

## Research Article

# Euler-Type Integral Operator Involving $\mathcal{S}$ -Function

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The object of the present paper is to establish some interested theorems on Euler-type integrals involving  $\mathcal{S}$ -function, which is defined by Saxena and Daiya. Further, we reduce some special cases involving various known functions like the  $k$ -Mittag-Leffler function,  $\mathcal{H}$ -function, and  $\mathcal{M}$ -series.

## 1. Introduction

In recent years, fractional calculus has become a significant instrument for modeling analysis and has assumed a significant role in different fields, for example, material science, science, mechanics, power, economy, and control theory. Additionally, a variety of researchers researched a selection of fractional calculus operators in-depth with a scope on properties, implementations, and complex extensions. Also, other analogous topics are very active and extensive around the world. Recently, Saxena and Daiya [1] defined and studied a special function called an  $\mathcal{S}$ -function and its relation with other special functions, which is a generalization of  $\mathcal{H}$ -function,  $\mathcal{M}$ -series,  $k$ -Mittag-Leffler function, Mittag-Leffler-type functions, and many other special functions. For a detailed account of the  $\mathcal{S}$ -function along with its properties and applications, one can read [2–5]. Motivated by these research findings, we tend to establish some of the theorems of concern for the Euler form integral concerning  $\mathcal{S}$ -function and its related special function. Such specific functions have recently been established as important applications for solving problems in biological sciences, genetics, physics, and engineering.

The  $\mathcal{S}$ -function is defined for  $\sigma, \eta, \varepsilon, \tau \in \mathbb{C}$ ,  $\Re(\sigma) > 0$ ,  $k \in \mathfrak{R}$ ,  $\Re(\sigma) > k\Re(\tau)$ ,  $l_i (i = 1, 2, 3, \dots, p)$ ,  $m_j (j = 1, 2, 3, \dots, q)$ , and  $p < q + 1$  as

$$\mathcal{S}_{(p,q)}^{\sigma, \eta, \varepsilon, \tau, k} [l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; x] = \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau, k} x^n}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta) n!}. \quad (1)$$

Here, the  $k$ -Pochhammer symbol

$$(\varepsilon)_{n,k} = \begin{cases} \frac{\Gamma_k(\varepsilon + nk)}{\Gamma_k(\varepsilon)}, & \left( k \in \mathfrak{R}, \varepsilon \in \frac{\mathbb{C}}{\{0\}} \right), \\ \varepsilon(\varepsilon + k) \cdots (\varepsilon + (n-1)k), & (n \in \mathbb{N}, \varepsilon \in \mathbb{C}). \end{cases} \quad (2)$$

The integral representation of the  $k$ -gamma function is

$$\Gamma_k(\varepsilon) = k^{(\varepsilon/k)-1} \Gamma\left(\frac{\varepsilon}{k}\right) = \int_0^{\infty} e^{(-t^k/k)} t^{\varepsilon-1} dt, \quad (3)$$

and defined  $k$ -beta function on the form:

$$\mathfrak{B}_k(x, y) = \frac{1}{k} \int_0^1 t^{x(k)-1} (1-t)^{y(k)-1} dt = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad (4)$$

where  $x > 0, y > 0, \varepsilon \in \mathbb{C}, k \in \mathfrak{R}$ , and  $n \in \mathbb{N}$ , introduced and defined by Díaz and Pariguan [6], Kokologiannaki [7], and Krasniqi [8].

