0^t a_1(\eta) d\eta} \int_0^t \|\nabla \Sigma\|_{L_2(\Omega_1)}^2 d\eta. \quad (69) \end{aligned}$$

Using (65) and (66) and choosing δ_1 in (69), we have

$$\int_0^t \|w\|_{L_2(\Omega_1)}^2 d\eta + \int_0^t \|w^m\|_{L_2(\Omega_2)}^2 d\eta \leq \frac{\sigma^2}{b_1b_2} \int_0^t P_2(s) e^{N_M^2 \int_s^t a_1(\eta) d\eta} ds, \quad (70)$$

$$\|\Sigma\|_{L_2(\Omega_1)}^2 + \|\Sigma^m\|_{L_2(\Omega_2)}^2 \leq \frac{\sigma^2}{b_1b_2} N_M^2 \int_0^t P_2(s) e^{N_M^2 \int_s^t a_1(\eta) d\eta} ds. \quad (71)$$

Inequalities (70) and (71) are a *priori* bounds demonstrating continuous dependence of the solution on the Forchheimer coefficient b .

Data Availability

All data generated or analyzed during this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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