Research Article

Inclusion Relation between Various Subclasses of Harmonic Univalent Functions Associated with Wright’s Generalized Hypergeometric Functions

Rajavadivelu Themangani,¹ Saurabh Porwal,² and Nanjundan Magesh³

¹Post-Graduate and Research Department of Mathematics, Voorhees College, Vellore, 632 001 Tamil Nadu, India
²Department of Mathematics, Ram Sahai Government Degree College, Bairi-Shivrajpur, Kanpur, 209205 (U.P.), India
³Post-Graduate and Research Department of Mathematics, Government Arts College (Men), Krishnagiri, 635 001 Tamil Nadu, India

Correspondence should be addressed to Saurabh Porwal; saurabhjcb@rediffmail.com

Received 11 September 2020; Revised 6 November 2020; Accepted 16 November 2020; Published 28 November 2020

Academic Editor: Jacek Dziok

Copyright © 2020 Rajavadivelu Themangani et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The purpose of the present paper is to obtain some inclusion relation between various subclasses of harmonic univalent functions by applying certain convolution operators associated with Wright’s generalized hypergeometric functions.

1. Introduction

A continuous complex-valued function \( f = u + iv \) defined in a simply connected domain \( D \) is said to be harmonic in \( D \) if both \( u \) and \( v \) are real harmonic in \( D \). In any simply connected domain \( D \), we can write \( f = h + \frac{g}{z} \), where \( h \) and \( g \) are analytic in \( D \). In 1984, Clunie and Sheil-Small [1] introduced a class \( \mathcal{SH} \) of complex-valued harmonic maps \( f \) which are univalent and sense-preserving in the open unit disk \( U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \). The function \( f \in \mathcal{SH} \) can be represented by

\[
h(z) = z + \sum_{n=1}^{\infty} h_n z^n, \quad g(z) = \sum_{n=1}^{\infty} g_n z^n, \quad |g_1| < 1,
\]

are analytic in the open unit disk \( U \). They also proved that the function \( f = h + g \in \mathcal{SH} \) is locally univalent and sense-preserving in \( U \), if and only if \( |h'(z)| > |g'(z)|, \forall z \in U \).

For more basic studies, one may refer to Duren [2] and Ahuja [3]. It is worthy to note that if \( g(z) \equiv 0 \) in (1), then the class \( \mathcal{SH} \) reduces to the familiar class \( \mathcal{S} \) of analytic functions. For this class, \( f(z) \) may be expressed as of the form

\[
f(z) = z + \sum_{n=2}^{\infty} h_n z^n. \quad (2)
\]

Further, we suppose \( \mathcal{SH}^{\varphi} \) subclass of \( \mathcal{SH} \) consisting of function \( f \in \mathcal{SH} \) of the form (1) with \( g_1 = 0 \). Now, we let \( \mathcal{K}^{\varphi}_{1\alpha}, \mathcal{S}^{\varphi \ast, \alpha}, \text{ and } \mathcal{C}^{\varphi}_{1\alpha} \) denote the subclasses of \( \mathcal{SH}^{\varphi}(\gamma) \) of harmonic functions which are, respectively, convex, starlike, and close-to-convex in \( U \). Also, let \( \mathcal{SH}^{\varphi} \) be the class of sense-preserving, typically real harmonic functions \( f = h + \bar{g} \in \mathcal{SH}^{\varphi} \). For a detailed study of these classes, one may refer to [1, 2].

A function \( f = h + \bar{g} \) of the form (1) is said to be in the class \( \mathcal{N}^{\varphi}(\gamma) \), if it satisfy the condition

\[
\Re \left\{ \frac{f'(z)}{z} \right\} \geq \gamma, \quad 0 \leq \gamma < 1, \quad z = re^{i\theta} \in U. \quad (3)
\]

Similarly, a function \( f = h + \bar{g} \) of the form (1) is said to be in the class \( \mathcal{G}^{\varphi}_{1\alpha}(\gamma) \), if it satisfy the condition

\[
\Re \left\{ \frac{f'(z)}{z} \right\} \geq \gamma, \quad 0 \leq \gamma < 1, \quad z = re^{i\theta} \in U. \quad (3)
\]
where 
\[ z' = (\partial/\partial \theta)(re^{i\theta}) \quad \text{and} \quad f'(z) = (\partial/\partial \theta)(f(ye^{i\theta})). \]