Several major special cases of the  $\mathcal{S}$ -function are described below:

- (i) For  $p = q = 0$ , the generalized  $k$ -Mittag-Leffler function is from Saxena et al. [9] (see [10, 11]).

$$\begin{aligned} \mathcal{E}_{k,\sigma,\eta}^{\sigma,\tau} (x) &= \mathcal{S}_{(0,0)}^{\sigma,\eta,\varepsilon,\tau,k} [-; -; x] \\ &= \sum_{n=0}^{\infty} \frac{(\varepsilon)_{n\tau,k} x^n}{\Gamma_k(n\sigma + \eta) n!}, \quad \Re\left(\left(\frac{\sigma}{k}\right) - \tau\right) > p - q \end{aligned} \tag{5}$$

(ii) For  $k = \tau = 1$ , the  $\mathcal{S}$ -function is the generalized  $\mathcal{K}$ -function, introduced by Sharma [12] (see also [13]).

$$\begin{aligned} \mathcal{K}_{(p,q)}^{\sigma,\eta,\varepsilon} [l_1, \dots, l_p; m_1, \dots, m_q; x] \\ &= \mathcal{S}_{(p,q)}^{\sigma,\eta,\varepsilon,1,1} [l_1, \dots, l_p; m_1, \dots, m_q; x] \\ &= \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_n x^n}{(m_1)_n \cdots (m_q)_n \Gamma(n\sigma + \eta) n!}, \quad \Re(\sigma) > p - q \end{aligned} \tag{6}$$

(iii) For  $\tau = k = \varepsilon = 1$ , the  $\mathcal{S}$ -function is reduced to generalized  $\mathcal{M}$ -series introduced by Sharma and Jain [14] (detail in [15]).

$$\begin{aligned} \mathcal{M}_{(p,q)}^{\sigma,\eta} [l_1, \dots, l_p; m_1, \dots, m_q; x] \\ &= \mathcal{S}_{(p,q)}^{\sigma,\eta,1,1,1} [l_1, \dots, l_p; m_1, \dots, m_q; x] \\ &= \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n x^n}{(m_1)_n \cdots (m_q)_n \Gamma(n\sigma + \eta)}, \quad \Re(\sigma) > p - q - 1 \end{aligned} \tag{7}$$

(iv) For  $\tau = k = \varepsilon = 1$  and  $p = q = 0$ , the  $\mathcal{S}$ -function is reduced to generalized Mittag-Leffler function, defined by Mittag-Leffler [16].

$$\begin{aligned} \mathcal{E}_{\sigma,\eta}^{\sigma,\eta} (x) &= \mathcal{S}_{(0,0)}^{\sigma,\eta,1,1,1} [-; -; x] \\ &= \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\sigma + \eta)}, \quad \Re(\sigma) > 0, \Re(\eta) > 0 \end{aligned} \tag{8}$$

Now, we mention the basic beta function indicated by  $\mathfrak{B}(a, b)$  that is described by Euler’s integral [17] as

$$\begin{aligned} \mathfrak{B}(a, b) &= \int_0^1 u^{a-1} (1-u)^{b-1} du \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (\Re(a) > 0, \Re(b) > 0). \end{aligned} \tag{9}$$

Euler has extended the factorial function from the natural number domain to the gamma function:

$$\Gamma(a) = \int_0^{\infty} u^{a-1} \exp(-u) du, \quad (\Re(a) > 0), \tag{10}$$

defined over the right half of the complex plane. Chaudhry and Zubair [18] expanded the scope of these functions to

the entire complex plane by adding an  $\exp(-A/u)$  regularization element in the integrand of equation (10). With  $A > 0$ , this element explicitly excludes the singularity resulting from the  $u = 0$  limit. For  $A = 0$ , this element is unity, and we get the gamma function originally used. We mention the relation below ([19], p. 20 (2)).

