Research Article

Maclaurin Coefficient Estimates for New Subclasses of Bi-univalent Functions Connected with a q -Analogue of Bessel Function

Sheza M. El-Deeb ^{1,2}

¹Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt

²Department of Mathematics, College of Science and Arts in Badaya, Al-Qassim University, Al-Badaya, Al-Qassim Province, Saudi Arabia

Correspondence should be addressed to Sheza M. El-Deeb; shezaeldeeb@yahoo.com

Received 10 January 2020; Revised 20 February 2020; Accepted 28 April 2020; Published 22 May 2020

Academic Editor: Wei-Shih Du

Copyright © 2020 Sheza M. El-Deeb. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we introduce new subclasses of the function class Σ of bi-univalent functions connected with a q -analogue of Bessel function and defined in the open unit disc. Furthermore, we find estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

1. Introduction, Definitions, and Preliminaries

The theory of q -calculus plays an important role in many areas of mathematical, physical, and engineering sciences. Jackson (see [1, 2]) was the first to have some applications of the q -calculus and introduced the q -analogue of the classical derivative and integral operators (see also [3]). Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}, \quad (1)$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ which are univalent functions in Δ . If $g \in \mathcal{A}$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad z \in \Delta, \quad (2)$$

then, the Hadamard (or convolution) product of f and g is defined by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in \Delta. \quad (3)$$

If f and F are analytic functions in Δ , we say that f is subordinate to F , written $f < F$, if there exists a Schwarz function w , which is analytic in Δ , with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \Delta$, such that $f(z) = F(w(z))$, $z \in \Delta$. Furthermore, if the function F is univalent in Δ , then we have the following equivalence (see [4, 5]):

$$f(z) < F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(\Delta) \subset F(\Delta). \quad (4)$$

The Bessel function of the first kind of order ν is defined by the infinite series (see [6])

$$J_{\nu}(z) := \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k! \Gamma(k+\nu+1)} \quad (z \in \mathbb{C}, \nu \in \mathbb{R}) \quad (5)$$

where Γ stands for the Gamma function. Recently, Szász and Kupán [7] investigated the univalence of the normalized Bessel function of the first kind $k_{\nu} : \Delta \rightarrow \mathbb{C}$ defined by (see also [8–10])

$$\begin{aligned} k_{\nu}(z) &:= 2^{\nu} \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_{\nu}\left(z^{\frac{1}{2}}\right) \\ &= z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1} \Gamma(\nu+1)}{4^{k-1} (k-1)! \Gamma(k+\nu)} z^k, \quad (z \in \Delta, \nu \in \mathbb{R}). \end{aligned} \quad (6)$$

For $0 < q < 1$, the q -derivative operator for k_ν is defined by

$$\begin{aligned}\partial_q k_\nu(z) &= \partial_q \left[z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1} \Gamma(\nu+1)}{4^{k-1} (k-1)! \Gamma(k+\nu)} z^k \right] := \frac{k_\nu(qz) - k_\nu(z)}{z(q-1)} \\ &= 1 + \sum_{k=2}^{\infty} \frac{(-1)^{k-1} \Gamma(\nu+1)}{4^{k-1} (k-1)! \Gamma(k+\nu)} [k, q] z^{k-1}, \quad z \in \Delta,\end{aligned}\quad (7)$$

where

$$[k, q] := \frac{1 - q^k}{1 - q} = 1 + \sum_{j=1}^{k-1} q^j, \quad [0, q] := 0. \quad (8)$$

Using the definition formula (8), we will define the next two products:

- (i) For any nonnegative integer k , the q -shifted factorial is given by

$$[k, q]! := \begin{cases} 1, & \text{if } k = 0, \\ [1, q][2, q][3, q] \cdots [k, q], & \text{if } k \in \mathbb{N}. \end{cases} \quad (9)$$