Now, we define the subclass \( \mathcal{S}_{\mathcal{F}}(y) \) of \( \mathcal{S}_{\mathcal{F}} \) consisting of functions \( f = h + g \), so that \( h \) and \( g \) are of the form

\[ h(z) = z - \sum_{n=2}^{\infty} |h_n| z^n, \]

\[ g(z) = \sum_{n=2}^{\infty} |g_n| z^n. \]  

Define \( \mathcal{N}_{\mathcal{F}}(y) = \mathcal{N}_{\mathcal{F}}(y) \cap \mathcal{F} \) and \( \mathcal{G}_{\mathcal{F}}(y) = \mathcal{G}_{\mathcal{F}}(y) \cap \mathcal{F}, \) where \( \mathcal{F} \) consists of the functions \( f = h + g \in \mathcal{S}_{\mathcal{F}} \).

The classes \( \mathcal{N}_{\mathcal{F}}(y), \mathcal{N}_{\mathcal{F}}(y), \mathcal{G}_{\mathcal{F}}(y), \) and \( \mathcal{G}_{\mathcal{F}}(y) \), were studied, respectively, by Akuh and Jahangiri [4] and Rosy et al. [5].

Let \( a_i \in \mathbb{C}, \) \((a_i/A_i) \neq 0, \) \(-1, -2, \cdots; i = 1, 2, \cdots, p)\) and \((b_j/B_j) \neq 0, \) \(-1, -2, \cdots; j = 1, 2, \cdots, q)\), for \( A_i > 0 (i = 1, \cdots, p)\) and \( B_j > 0 (j = 1, \cdots, q)\) with

\[ 1 + \sum_{i=1}^{q} B_i - \sum_{j=1}^{p} A_i \geq 0. \]  

Wright’s generalized hypergeometric functions [6] is defined by

\[ p_{\Psi_q}^r \left[ \frac{(a_i, A_i)_{1, p}}{(b_j, B_j)_{1, q}} ; z \right] = \sum_{n=0}^{\infty} \prod_{n=1}^{p} \Gamma(a_i + nA_i) \prod_{n=1}^{q} \Gamma(b_j + nB_j) \frac{z^n}{n!}, \]  

which is analytic for suitable bounded values of \( |z| \) (see also [7, 8]). The generalized Mittag-Leffler, Bessel-Maitland, and generalized hypergeometric functions are some of the important special cases of Wright’s generalized hypergeometric functions, and for their details, one may refer to [8].

For \( A_i > 0 (i = 1, \cdots, p)\), \( B_j > 0 (j = 1, \cdots, q)\) with \( 1 + \sum_{i=1}^{q} B_i - \sum_{j=1}^{p} A_i \geq 0 \) and \( C_j > 0 (i = 1, \cdots, r), \) \( D_j > 0, \) \( d_j > 0 (j = 1, \cdots, s)\) with \( 1 + \sum_{i=1}^{r} D_j - \sum_{j=1}^{s} C_j \geq 0 \), we define Wright’s generalized hypergeometric functions:

\[ p_{\Psi_q}^r \left[ \frac{(a_i, A_i)_{1, p}}{(b_j, B_j)_{1, q}} ; z \right] = \sum_{n=0}^{\infty} \prod_{n=1}^{p} \Gamma(a_i + nA_i) \prod_{n=1}^{q} \Gamma(b_j + nB_j) \frac{z^n}{n!}, \]  

\[ r_{\Psi_s}^s \left[ \frac{(c_j, C_j)_{1, r}}{(d_j, D_j)_{1, s}} ; z \right] = \sum_{n=0}^{\infty} \prod_{n=1}^{r} \Gamma(c_j + nC_j) \prod_{n=1}^{s} \Gamma(d_j + nD_j) \frac{z^n}{n!}, \]  

with

\[ \frac{\prod_{n=1}^{r} \Gamma(c_j + nC_j) \Gamma(d_j) \Gamma(c_j) \Gamma(d_j)}{\prod_{n=1}^{s} \Gamma(d_j + nD_j) \Gamma(d_j)} < 1. \]  