$$\begin{aligned} \Gamma_A(a) &= \int_0^{\infty} u^{a-1} \exp\left(-u - \frac{A}{u}\right) du \\ &= 2 A^{a/2} K_a\left(2\sqrt{A}\right), \quad (\Re(A) > 0), \end{aligned} \tag{11}$$

where  $K_a(x)$  is the altered Bessel function of the second kind of order  $n$ . We consider Riemann’s zeta function  $\zeta(x)$  described by the series ([20], p. 102 (2.101))

$$\zeta(u) = \sum_{m=1}^{\infty} \frac{1}{m^u}, \quad (u > 1), \tag{12}$$

is useful for comparison testing to provide convergence or divergence of certain series. Zeta function is directly connected to the gamma function logarithm and the polygamma functions. The  $\exp(-A/u)$  regularizer has also proven to be very useful in expanding the zeta function of Riemann’s domain, thereby supplying connections that could not have been achieved with the original zeta function. Considering the usefulness of the above regularizer for gamma and zeta functions, Chaudhry et al. [19] proposed the following expansion of Euler’s beta function as follows:

$$\mathfrak{B}_A(a, b) = \int_0^1 u^{a-1} (1-u)^{b-1} \exp\left[-\frac{A}{u(1-u)}\right] du. \tag{13}$$

Lee et al. [21] presented the extended Euler beta functions in the continuation of his research and described it as

$$\begin{aligned} \mathfrak{B}(a, b; A; m) &= \int_0^1 u^{a-1} (1-u)^{b-1} \\ &\cdot \exp\left[-\frac{A}{u^m(1-u)^m}\right] du, \quad (\Re(p) > \Re(m) > 0). \end{aligned} \tag{14}$$

In addition, new Euler generalizations of  $k$ -beta functions are described by Khan et al. [22] as follows:

$$\begin{aligned} \mathfrak{B}_k(a, b; A; m) &= \int_0^1 u^{(a/k)-1} (1-u)^{(b/k)-1} \\ &\exp\left[-\frac{A}{u^{m/k}(1-u)^{m/k}}\right] du, \end{aligned} \tag{15}$$

where  $k \in \mathfrak{R}, (\Re(p) > \Re(m) > 0)$ .

Obviously, if  $m = k = 1$ , equation (15) is reduced to (13) and then, by taking  $A = 0$  in (15), we get (9).

In this article, several Euler-type integral operator theorems concerning  $\mathcal{S}$ -function were obtained and some specific cases were addressed.

### 2. Euler-Type Integral Operator Involving $\mathcal{S}$ -Function

**Theorem 1.** *If  $\sigma, \eta, \varepsilon, \tau, \beta \in \mathbb{C}$ ,  $k \in \mathfrak{R}$ ,  $\Re(\sigma) > 0$ ,  $\Re(\delta) > 0$ ,  $\Re(A) > 0$ ,  $\Re(\sigma) > k\Re(\tau)$ ,  $l_i (i = 1, 2, 3, \dots, p)$ ,  $m_j (j = 1, 2, 3, \dots, q)$ , and  $p < q + 1$ , then*

$$\int_0^1 u^{\delta/k-1} (1-u)^{(\beta/k)-1} \exp\left(\frac{-A}{u^{m/k}(1-u)^{m/k}}\right) \times \mathcal{S}_{(p,q)}^{\sigma,\eta,\varepsilon,\tau,k} [l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; zu^{\sigma/k}] du = \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta)} \frac{z^n}{n!} \mathfrak{B}_k(\delta + \sigma n, \beta; A; m). \tag{16}$$

*Proof.* To derive (16), we denote (16) by  $I_1$  to L.H.S. and by using (1), to obtain

$$I_1 = \int_0^1 u^{(\delta/k)-1} (1-u)^{(\beta/k)-1} \exp\left(\frac{-A}{u^{m/k}(1-u)^{m/k}}\right) \times \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta)} \frac{z^n u^{\sigma n/k}}{n!} du. \tag{17}$$

Changing the summation and integration order (which is assured under the stated conditions), we now get

$$I_1 = \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta)} \frac{z^n}{n!} \times \int_0^1 u^{(\delta+\sigma n)/k-1} (1-u)^{\beta/k-1} \exp\left(\frac{-A}{u^{m/k}(1-u)^{m/k}}\right) du. \tag{18}$$

We obtain the necessary result by using (15) as in the equation (16) above.