- (ii) For any positive number r , the q -generalized Pochhammer symbol is defined by

$$[r, q]_k := \begin{cases} 1, & \text{if } k = 0, \\ [r, q][r+1, q] \cdots [r+k-1, q], & \text{if } k \in \mathbb{N}. \end{cases} \quad (10)$$

For $\nu > 0$, $\lambda > -1$, and $0 < q < 1$, El-Deeb and Bulboacă [11] define the function $\mathcal{J}_{\nu, q}^\lambda : \Delta \longrightarrow \mathbb{C}$ by

$$\mathcal{J}_{\nu, q}^\lambda(z) := z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1} \Gamma(\nu+1)}{4^{k-1} (k-1)! \Gamma(k+\nu)} \frac{[k, q]!}{[\lambda+1, q]_{k-1}} z^k, \quad z \in \Delta. \quad (11)$$

A simple computation shows that

$$\mathcal{J}_{\nu, q}^\lambda(z) * \mathcal{M}_{q, \lambda+1}(z) = z \partial_q k_\nu(z), \quad z \in \Delta, \quad (12)$$

where the function $\mathcal{M}_{q, \lambda+1}$ is given by

$$\mathcal{M}_{q, \lambda+1}(z) := z + \sum_{k=2}^{\infty} \frac{[\lambda+1, q]_{k-1}}{[k-1, q]!} z^k, \quad z \in \Delta. \quad (13)$$

Using the definition of q -derivative along with the idea of convolutions, El-Deeb and Bulboacă [11] introduce the linear operator $\mathcal{N}_{\nu, q}^\lambda : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$\begin{aligned}\mathcal{N}_{\nu, q}^\lambda f(z) &:= \mathcal{J}_{\nu, q}^\lambda(z) * f(z) \\ &= z + \sum_{k=2}^{\infty} \psi_k a_k z^k, \quad z \in \Delta, \quad (\nu > 0, \lambda > -1, 0 < q < 1),\end{aligned}\quad (14)$$

where

$$\psi_k := \frac{(-1)^{k-1} \Gamma(\nu+1)}{4^{k-1} (k-1)! \Gamma(k+\nu)} \cdot \frac{[k, q]!}{[\lambda+1, q]_{k-1}}. \quad (15)$$

Remark 1. From the definition relation (14), we can easily verify that the next relations hold for all $f \in \mathcal{A}$:

$$\begin{aligned}\text{(i)} \quad & [\lambda+1, q] \mathcal{N}_{\nu, q}^\lambda f(z) = [\lambda, q] \mathcal{N}_{\nu, q}^{\lambda+1} f(z) \\ & \quad + q^\lambda z \partial_q \left(\mathcal{N}_{\nu, q}^{\lambda+1} f(z) \right), \quad z \in \Delta; \\ \text{(ii)} \quad & \lim_{q \rightarrow 1} \mathcal{N}_{\nu, q}^\lambda f(z) = \mathcal{J}_{\nu, 1}^\lambda * f(z) =: \mathcal{J}_\nu^\lambda f(z) \\ & = z + \sum_{k=2}^{\infty} \frac{k!}{(\lambda+1)_{k-1}} \frac{(-1)^{k-1} \Gamma(\nu+1)}{4^{k-1} (k-1)! \Gamma(k+\nu)} \\ & \quad \cdot a_k z^k, \quad z \in \Delta.\end{aligned}\quad (16)$$

The Koebe one quarter theorem (see [12]) proves that the image of Δ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Therefore, every function $f \in \mathcal{S}$ has an inverse f^{-1} satisfying

$$\begin{aligned}f^{-1}(f(z)) &= z \quad (z \in \Delta), \\ f(f^{-1}(w)) &= w \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right),\end{aligned}\quad (17)$$

where

$$\begin{aligned}f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 - a_3) w^3 \\ &\quad - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots.\end{aligned}\quad (18)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both $f(z)$ and $f^{-1}(z)$ are univalent in Δ . Let Σ denote the class of bi-univalent functions in Δ given by (1). For a brief history and interesting examples in the class Σ , see [13]. Brannan and Taha [14] (see also [15–17]) introduced certain subclasses of the bi-univalent functions class Σ similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order α ($0 \leq \alpha < 1$), respectively (see [13]). Thus, following Brannan and Taha [14], a function $f \in \mathcal{A}$ is