We consider a harmonic univalent function

\[ W(z) = H(z) + G(z) \in \mathcal{S}_{\mathcal{F}}, \]  

where

\[ H(z) = z \frac{\prod_{i=1}^{p} \Gamma(b_i + (n-1)A_i) \Gamma(c_i) \Gamma(d_i)}{\prod_{i=1}^{r} \Gamma(b_i + (n-1)B_i) \Gamma(c_i) \Gamma(d_i)} (n-1)! \],

\[ G(z) = \sigma z \frac{\prod_{i=1}^{r} \Gamma(c_i + (n-1)C_i) \Gamma(d_i) \Gamma(c_i)}{\prod_{i=1}^{s} \Gamma(c_i + (n-1)D_i) \Gamma(d_i) \Gamma(d_i)} (n-1)!. \]  

From (12), we have for \( n \in \mathbb{N} = \{1, 2, \cdots\} \)

\[ |h_n| \leq \frac{\prod_{i=1}^{p} \Gamma(b_i + (n-1)A_i) \Gamma(c_i)}{\prod_{i=1}^{r} \Gamma(b_i + (n-1)B_i) \Gamma(c_i)} (n-1)! = \nu_n, \]  

\[ |k_n| \leq \frac{\prod_{i=1}^{r} \Gamma(c_i + (n-1)C_i) \Gamma(d_i)}{\prod_{i=1}^{s} \Gamma(c_i + (n-1)D_i) \Gamma(d_i)} (n-1)! = \eta_n. \]  

For some fixed value of \( j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and for

\[ \prod_{i=1}^{q} B_i \geq \prod_{i=1}^{p} A_i, \]  

\[ \prod_{i=1}^{r} D_i \geq \prod_{i=1}^{s} C_i, \]  

we denote

\[ p_{\Psi_q}^r \left[ \frac{(a_i + jA_i, A_i)_{1, p}}{(b_j + jB_j, B_j)_{1, q}} ; 1 \right] = p_{\Psi_q}^r, \]  

\[ r_{\Psi_s}^s \left[ \frac{(c_j + jC_j, C_j)_{1, r}}{(d_j + jD_j, D_j)_{1, s}} ; 1 \right] = r_{\Psi_s}^s. \]
provided that
\[ \sum_{i=1}^{q} b_{i} - \sum_{i=1}^{p} |a_{i}| + \frac{p-q}{2} > \frac{1}{2} + j, \]
and
\[ \sum_{i=1}^{s} d_{i} - \sum_{i=1}^{r} |c_{i}| + \frac{r-s}{2} > \frac{1}{2} + j. \]

Making use of (13) and (15), we have
\[ \sum_{n=1+j}^{\infty} (n-j) \nu_{n} = \frac{\prod_{s=1}^{q} \Gamma(b_{s})}{\prod_{s=1}^{p} \Gamma(|a_{s}|)} h_{q} \psi_{p}, \]
\[ \sum_{n=1+j}^{\infty} (n-j) \eta_{n} = \frac{\prod_{s=1}^{r} \Gamma(d_{s})}{\prod_{s=1}^{r} \Gamma(|c_{s}|)} g_{r} \psi_{s}, \]
provided that (16) holds true.

The convolution of two functions \( f(z) \) of the form (1) and \( F(z) \) of the form
\[ F(z) = z + \sum_{n=2}^{\infty} H_{n} z^{n} + \sum_{n=1}^{\infty} G_{n} z^{n}, \]
is given by
\[ (f * F)(z) = f(z) * F(z) = z + \sum_{n=2}^{\infty} h_{n} H_{n} z^{n} + \sum_{n=1}^{\infty} \tilde{g}_{n} G_{n} z^{n}. \]

Now, we introduce a convolution operator \( \Omega(p, q, r, s) \) as
\[ \Omega(p, q, r, s)f(z) = f(z) * W(z) = h(z) * H(z) + g(z) * G(z), \]
where \( f = h + \tilde{g} \) and \( W(z) = H(z) + G(z) \) given by (1) and (10), respectively. Hence
\[ \Omega(p, q, r, s)f(z) = z + \sum_{n=2}^{\infty} \theta_{n} h_{n} z^{n} + \sum_{n=1}^{\infty} \tilde{\xi}_{n} g_{n} z^{n}. \]

The application of the special functions on the geometric function theory always attracts researchers with various kinds of special functions, for example, hypergeometric functions [9–11], confluent hypergeometric functions [12], generalized hypergeometric functions [6, 13], Bessel functions [14], generalized Bessel functions [15–17], Wright functions [18–21], Fox-Wright functions [6, 22], and Mittag-Leffler functions [23] that have rich applications in analytic and harmonic univalent functions. By using special functions, some researchers introduce operators, for example, Carlson-Shaffer operator [24], Hohlov operator [25], and Dziok-Srivastava operator [26, 27], and obtain interesting results. Motivated with the work of [20], we obtain some inclusion relation between the classes \( \mathcal{S}_{\mathcal{F}}(\gamma) \), \( \mathcal{K}_{\mathcal{F}} \), \( \mathcal{E}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}}^{0}, \mathcal{D}_{\mathcal{F}}^{0} \), and \( \mathcal{N}_{\mathcal{F}}(\beta) \) by applying the convolution operator \( \Omega \).