**Corollary 2.** *For  $A = 0, \delta = \eta$  in Theorem 1, we deduce the following result:*

$$\int_0^1 u^{n/k-1} (1-u)^{\beta/k-1} \mathcal{S}_{(p,q)}^{\sigma,\eta,\varepsilon,\tau,k} [l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; zu^{\sigma/k}] du = k \Gamma_k(\beta) \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \beta + \eta)} \frac{z^n}{n!}. \tag{19}$$

**Theorem 3.** *If  $\sigma, \eta, \varepsilon, \tau, \beta \in \mathbb{C}$ ,  $k \in \mathfrak{R}$ ,  $\Re(\sigma) > 0$ ,  $\Re(\delta) > 0$ ,  $\Re(A) > 0$ ,  $\Re(\sigma) > k\Re(\tau)$ ,  $l_i (i = 1, 2, 3, \dots, p)$ ,  $m_j (j = 1, 2, 3, \dots, q)$ , and  $p < q + 1$ , then*

$$\int_u^s (x-u)^{\beta/k-1} (s-x)^{\delta/k-1} \exp\left(\frac{-A}{(x-u)^{m/k}(s-x)^{m/k}}\right) \times \mathcal{S}_{(p,q)}^{\sigma,\eta,\varepsilon,\tau,k} [l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; z(s-x)^{\sigma/k}] dx = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k} (-A)^r}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta)} \frac{z^n (s-u)^{(\sigma n + \beta + \delta - 2mr)/k-1}}{r! n!} \times k \mathfrak{B}_k(\delta + \sigma n - mr, \beta - mr). \tag{20}$$

*Proof.* We denote L.H.S. of (20) by  $I_2$  to derive (20), and then by changing the variable  $x$  to  $t = (x-u)/(s-u)$ , we get

$$I_2 = \int_0^1 (1-t)^{\delta/k-1} (s-u)^{\delta/k-1} (t)^{\beta/k-1} (s-u)^{\beta/k-1} \exp\left(\frac{-A}{((1-t)(s-u))^{m/k} (t(s-u))^{m/k}}\right) \times \mathcal{S}_{(p,q)}^{\sigma,\eta,\varepsilon,\tau,k} [l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; zu^{\sigma/k} (s-u)^{\sigma/k}] (s-u) dt. \tag{21}$$

Expanding the exponential function and  $\mathcal{S}$ -function in their respective series, we achieve

$$I_2 = \int_0^1 (1-t)^{\delta/k-1} (t)^{\beta/k-1} (s-u)^{\delta/k+\beta/k-2} \sum_{r=0}^{\infty} \left(\frac{(-A)^r}{(t)^{mr/k} (1-t)^{mr/k} (s-u)^{2mr/k} r!}\right) \times \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta)} \frac{z^n (1-t)^{\sigma n/k} (s-u)^{\sigma n/k+1}}{n!} dt. \tag{22}$$

Changing the summation and integration order (which is assured under the stated conditions), we now get

$$I_2 = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{(-A)^r}{r!}\right) \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta)} \frac{z^n (s-u)^{(\sigma n + \beta + \delta - 2mr)/k-1}}{n!} \times \int_0^1 t^{(\beta-mr)/k-1} (1-t)^{(\delta+\sigma n-mr)/k-1} dt. \tag{23}$$

And result ((20)) is needed further by using the integral ((4)).