Definition 5. Let $f(z)$ be given by (1), $h(z)$ and $p(w)$ in P have the forms

$$h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \dots, \quad (27)$$

$$p(w) = 1 + u_1 w + u_2 w^2 + u_3 w^3 + \dots, \quad (28)$$

then $f(z)$ is said to be in the class $Q_\Sigma(h, p, \gamma, v, q, \lambda)$ if the following conditions are satisfied:

$$\begin{aligned} f \in \Sigma \text{ and } \frac{z \left(\mathcal{N}_{v,q}^\lambda f(z) \right)' + \gamma z^2 \left(\mathcal{N}_{v,q}^\lambda f(z) \right)''}{(1-\gamma) \left(\mathcal{N}_{v,q}^\lambda f(z) \right) + \gamma z \left(\mathcal{N}_{v,q}^\lambda f(z) \right)'} &\in h(\Delta) (v > 0; \\ \lambda > -1; 0 < q < 1; 0 < \alpha \leq 1; 0 \leq \gamma \leq 1, \quad z \in \Delta), \end{aligned} \quad (29)$$

$$\begin{aligned} \frac{w \left(\mathcal{N}_{v,q}^\lambda g(w) \right)' + \gamma w^2 \left(\mathcal{N}_{v,q}^\lambda g(w) \right)''}{(1-\gamma) \left(\mathcal{N}_{v,q}^\lambda g(w) \right) + \gamma w \left(\mathcal{N}_{v,q}^\lambda g(w) \right)'} &\in p(\Delta) (v > 0; \\ \lambda > -1; 0 < q < 1; 0 < \alpha \leq 1; 0 \leq \gamma \leq 1, \quad w \in \Delta), \end{aligned} \quad (30)$$

where the function g is given by (2).

2. Coefficient Bounds for the Function

Class $Q_\Sigma^q(\alpha, \gamma, \lambda, v)$

Unless otherwise mentioned, we assume throughout this paper that

$$\begin{aligned} 0 < \alpha \leq 1, \\ 0 \leq \beta < 1, \\ 0 \leq \gamma \leq 1, \\ v > 0, \\ \lambda > -1, \\ 0 < q < 1, \\ z \in \Delta. \end{aligned} \quad (31)$$

Theorem 6. Let $f(z)$ be given by (1) which belongs to the class $Q_\Sigma^q(\alpha, \gamma, \lambda, v)$, then

$$|a_2| \leq \frac{2\alpha}{\sqrt{4\alpha(1+2\gamma)\psi_3 + (1-3\alpha)(1+\gamma)^2\psi_2^2}}, \quad (32)$$

$$|a_3| \leq \frac{\alpha}{(1+2\gamma)\psi_3} + \frac{4\alpha^2}{(1+\gamma)^2\psi_2^2}. \quad (33)$$

where ψ_k , $k \in \{2, 3\}$, are given by (15).

Proof. It follows from (21) and (22) that

$$\frac{z \left(\mathcal{N}_{v,q}^\lambda f(z) \right)' + \gamma z^2 \left(\mathcal{N}_{v,q}^\lambda f(z) \right)''}{(1-\gamma) \left(\mathcal{N}_{v,q}^\lambda f(z) \right) + \gamma z \left(\mathcal{N}_{v,q}^\lambda f(z) \right)'} = [p(z)]^\alpha, \quad (34)$$

$$\frac{w \left(\mathcal{N}_{v,q}^\lambda g(w) \right)' + \gamma w^2 \left(\mathcal{N}_{v,q}^\lambda g(w) \right)''}{(1-\gamma) \left(\mathcal{N}_{v,q}^\lambda g(w) \right) + \gamma w \left(\mathcal{N}_{v,q}^\lambda g(w) \right)'} = [q(w)]^\alpha, \quad (35)$$

where $p(z)$ and $q(w)$ in P and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots, \quad (36)$$