2. Main Results

In order to establish our main results, we shall require the following lemmas.

Lemma 1 [1]. If \( f = h + \tilde{g} \in \mathcal{K}_{\mathcal{F}}^{0} \), where \( h \) and \( g \) are given by (5) with \( g_{i} = 0 \), then
\[ |h_{n}| \leq \frac{n + 1}{2}, \]
\[ |g_{n}| \leq \frac{n - 1}{2}. \]

Lemma 2 [1]. Let \( f = h + \tilde{g} \in \mathcal{E}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}}^{0} \), where \( h \) and \( g \) are given by (1) with \( g_{i} = 0 \). Then
\[ |h_{n}| \leq \frac{(2n + 1)(n + 1)}{6}, \]
\[ |g_{n}| \leq \frac{(2n - 1)(n - 1)}{6}. \]

Lemma 3 [5]. Let \( f = h + \tilde{g} \) be given by (5). If \( 0 \leq \gamma < 1 \) and
\[ \sum_{n=2}^{\infty} (2n - 1 - \gamma)|h_{n}| + \sum_{n=1}^{\infty} (2n + 1 + \gamma)|g_{n}| \leq 1 - \gamma, \]
then \( f \) is a sense-preserving Goodman-Rønning-type harmonic univalent function in \( U \) and \( f \in \mathcal{S}_{\mathcal{F}}(\gamma) \).

Remark 4. In [5], it is also shown that \( f = h + \tilde{g} \) given by (5) is in the family \( \mathcal{S}_{\mathcal{F}}(\gamma) \), if and only if the coefficient condition (24) holds. Moreover, if \( f \in \mathcal{S}_{\mathcal{F}}(\gamma) \), then
\[ |h_{n}| = \frac{1 - \gamma}{2n - 1 - \gamma}, \quad n \geq 2, \]
\[ |g_{n}| = \frac{1 - \gamma}{2n + 1 + \gamma}, \quad n \geq 1. \]

Theorem 5. Let \( \sum_{i=1}^{q} b_{i} - \sum_{i=1}^{p} |a_{i}| + ((p - q)/2) > 5/2 \) and \( \sum_{i=1}^{r} d_{i} - \sum_{i=1}^{s} |c_{i}| + ((r - s)/2) > 5/2 \), and if the inequality
\[ \frac{\prod_{s=1}^{q} \Gamma(b_{s})}{\prod_{s=1}^{p} \Gamma(|a_{s}|)} \left\{ \frac{\gamma_{2} \psi_{2}^{2} + (\gamma - \gamma_{2}) \psi_{1}^{2} + 2(1 - \gamma) \left( \frac{p \psi_{0} - 1}{p \psi_{0} - q} \right)}{\prod_{s=1}^{r} \Gamma(|c_{s}|)} \left\{ \frac{\gamma_{2} \psi_{2}^{2} + (\gamma + 5 \gamma_{2}) \psi_{3}^{1} + 2(1 - \gamma) \left( \frac{p \psi_{0} - 1}{p \psi_{0} - q} \right)}{\prod_{s=1}^{r} \Gamma(|c_{s}|)} \right\} \right\} \]
holds, then \( \Omega(K_{\mathcal{F}}^{0}) \subset \mathcal{S}_{\mathcal{F}}(\gamma) \).