**Corollary 4.** For  $A = 0, \beta = \eta$  in Theorem 3, the following conclusion is deduced:

$$\begin{aligned} & \int_u^s (x-u)^{\eta/k-1} (s-x)^{\delta/k-1} \mathcal{S}_{(p,q)}^{\sigma,\eta,\varepsilon,\tau,k} \\ & \cdot [l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; z(s-x)^{\sigma/k}] dx \\ & = k \Gamma_k(\eta) \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta + \delta)} \\ & \cdot \frac{z^n (s-u)^{(\sigma+\eta+\delta)/k-1}}{n!}. \end{aligned} \tag{24}$$

**Theorem 5.** If  $\sigma, \eta, \varepsilon, \tau, \beta \in \mathbb{C}, k \in \mathfrak{R}, \Re(\sigma) > 0, \Re(\delta) > 0, \Re(A) > 0, \Re(\sigma) > k\Re(\tau), l_i (i = 1, 2, 3, \dots, p), m_j (j = 1, 2, 3, \dots, q), \alpha, \rho \geq 0,$  and  $p < q + 1,$  then

$$\begin{aligned} & \int_0^1 u^{\delta/k-1} (1-u)^{(\beta-\delta)/k-1} (1-wu^{\alpha/k}(1-u)^{\rho/k})^{-\vartheta} \\ & \cdot \exp\left(\frac{-A}{u^{m/k}(1-u)^{m/k}}\right) \\ & \times \mathcal{S}_{(p,q)}^{\sigma,\eta,\varepsilon,\tau,k} [l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; zu^{\sigma/k}] du \\ & = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\vartheta)_r w^r}{r!} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta) n!} \frac{z^n}{n!} \\ & \mathfrak{B}_k(\delta + \alpha r + \sigma n, \beta - \delta + \rho r; A; m). \end{aligned} \tag{25}$$

*Proof.* In order to obtain (25), we represent L.H.S. of (25) by  $I_3$  and by using (1), to get

$$\begin{aligned} I_3 & = \int_0^1 u^{\delta/k-1} (1-u)^{(\beta-\delta)/k-1} \sum_{r=0}^{\infty} \frac{(\vartheta)_r w^r u^{\alpha r/k} (1-u)^{\rho r/k}}{r!} \\ & \cdot \exp\left(\frac{-A}{u^{m/k}(1-u)^{m/k}}\right) \\ & \times \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta) n!} \frac{z^n u^{\sigma n/k}}{n!} du. \end{aligned} \tag{26}$$

Now changing the order of summation and integration (which is assured under the specified conditions), we get

$$\begin{aligned} I_3 & = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\vartheta)_r w^r}{r!} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta) n!} \frac{z^n}{n!} \\ & \times \int_0^1 u^{(\delta+\alpha r+\sigma n)/k-1} (1-u)^{(\beta-\delta+\rho r)/k-1} \exp\left(\frac{-A}{u^{m/k}(1-u)^{m/k}}\right) du. \end{aligned} \tag{27}$$

By using (15) as seen in the above equation, we achieve the correct result.

**Corollary 6.** For  $\vartheta = 0$  in Theorem 5, we deduce the following result:

$$\begin{aligned} & \int_0^1 u^{\delta/k-1} (1-u)^{(\beta-\delta)/k-1} \exp\left(\frac{-A}{u^{m/k}(1-u)^{m/k}}\right) \\ & \times \mathcal{S}_{(p,q)}^{\sigma,\eta,\varepsilon,\tau,k} [l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; zu^{\sigma/k}] du \\ & = \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta) n!} \frac{z^n}{n!} \mathfrak{B}_k(\delta + \sigma n, \beta - \delta; A; m). \end{aligned} \tag{28}$$

**Corollary 7.** For  $A = 0, \vartheta = 0$  in Theorem 5, we deduce the subsequent result:

$$\begin{aligned} & \int_0^1 u^{\delta/k-1} (1-u)^{(\beta-\delta)/k-1} \mathcal{S}_{(p,q)}^{\sigma,\eta,\varepsilon,\tau,k} \\ & \cdot [l_1, l_2, \dots, l_p; m_1, m_2, \dots, m_q; zu^{\sigma/k}] du \\ & = \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma_k(n\sigma + \eta) n!} \frac{z^n}{n!} k \mathfrak{B}_k(\delta + \sigma n, \beta - \delta). \end{aligned} \tag{29}$$