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \dots. \quad (37)$$

Now, equating the coefficients in (34) and (35), we get

$$(1+\gamma)\psi_2 a_2 = p_1 \alpha, \quad (38)$$

$$2(1+2\gamma)\psi_3 a_3 - (1+\gamma)^2 \psi_2^2 a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2, \quad (39)$$

$$-(1+\gamma)\psi_2 a_2 = q_1 \alpha, \quad (40)$$

$$2(1+2\gamma)(2a_2^2 - a_3)\psi_3 - (1+\gamma)^2 \psi_2^2 a_2^2 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2. \quad (41)$$

From (38) and (40), we get

$$p_1 = -q_1, \quad (42)$$

$$2(1+\gamma)^2 a_2^2 \psi_2^2 = \alpha^2 (p_1^2 + q_1^2). \quad (43)$$

From (39), (41), and (43), we obtain

$$a_2^2 = \frac{\alpha^2 (p_2 + q_2)}{4\alpha(1+2\gamma)\psi_3 + (1-3\alpha)(1+\gamma)^2\psi_2^2}. \quad (44)$$

Applying Lemma 4 for the coefficients p_2 and q_2 , we immediately have

$$|a_2| \leq \frac{2\alpha}{\sqrt{4\alpha(1+2\gamma)\psi_3 + (1-3\alpha)(1+\gamma)^2\psi_2^2}}. \quad (45)$$

This gives the bound on $|a_2|$ as asserted in (32).

Next, in order to find the bound on $|a_3|$, by subtracting (41) from (39), we get

$$\begin{aligned} 4(1+2\gamma)a_3\psi_3 - 4(1+2\gamma)\psi_3 a_2^2 \\ = \alpha(p_2 - q_2) + \frac{\alpha(\alpha-1)}{2} (p_1^2 - q_1^2). \end{aligned} \quad (46)$$

It follows from (42), (43), and (46) that

$$a_3 = \frac{\alpha^2(p_1^2 + q_1^2)}{2(1+\gamma)^2\psi_2^2} + \frac{\alpha(p_2 - q_2)}{4(1+2\gamma)\psi_3}. \quad (47)$$

Applying Lemma 4 once again for the coefficients p_1 , p_2 , q_1 , and q_2 , we immediately have

$$|a_3| \leq \frac{4\alpha^2}{(1+\gamma)^2\psi_2^2} + \frac{\alpha}{(1+2\gamma)\psi_3}. \quad (48)$$

This completes the proof of Theorem 6.

3. Coefficient Bounds for the Function

Class $\mathcal{P}_\Sigma^q(\beta, \gamma, \lambda, \nu)$

Theorem 7. Let $f(z)$ be given by (1) which belongs to the class $\mathcal{P}_\Sigma^q(\beta, \gamma, \lambda, \nu)$, then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{|2(1+2\gamma)\psi_3 - (1+\gamma)^2\psi_2^2|}}. \quad (49)$$

$$|a_3| \leq \frac{(1-\beta)}{(1+2\gamma)\psi_3} + \frac{4(1-\beta)^2}{(1+\gamma)^2\psi_2^2}, \quad (50)$$

where ψ_k , $k \in \{2, 3\}$, are given by (15).