Proof. Let \( f = h + \tilde{g} \in \mathcal{K}_{\mathcal{F}}^{0} \), where \( h \) and \( g \) are given by (1) with \( g_{i} = 0 \). We have to prove that \( \Omega(f) \in \mathcal{S}_{\mathcal{F}}(\gamma) \), where \( \Omega(f) \) is
defined by (21). To prove \( \Omega(f) \in \mathcal{S}_e^\alpha(\gamma) \), in view of Lemma 3, it is sufficient to prove that \( P_1 \leq 1 - \gamma \), where

\[
P_1 = \frac{\sum_{n=2}^{\infty} (2n - 1 - \gamma) |\theta_n| h_n + \sum_{n=2}^{\infty} (2n + 1 + \gamma) |\xi_n g_n|}{2}.
\]

(27)

By using Lemma 1,

\[
P_1 \leq \frac{1}{2} \left[ \sum_{n=2}^{\infty} (2n - 1)(2n - 2) + (7 - \gamma)(n - 1) + 2(1 - \gamma) \right] v_n
\]

+ \begin{equation}
\sigma \left[ \sum_{n=2}^{\infty} (2n - 2) + (5 + \gamma) \right] \eta_n
\end{equation}

\[
= \frac{1}{2} \left[ \prod_{i=1}^{p} \Gamma(b_i) \prod_{i=1}^{q} \Gamma(c_i) \right] \left\{ 2 + (7 - \gamma) \Psi^1 + 2(1 - \gamma) \left( \Psi_2^3 - 1 \right) \right\}
\]

+ \begin{equation}
\sigma \prod_{i=1}^{p} \Gamma(d_i) \prod_{i=1}^{q} \Gamma(c_i) \left\{ 2, \Psi^2 + (5 + \gamma) \Psi^1 \right\} \leq 1 - \gamma,
\end{equation}

(28)

by the given hypothesis. This completes the proof of Theorem 5.

The result is sharp for the function

\[
L(z) = z + \sum_{n=2}^{\infty} \left( \frac{n + 1}{2} \right) z^n - \sum_{n=2}^{\infty} \left( \frac{n - 1}{2} \right) z^n.
\]

(29)

Theorem 6. Let \( \sum_{i=1}^{p} b_i - \sum_{i=1}^{q} a_i = \left( \frac{p - q}{2} \right) \right) > 7/2 \) and \( \sum_{i=1}^{p} d_i - \sum_{i=1}^{q} c_i = \left( \frac{(r - s)}{2} \right) > 7/2 \), and if the inequality

\[
\prod_{i=1}^{p} \Gamma(b_i) \prod_{i=1}^{q} \Gamma(a_i) \left\{ 4, \Psi^3 + (28 - 2\gamma) \Psi_2^3 + (39 - 9\gamma) \Psi_2^1 \right\}
\]

+ \begin{equation}
6 \left( 1 - \gamma \right) \left( \Psi_2^3 - 1 \right) \right) + \sigma \prod_{i=1}^{p} \Gamma(d_i) \prod_{i=1}^{q} \Gamma(c_i) \left\{ 4, \Psi^3 + 2(10 + \gamma) \Psi_2^1 + 3(5 + \gamma) \Psi_2^1 \right\} \leq 6 \left( 1 - \gamma \right),
\end{equation}

(30)

holds, then \( \Omega(S^\alpha_{e^\alpha}(\gamma)) \in \mathcal{S}_e^\alpha(\gamma) \) and \( \Omega(G_{e^\alpha}(\gamma)) \in \mathcal{S}_e^\alpha(\gamma) \).

Proof. Let \( f = h + \bar{g} \in S^\alpha_{e^\alpha}(\gamma) \) or \( G_{e^\alpha}(\gamma) \), where \( h \) and \( g \) are given by (1) with \( g_1 = 0 \); we need to prove that \( \Omega(f) \in \mathcal{S}_e^\alpha(\gamma) \), where \( \Omega(f) \) is defined by (21). In view of Lemma 3, it is sufficient to prove that \( P_1 \leq 1 - \gamma \), where \( P_1 \) is given by (27).

Now using Lemma 2, we have

\[
P_1 \leq \frac{1}{6} \left[ \sum_{n=2}^{\infty} (n + 1)(2n + 1)(2n - 1 - \gamma) |\theta_n| \right]
\]

+ \begin{equation} \sigma \sum_{n=2}^{\infty} (n + 1)(2n - 1 + \gamma) |\xi_n| \right]
\end{equation}

\[
= \frac{1}{6} \left[ \sum_{n=2}^{\infty} (4n - 1)(n - 2)(n - 3) + (28 - 2\gamma)(n - 1)(n - 2)
\right]
\]

+ \begin{equation} (39 - 9\gamma)(n - 1) + 6(1 - \gamma) \right] v_n \right]
\end{equation}

+ \begin{equation} \sigma \sum_{n=2}^{\infty} (4n - 1)(n - 2)(n - 3)
\right]
\end{equation}