### 3. Special Cases

In this section, we define as special cases of our key results the following potentially useful integral operators that include generalized  $k$ -Beta type functions and generalized Beta type functions:

- (1) On setting  $p = q = 0$  in Theorem 1, we get

$$\begin{aligned} & \int_0^1 u^{\delta/k-1} (1-u)^{\beta/k-1} \exp\left(\frac{-A}{u^{m/k}(1-u)^{m/k}}\right) \\ & \cdot \mathcal{E}_{k,\sigma,\eta}^{\varepsilon,\tau} [zu^{\sigma/k}] du \\ & = \sum_{n=0}^{\infty} \frac{(\varepsilon)_{n\tau,k}}{\Gamma_k(n\sigma + \eta) n!} \frac{z^n}{n!} \mathfrak{B}_k(\delta + \sigma n, \beta; A; m), \end{aligned} \tag{30}$$

where  $\mathcal{E}_{k,\sigma,\eta}^{\varepsilon,\tau}(z)$  is a  $k$ -Mittag-Leffler function (5)

- (2) On setting  $k = \tau = 1$  in Theorem 1, we get

$$\begin{aligned} & \int_0^1 u^{\delta-1} (1-u)^{\beta-1} \exp\left(\frac{-A}{u^m(1-u)^m}\right) \\ & \mathcal{H}_{(p,q)}^{\sigma,\eta,\varepsilon} [l_1, \dots, l_p; m_1, \dots, m_q; zu^{\sigma}] du \\ & = \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_n}{(m_1)_n \cdots (m_q)_n \Gamma(n\sigma + \eta) n!} \frac{z^n}{n!} \\ & \cdot \mathfrak{B}(\delta + \sigma n, \beta; A; m), \end{aligned} \tag{31}$$

where  $\mathcal{H}_{(p,q)}^{\sigma,\eta,\varepsilon}(z)$  is a  $\mathcal{H}$ -function (6)

(3) On setting  $\tau = k = \varepsilon = 1$  in Theorem 1, we get

$$\int_0^1 u^{\delta-1} (1-u)^{\beta-1} \exp\left(\frac{-A}{u^m(1-u)^m}\right) \cdot \mathcal{M}_{(p,q)}^{\sigma,\eta}[l_1, \dots, l_p; m_1, \dots, m_q; zu^\sigma] du = \sum_{n=0}^\infty \frac{(l_1)_n \cdots (l_p)_n z^n}{(m_1)_n \cdots (m_q)_n \Gamma(n\sigma + \eta)} \cdot \mathfrak{B}(\delta + \sigma n, \beta; A; m), \tag{32}$$

where  $\mathcal{M}_{(p,q)}^{\sigma,\eta}(z)$  is a  $\mathcal{M}$ -series (7)

(4) On setting  $\tau = k = \varepsilon = 1$ , and  $p = q = 0$  in Theorem 1, we get

$$\int_0^1 u^{\delta-1} (1-u)^{\beta-1} \exp\left(\frac{-A}{u^m(1-u)^m}\right) \mathcal{E}_{\sigma,\eta}[zu^\sigma] du = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\sigma + \eta)} \mathfrak{B}(\delta + \sigma n, \beta; A; m), \tag{33}$$

where  $\mathcal{E}_{\sigma,\eta}(z)$  is a generalized Mittag-Leffler function (8)

(5) On setting  $p = q = 0$  in Theorem 3, we get

$$\int_u^s (x-u)^{\beta/k-1} (s-x)^{\delta/k-1} \exp\left(\frac{-A}{(x-u)^{m/k}(s-x)^{m/k}}\right) \cdot \mathcal{E}_{k,\sigma,\eta}^{\varepsilon,\tau}[z(s-x)^{\sigma/k}] dx = \sum_{r=0}^\infty \sum_{n=0}^\infty \frac{(\varepsilon)_{nr,k} (-A)^r z^n (s-u)^{(\sigma n + \beta + \delta - 2mr)/k-1}}{k \Gamma_k(n\sigma + \eta) r! n!} \cdot k \mathfrak{B}_k(\delta + \sigma n - mr, \beta - mr), \tag{34}$$

where  $\mathcal{E}_{k,\sigma,\eta}^{\varepsilon,\tau}(z)$  is a  $k$ -Mittag-Leffler function (5)