Proof. It follows from (24) and (25) that

$$\frac{z\left(\mathcal{N}_{\nu,q}^\lambda f(z)\right)' + \gamma z^2\left(\mathcal{N}_{\nu,q}^\lambda f(z)\right)''}{(1-\gamma)\left(\mathcal{N}_{\nu,q}^\lambda f(z)\right) + \gamma z\left(\mathcal{N}_{\nu,q}^\lambda f(z)\right)'} = \beta + (1-\beta)p(z) \quad (51)$$

$$\frac{w\left(\mathcal{N}_{\nu,q}^\lambda g(w)\right)' + \gamma w^2\left(\mathcal{N}_{\nu,q}^\lambda g(w)\right)''}{(1-\gamma)\left(\mathcal{N}_{\nu,q}^\lambda g(w)\right) + \gamma w\left(\mathcal{N}_{\nu,q}^\lambda g(w)\right)'} = \beta + (1-\beta)q(w), \quad (52)$$

where $p(z)$ and $q(w)$ have the forms (36) and (37), respectively. Equating the coefficients in (51) and (52), we get

$$(1+\gamma)\psi_2 a_2 = (1-\beta)p_1, \quad (53)$$

$$2(1+2\gamma)\psi_3 a_3 - (1+\gamma)^2\psi_2^2 a_2^2 = (1-\beta)p_2, \quad (54)$$

$$-(1+\gamma)\psi_2 a_2 = (1-\beta)q_1, \quad (55)$$

$$2(1+2\gamma)(2a_2^2 - a_3)\psi_3 - (1+\gamma)^2\psi_2^2 a_2^2 = (1-\beta)q_2. \quad (56)$$

From (53) and (55), we get

$$p_1 = -q_1, \quad (57)$$

$$2(1+\gamma)^2\psi_2^2 a_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \quad (58)$$

From (54) and (56), we obtain

$$a_2^2 = \frac{(1-\beta)(p_2 + q_2)}{4(1+2\gamma)\psi_3 - 2(1+\gamma)^2\psi_2^2}. \quad (59)$$

Applying Lemma 4 for the coefficients p_2 and q_2 , we immediately have

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{|2(1+2\gamma)\psi_3 - (1+\gamma)^2\psi_2^2|}}. \quad (60)$$

This gives the bound on $|a_2|$ as asserted in (49).

Next, in order to find the bound on $|a_3|$, by subtracting (56) from (54), we get

$$4(1+2\gamma)\psi_3 a_3 - 4(1+2\gamma)\psi_2^2 a_2^2 = (1-\beta)(p_2 - q_2). \quad (61)$$

It follows from (58) and (61) that

$$a_3 = \frac{(1-\beta)(p_2 - q_2)}{4(1+2\gamma)\psi_3} + \frac{(1-\beta)^2(p_1^2 + q_1^2)}{2(1+\gamma)^2\psi_3}. \quad (62)$$

Applying Lemma 4 once again for the coefficients p_1 , p_2 , q_1 , and q_2 , we immediately have

$$|a_3| \leq \frac{(1-\beta)}{(1+2\gamma)\psi_3} + \frac{4(1-\beta)^2}{(1+\gamma)^2\psi_2^2}. \quad (63)$$

This completes the proof of Theorem 6.

4. General Coefficient Bounds for the Function

Class $Q_\Sigma^q(h, p, \gamma, \lambda, \nu)$

This section begins by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $Q_\Sigma^q(h, p, \gamma, \lambda, \nu)$.

Theorem 8. Let $f(z)$ be given by (1) which belongs to the class $Q_\Sigma^q(h, p, \gamma, \lambda, \nu)$, then

$$|a_2| \leq \min \left\{ \sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{2(1+\gamma)^2\psi_2^2}}, \sqrt{\frac{|h''(0)| + |u''(0)|}{|4(1+2\gamma)\psi_3 - 2(1+\gamma)^2\psi_2^2|}} \right\}, \quad (64)$$

$$|a_3| \leq \min \left\{ \frac{|h'(0)|^2 + |u'(0)|^2}{2(1+\gamma)^2\psi_2^2} + \frac{|h''(0)| + |u''(0)|}{4(1+2\gamma)\psi_3}, \frac{|4(1+2\gamma)\psi_3 - (1+\gamma)^2\psi_2^2||h''(0)| + (1+\gamma)^2\psi_2^2|u''(0)|}{|4(1+2\gamma)\psi_3[2(1+2\gamma)\psi_3 - (1+\gamma)^2\psi_2^2]|} \right\}, \quad (65)$$

where ψ_k , $k \in \{2, 3\}$, are given by (15).