+ \begin{equation} (20 + 2\gamma)(n - 1)(n - 2) + (15 + 3\gamma)(n - 1) \right] \eta_n \right]
\end{equation}

\[
= \frac{1}{6} \left[ \prod_{i=1}^{p} \Gamma(b_i) \prod_{i=1}^{q} \Gamma(a_i) \left\{ 4, \Psi^3 + (28 - 2\gamma) \Psi_2^3 + (39 - 9\gamma) \Psi_2^1 \right\}
\right]
\]

+ \begin{equation} \sigma \prod_{i=1}^{p} \Gamma(d_i) \prod_{i=1}^{q} \Gamma(c_i) \left\{ 4, \Psi^3 + 2(10 + \gamma) \Psi_2^1 + 3(5 + \gamma) \Psi_2^1 \right\} \right] \leq 1 - \gamma,
\end{equation}

(31)

by the given hypothesis. Thus, the proof of Theorem 6 is established.

The result is sharp for the function

\[
f(z) = H(z) + G(z),
\]

(32)

where

\[
H(z) = \frac{z - (1/2)z^2 + (1/6)z^3}{(1 - z)^3},
\]

(33)

\[
G(z) = \frac{(1/2)z^2 + (1/6)z^3}{(1 - z)^3}.
\]

In our next theorem, we establish connections between \( \mathcal{S}_e(\gamma) \) and \( \mathcal{S}_e(\gamma) \).

Theorem 7. Let \( \sum_{i=1}^{p} b_i - \sum_{i=1}^{q} a_i = \left( \frac{p - q}{2} \right) \right) > 1/2 \) and \( \sum_{i=1}^{p} d_i - \sum_{i=1}^{q} c_i = \left( \frac{(r - s)}{2} \right) > 1/2 \), and if the inequality

\[
\prod_{i=1}^{p} \Gamma(b_i) \prod_{i=1}^{q} \Gamma(a_i) \left\{ \Psi^3 q_1 - 1 \right\} + \sigma \prod_{i=1}^{p} \Gamma(d_i) \prod_{i=1}^{q} \Gamma(c_i) \Psi^1 \leq 1,
\]

(34)

holds, then \( \Omega(F_{e}(\gamma)) \in \mathcal{S}_e(\gamma) \).
Proof. Let \( f = h + g \in \mathcal{F} G_{\mathcal{P}}(y) \) be given by (1). We have to prove that \( P_2 \leq 1 - \gamma \), where

\[
P_2 = \sum_{n=2}^{\infty} (2n - 1 - \gamma) |\theta_n h_n| + |\sigma| \sum_{n=1}^{\infty} (2n + 1 + \gamma) |\zeta_n g_n|.
\]

(35)

Now, using Remark 4, we have

\[
P_2 \leq (1 - \gamma) \sum_{n=2}^{\infty} v_n + (1 - \gamma) \sigma \sum_{n=1}^{\infty} \eta_n = (1 - \gamma) \left( \prod_{n=1}^{\infty} \Gamma(b_n) \right) \left( p \Psi^0 q - 1 \right) + |\sigma| \left( \prod_{n=1}^{\infty} \Gamma(c_n) \right) \Psi^0 s,
\]

\[
\leq 1 - \gamma,
\]

(36)

by the given hypothesis. This completes the proof of Theorem 7.

The result is sharp for the function

\[
f(z) = z - \sum_{n=2}^{\infty} \left( \frac{1 - \gamma}{2n - 1 - \gamma} \right) |x_n|z^n + \sum_{n=1}^{\infty} \left( \frac{1 - \gamma}{2n + 1 + \gamma} \right) |y_n|z^n,
\]

(37)

where

\[
\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1.
\]

(38)

3. Some Consequences of the Main Results

If we let \( p = q = r = s = 1 \) and \( a_1 = A_1 = c_1 = C_1 = 1 \) in (10), then \( W(z) \) reduces to a harmonic univalent function \( E(z) \) involving the following generalized Mittag-Leffler functions as

\[
E(z) = z \Gamma(b_1) E_{b_1, b_1}^{1,1}[z] + \sigma z \Gamma(d_1) E_{d_1, d_1}^{1,1}[z],
\]

(39)

where

\[
E_{b_1, b_1}^{1,1}[z] = \Psi^1 \left[ \frac{(1, 1)}{(b_1, b_1)} ; z \right] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(b_1 + n b_1)},
\]