(6) On setting  $k = \tau = 1$  in Theorem 3, we get

$$\int_u^s (x-u)^{\beta-1} (s-x)^{\delta-1} \exp\left(\frac{-A}{(x-u)^m(s-x)^m}\right) \times \mathcal{K}_{(p,q)}^{\sigma,\eta,\varepsilon}[l_1, \dots, l_p; m_1, \dots, m_q; z(s-x)^\sigma] dx = \sum_{r=0}^\infty \sum_{n=0}^\infty \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_n (-A)^r z^n (s-u)^{\sigma n + \beta + \delta - 2mr-1}}{(m_1)_n \cdots (m_q)_n \Gamma(n\sigma + \eta) r! n!} \cdot \mathfrak{B}(\delta + \sigma n - mr, \beta - mr), \tag{35}$$

where  $\mathcal{K}_{(p,q)}^{\sigma,\eta,\varepsilon}(z)$  is a  $\mathcal{K}$ -function (6)

(7) On setting  $k = \tau = \varepsilon = 1$  in Theorem 3, we get

$$\int_u^s (x-u)^{\beta-1} (s-x)^{\delta-1} \exp\left(\frac{-A}{(x-u)^m(s-x)^m}\right) \times \mathcal{M}_{(p,q)}^{\sigma,\eta}[l_1, \dots, l_p; m_1, \dots, m_q; z(s-x)^\sigma] dx = \sum_{r=0}^\infty \sum_{n=0}^\infty \frac{(l_1)_n \cdots (l_p)_n (-A)^r z^n (s-u)^{\sigma n + \beta + \delta - 2mr-1}}{(m_1)_n \cdots (m_q)_n \Gamma(n\sigma + \eta) r!} \cdot \mathfrak{B}(\delta + \sigma n - mr, \beta - mr), \tag{36}$$

where  $\mathcal{M}_{(p,q)}^{\sigma,\eta}(z)$  is a  $\mathcal{M}$ -series (7)

(8) On setting  $p = q = 0$  and  $k = \tau = \varepsilon = 1$  in Theorem 3, we get

$$\int_u^s (x-u)^{\beta-1} (s-x)^{\delta-1} \exp\left(\frac{-A}{(x-u)^m(s-x)^m}\right) \cdot \mathcal{E}_{\sigma,\eta}[z(s-x)^\sigma] dx = \sum_{r=0}^\infty \sum_{n=0}^\infty \frac{(-A)^r z^n (s-u)^{\sigma n + \beta + \delta - 2mr-1}}{\Gamma(n\sigma + \eta) r!} \cdot \mathfrak{B}(\delta + \sigma n - mr, \beta - mr), \tag{37}$$

where  $\mathcal{E}_{\sigma,\eta}(z)$  is a generalized Mittag-Leffler function (8)

(9) On setting  $p = q = 0$  in Theorem 5, we get

$$\int_0^1 u^{\delta/k-1} (1-u)^{(\beta-\delta)/k-1} (1-wu^{\alpha/k}(1-u)^{\rho/k})^{-\vartheta} \cdot \exp\left(\frac{-A}{u^{m/k}(1-u)^{m/k}}\right) \times \mathcal{E}_{k,\sigma,\eta}^{\varepsilon,\tau}[zu^{\sigma/k}] du = \sum_{r=0}^\infty \sum_{n=0}^\infty \frac{(\vartheta)_r w^r (\varepsilon)_{nr,k} z^n}{\Gamma_k(n\sigma + \eta) r! n!} \cdot \mathfrak{B}_k(\delta + \alpha r + \sigma n, \beta - \delta + \rho r; A; m), \tag{38}$$

where  $\mathcal{E}_{k,\sigma,\eta}^{\varepsilon,\tau}(z)$  is a  $k$ -Mittag-Leffler function (5)