Proof. It follows from (29) and (30) that

$$\frac{z\left(\mathcal{N}_{v,q}^\lambda f(z)\right)' + \gamma z^2\left(\mathcal{N}_{v,q}^\lambda f(z)\right)''}{(1-\gamma)\left(\mathcal{N}_{v,q}^\lambda f(z)\right) + \gamma z\left(\mathcal{N}_{v,q}^\lambda f(z)\right)'} = h(\Delta), \quad (66)$$

$$\frac{w\left(\mathcal{N}_{v,q}^\lambda g(w)\right)' + \gamma w^2\left(\mathcal{N}_{v,q}^\lambda g(w)\right)''}{(1-\gamma)\left(\mathcal{N}_{v,q}^\lambda g(w)\right) + \gamma w\left(\mathcal{N}_{v,q}^\lambda g(w)\right)'} = p(\Delta), \quad (67)$$

where $h(z)$ and $p(w)$ are given by (27) and (28). Now, equating the coefficients of z^2 and z^3 in (66) and (67), we get

$$(1+\gamma)\psi_2 a_2 = h_1, \quad (68)$$

$$2(1+2\gamma)\psi_3 a_3 - (1+\gamma)^2 \psi_2^2 a_2^2 = h_2, \quad (69)$$

$$-(1+\gamma)\psi_2 a_2 = u_1, \quad (70)$$

$$2(1+2\gamma)(2a_2^2 - a_3)\psi_3 - (1+\gamma)^2 \psi_2^2 a_2^2 = u_2. \quad (71)$$

From (68) and (70), we get

$$h_1 = -u_1, \quad (72)$$

$$a_2^2 = \frac{h_1^2 + u_1^2}{2(1+\gamma)^2 \psi_2^2}. \quad (73)$$

From (69) and (71), we obtain

$$a_2^2 = \frac{h_2 + u_2}{4(1+2\gamma)\psi_3 - 2(1+\gamma)^2 \psi_2^2}. \quad (74)$$

Therefore, we obtain from the equations (73) and (74) that

$$|a_2| \leq \sqrt{\frac{|h'(0)|^2 + |u'(0)|^2}{2(1+\gamma)^2 \psi_2^2}}, \quad (75)$$

$$|a_2| \leq \sqrt{\frac{|h''(0)| + |u''(0)|}{|4(1+2\gamma)\psi_3 - 2(1+\gamma)^2 \psi_2^2|}},$$

respectively, this gives the bound on $|a_2|$ as asserted in (64).

Next, in order to find the bound on $|a_3|$, by subtracting (71) from (69), we get

$$4(1+2\gamma)a_3\psi_3 - 4(1+2\gamma)\psi_3 a_2^2 = h_2 - u_2. \quad (76)$$

It follows from (73) into (76) that

$$a_3 = \frac{h_1^2 + u_1^2}{2(1+\gamma)^2 \psi_2^2} + \frac{h_2 - u_2}{4(1+2\gamma)\psi_3}. \quad (77)$$

We immediately have

$$|a_3| \leq \frac{|h'(0)|^2 + |u'(0)|^2}{2(1+\gamma)^2 \psi_2^2} + \frac{|h''(0)| + |u''(0)|}{4(1+2\gamma)\psi_3}. \quad (78)$$

On the other hand, upon substituting the value of a_2^2 from (74) into (76), we get

$$a_3 = \frac{[4(1+2\gamma)\psi_3 - (1+\gamma)^2 \psi_2^2]h_2 + (1+\gamma)^2 \psi_2^2 u_2}{4(1+2\gamma)\psi_3 [2(1+2\gamma)\psi_3 - (1+\gamma)^2 \psi_2^2]}. \quad (79)$$