(40)

\[
E_{d_1, d_1}^{1,1}[z] = \Psi^1 \left[ \frac{(1, 1)}{(d_1, d_1)} ; z \right] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(d_1 + n d_1)}.
\]

With these specializations, the convolution operator \( \Omega(p, q, r, s) \) reduces to the operator \( \Phi(b_1; B_1; d_1; D_1) \), which is defined as

\[
\Phi(b_1; B_1; d_1; D_1) f(z) = f(z) * E(z) = h(z) * z \Gamma(b_1) E_{b_1, b_1}^{1,1}[z] + \sigma g(z) * z \Gamma(d_1) E_{d_1, d_1}^{1,1}[z].
\]

(41)

For these specific values of \( p = q = r = s = 1 \) and \( a_1 = A_1 = c_1 = C_1 = 1 \), Theorems 5–7 yield the following results.

**Corollary 8. If the inequality**

\[
\Gamma(b_1) \left\{ 2 E_{b_1, b_1}^{1,1}(1) + (7 - \gamma) E_{b_1, b_1}^{1,1}(1) + 2(1 - \gamma) \left( E_{b_1, b_1}^{1,1} - 1 \right) \right\} + |\sigma| \Gamma(d_1) \left\{ 2 E_{d_1, d_1}^{1,1}(1) + (5 + \gamma) E_{d_1, d_1}^{1,1}(1) \right\} \leq 2 (1 - \gamma),
\]

holds, then \( \Phi(K_0^1) \subset \mathcal{G}(\mathcal{P}) \).

**Corollary 9. If the inequality**

\[
\Gamma(b_1) \left\{ 4 E_{b_1, b_1}^{1,1}(1) + 2(28 - 2\gamma) E_{b_1, b_1}^{1,1}(1) + (39 - 9\gamma) E_{b_1, b_1}^{1,1}(1) + 2(1 - \gamma) \left( E_{b_1, b_1}^{1,1} - 1 \right) \right\} + |\sigma| \Gamma(d_1) \left\{ 4 E_{d_1, d_1}^{1,1}(1) + 2(10 + \gamma) E_{d_1, d_1}^{1,1}(1) + 3(5 + \gamma) E_{d_1, d_1}^{1,1}(1) \right\} \leq 6 (1 - \gamma),
\]

holds, then \( \Phi(S_{\mathcal{P}}^0) \subset \mathcal{G}(\mathcal{P}) \) and \( \Phi(C_{\mathcal{P}}^0) \subset \mathcal{G}(\mathcal{P}) \).

**Corollary 10. If the inequality**

\[
\Gamma(b_1) \left\{ \left( E_{b_1, b_1}^{1,1} - 1 \right) \right\} + |\sigma| \Gamma(d_1) \left\{ E_{d_1, d_1}^{1,1} - 1 \right\} \leq 1,
\]

holds, then \( \Phi(\mathcal{G}(\mathcal{P})) \subset \mathcal{G}(\mathcal{P}) \).

**Remark 11.** If we put \( p = q = r = s = 1 \), \( a_1 = c_1 = 1 \), \( A_1 = C_1 = 0 \), and \( \sigma = 1 \), then

\[
W(z) = z + \sum_{n=2}^{\infty} \Gamma(b_n) \Gamma(b_1 + B_n(n - 1))(n - 1)! z^n + \sum_{n=1}^{\infty} \frac{\Gamma(d_1)}{\Gamma(d_1 + D_1(n - 1))(n - 1)!} z^n,
\]

(45)

and results of Theorems 5–7 reduce to corresponding results of Maharana and Sahoo [28].

**Remark 12.** If we put \( p = r = 2 \), \( q = s = 1 \), \( A_1 = A_2 = B_1 = B_2 = C_2 = D_2 = 1 \), and \( \sigma = 1 \), then

\[
W(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_n}{(b_1)_n} \frac{(a_2)_n}{(b_2)_n} \frac{z^n}{(n - 1)!} + \sum_{n=2}^{\infty} \frac{(c_1)_n (c_2)_n}{(d_1)_n (d_2)_n} \frac{z^n}{(n - 1)!},
\]

(46)

and results of Theorems 5–7 reduce to corresponding results of Porwal and Dixit [11].
Data Availability
No data is required.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Authors’ Contributions
All authors equally worked on the results, and they read and approved the final manuscript.

References