(10) On setting  $k = \tau = 1$  in Theorem 5, we get

$$\int_0^1 u^{\delta-1} (1-u)^{\beta-\delta-1} (1-wu^\alpha(1-u)^\rho)^{-\vartheta} \cdot \exp\left(\frac{-A}{u^m(1-u)^m}\right) \times \mathcal{K}_{(p,q)}^{\sigma,\eta,\varepsilon}[l_1, \dots, l_p; m_1, \dots, m_q; zu^\sigma] du = \sum_{r=0}^\infty \sum_{n=0}^\infty \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_n (\vartheta)_r w^r z^n}{(m_1)_n \cdots (m_q)_n \Gamma(n\sigma + \eta) r! n!} \cdot \mathfrak{B}(\delta + \alpha r + \sigma n, \beta - \delta + \rho r; A; m), \tag{39}$$

where  $\mathcal{K}_{(p,q)}^{\sigma,\eta,\varepsilon}(z)$  is a  $\mathcal{K}$ -function (6)

(11) On setting  $\varepsilon = k = \tau = 1$  in Theorem 5, we get

$$\begin{aligned} & \int_0^1 u^{\delta-1} (1-u)^{\beta-\delta-1} (1-wu^\alpha(1-u)^\rho)^{-\vartheta} \\ & \quad \cdot \exp\left(\frac{-A}{u^m(1-u)^m}\right) \\ & \quad \times \mathcal{M}_{(p,q)}^{\sigma,\eta}[l_1, \dots, l_p; m_1, \dots, m_q; zu^\sigma] du \quad (40) \\ & = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\vartheta)_r w^r z^n}{(m_1)_n \cdots (m_q)_n \Gamma(n\sigma + \eta)r!} \\ & \quad \cdot \mathfrak{B}(\delta + \alpha r + \sigma n, \beta - \delta + \rho r; A; m), \end{aligned}$$

where  $\mathcal{M}_{(p,q)}^{\sigma,\eta}(z)$  is a  $\mathcal{M}$ -series (7)

(12) On setting  $p = q = 0$  and  $\varepsilon = k = \tau = 1$  in Theorem 5, we get

$$\begin{aligned} & \int_0^1 u^{\delta-1} (1-u)^{\beta-\delta-1} (1-wu^\alpha(1-u)^\rho)^{-\vartheta} \\ & \quad \cdot \exp\left(\frac{-A}{u^m(1-u)^m}\right) \mathcal{E}_{\sigma,\eta}[zu^\sigma] du \quad (41) \\ & = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\vartheta)_r w^r z^n}{\Gamma(n\sigma + \eta)r!} \\ & \quad \cdot \mathfrak{B}(\delta + \alpha r + \sigma n, \beta - \delta + \rho r; A; m), \end{aligned}$$

where  $\mathcal{E}_{\sigma,\eta}(z)$  is a generalized Mittag-Leffler function (8)

#### 4. Concluding Remark and Discussion

In this paper, we presented Euler-type integrals including the  $\mathcal{S}$ -function defined by [1]. Various special cases with similar outcomes of the report may be evaluated by taking acceptable values of the parameters concerned. For example, given in remark (i), [22, 23] give us the undeniable result. We refer to [24, 25] for a variety of other special cases and give the results to interested readers. Our paper ends with the remark that the stated outcome is important and can result in the yield of the number of other integral Euler forms involving various types of Wiman function, Prabhakar function, and exponential and binomial functions.

#### Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

#### Conflicts of Interest

The authors declare to have no competing interests.

#### Authors' Contributions

All authors contributed equally to the present investigation. All authors read and approved the final manuscript.

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