Consequently, we get

$$|a_3| \leq \frac{|4(1+2\gamma)\psi_3 - (1+\gamma)^2 \psi_2^2| |h''(0)| + (1+\gamma)^2 \psi_2^2 |u''(0)|}{|4(1+2\gamma)\psi_3 [2(1+2\gamma)\psi_3 - (1+\gamma)^2 \psi_2^2]|}. \quad (80)$$

This completes the proof of Theorem 8.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

References

- [1] F. H. Jackson, "XI.—On q -functions and a certain difference operator," *Transactions of the Royal Society of Edinburgh*, vol. 46, no. 2, pp. 253–281, 1909.
- [2] F. H. Jackson, "On q -definite integrals," *Quarterly Journal of Pure and Applied Mathematics*, vol. 41, pp. 193–203, 1910.
- [3] M. H. Abu Risha, M. H. Annaby, M. E. H. Ismail, and Z. S. Mansour, "Linear q -difference equations," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 26, pp. 481–494, 2007.
- [4] T. Bulboacă, *Differential subordinations and superordinations. Recent results*, House of Scientific Book Publishers, Cluj-Napoca, 2005.
- [5] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics*, vol. 225, Marcel Dekker Inc., New York and Basel, 2000.
- [6] M. Naeem, S. Hussain, F. Muge Sakar, T. Mahmood, and A. Rasheed, "Subclasses of uniformly convex and starlike functions associated with Bessel functions," *Turkish Journal of Mathematics*, vol. 43, no. 5, pp. 2433–2443, 2019.
- [7] R. Szász and P. A. Kupán, "About the univalence of the Bessel functions," *Studia Universitatis Babeş-Bolyai, Mathematica*, vol. LIV, pp. 127–132, 2009.
- [8] Á. Baricz, "Geometric properties of generalized Bessel functions," *Publicationes Mathematicae*, vol. 73, pp. 155–178, 2008.
- [9] F. H. Jackson, "The application of basic numbers to Bessel's and Legendre's functions," *Proceedings of the London Mathematical Society*, vol. s2-2, no. 1, pp. 192–220, 1905.

- [10] K. A. Selvakumaran and R. Szász, "Certain geometric properties of an integral operator involving Bessel functions," *Kyungpook National University*, vol. 58, pp. 507–517, 2018.
- [11] S. M. el-Deeb and T. Bulboacă, "Fekete-Szegő inequalities for certain class of analytic functions connected with q -analogue of Bessel function," *Journal of the Egyptian Mathematical Society*, vol. 27, no. 1, 2019.
- [12] P. L. Duren, *Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259*, Springer-Verlage, New York, Berlin, Heidelberg and Tokyo, 1983.
- [13] D. A. Brannan, J. Clunie, and W. E. Kirwan, "Coefficient estimates for a class of starlike functions," *Canadian Journal of Mathematics*, vol. 22, no. 3, pp. 476–485, 1970.
- [14] D. A. Brannan and T. S. Taha, "On some classes of bi-univalent functions," *Mathematical Analysis and Its Applications, Kuwait; February 18-21, 1985, Proceedings of the International Conference on Mathematical Analysis and its Applications*, vol. 3, S. M. Mazhar, A. Hamoui, and N. S. Faour, Eds., , pp. 53–60, Pergamon Press(Elsevier Science Limited), Oxford, 1988.
- [15] A. Akgül and F. Müge Sakar, "A certain subclass of bi-univalent analytic functions introduced by means of the q -analogue of Noor integral operator and Horadam polynomials," *Turkish Journal of Mathematics*, vol. 43, no. 5, pp. 2275–2286, 2019.
- [16] S. M. el-Deeb, T. Bulboacă, and B. M. el-Matary, "Maclaurin coefficient estimates of bi-univalent functions connected with the q -derivative," *Mathematics*, vol. 8, no. 3, p. 418, 2020.
- [17] T. S. Taha, *Topics in univalent function theory*, Ph.D. thesis, University of London, 1981.
- [18] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1188–1192, 2010.
- [19] C. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Gottingen, 1975.