Hindawi
Abstract and Applied Analysis
Volume 2020, Article ID 8811810, 6 pages
https://doi.org/10.1155/2020/8811810

Research Article

Inclusion Relation between Various Subclasses of Harmonic Univalent Functions Associated with Wright's Generalized Hypergeometric Functions

Rajavadivelu Themangani,[1] **Saurabh Porwal**,[2] **and Nanjundan Magesh**[3]

[1]*Post-Graduate and Research Department of Mathematics, Voorhees College, Vellore, 632 001 Tamil Nadu, India*
[2]*Department of Mathematics, Ram Sahai Government Degree College, Bairi-Shivrajpur, Kanpur, 209205 (U.P.), India*
[3]*Post-Graduate and Research Department of Mathematics, Government Arts College (Men), Krishnagiri, 635 001 Tamil Nadu, India*

Correspondence should be addressed to Saurabh Porwal; saurabhjcb@rediffmail.com

Received 11 September 2020; Revised 6 November 2020; Accepted 16 November 2020; Published 28 November 2020

Academic Editor: Jacek Dziok

The purpose of the present paper is to obtain some inclusion relation between various subclasses of harmonic univalent functions by applying certain convolution operators associated with Wright's generalized hypergeometric functions.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if both u and v are real harmonic in $\mathbb{D}$. In any simply connected domain $\mathbb{D}$, we can write $f = h + \bar{g}$, where h and g are analytic in $\mathbb{D}$. In 1984, Clunie and Sheil-Small [1] introduced a class $\mathcal{S}_{\mathcal{H}}$ of complex-valued harmonic maps f which are univalent and sense-preserving in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}$ and $|z| < 1\}$. The function $f \in \mathcal{S}_{\mathcal{H}}$ can be represented by $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} g_n z^n, \quad |g_1| < 1, \tag{1}$$

are analytic in the open unit disk $\mathbb{U}$. They also proved that the function $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}$ is locally univalent and sense-preserving in $\mathbb{U}$, if and only if $|h'(z)| > |g'(z)|$, $\forall\, z \in \mathbb{U}$. For more basic studies, one may refer to Duren [2] and Ahuja [3]. It is worthy to note that if $g(z) \equiv 0$ in (1), then the class $\mathcal{S}_{\mathcal{H}}$ reduces to the familiar class $\mathcal{S}$ of analytic functions. For this class, $f(z)$ may be expressed as of the form

$$f(z) = z + \sum_{n=2}^{\infty} h_n z^n. \tag{2}$$

Further, we suppose $\mathcal{S}_{\mathcal{H}}^0$ subclass of $\mathcal{S}_{\mathcal{H}}$ consisting of function $f \in \mathcal{S}_{\mathcal{H}}$ of the form (1) with $g_1 = 0$. Now, we let K_H^0, $\mathcal{S}_{\mathcal{H}}^{*,0}$, and C_H^0 denote the subclasses of $\mathcal{S}_{\mathcal{H}}^0$ of harmonic functions which are, respectively, convex, starlike, and close-to-convex in $\mathbb{U}$. Also, let $\mathcal{T}_{\mathcal{H}}^0$ be the class of sense-preserving, typically real harmonic functions $f = h + \bar{g}$ in $\mathcal{S}_{\mathcal{H}}$. For a detailed study of these classes, one may refer to [1, 2].

A function $f = h + \bar{g}$ of the form (1) is said to be in the class $\mathcal{N}_{\mathcal{H}}(\gamma)$, if it satisfy the condition

$$\Re\left\{\frac{f'(z)}{z'}\right\} \geq \gamma, \quad 0 \leq \gamma < 1, \ z = re^{i\theta} \in \mathbb{U}. \tag{3}$$

Similarly, a function $f = h + \bar{g}$ of the form (1) is said to be in the class $G_H(\gamma)$, if it satisfy the condition

$$\Re\left\{\left(1+e^{i\alpha}\right)\frac{zf'(z)}{f(z)}-e^{i\alpha}\right\}\geq\gamma,\quad 0\leq\gamma<1,\ \alpha\in\mathbb{R},\ z=re^{i\theta}\in\mathbb{U},\tag{4}$$

where $z'=(\partial/\partial\theta)(re^{i\theta})$ and $f'(z)=(\partial/\partial\theta)(f(\gamma e^{i\theta}))$.

Now, we define the subclass $\mathcal{T}\mathcal{S}_{\mathcal{H}}$ of $\mathcal{S}_{\mathcal{H}}$ consisting of functions $f=h+\bar{g}$, so that h and g are of the form

$$h(z)=z-\sum_{n=2}^{\infty}|h_n|z^n,\qquad g(z)=\sum_{n=1}^{\infty}|g_n|z^n.\tag{5}$$

Define $\mathcal{T}\mathcal{N}_{\mathcal{H}}(\gamma)=\mathcal{N}_{\mathcal{H}}(\gamma)\cap\mathcal{T}$ and $\mathcal{T}\mathcal{G}_{\mathcal{H}}(\gamma)=\mathcal{G}_{\mathcal{H}}(\gamma)\cap\mathcal{T}$, where $\mathcal{T}$ consists of the functions $f=h+\bar{g}$ in $\mathcal{S}_{\mathcal{H}}$. The classes $\mathcal{N}_{\mathcal{H}}(\gamma)$, $\mathcal{T}\mathcal{N}_{\mathcal{H}}(\gamma)$, $\mathcal{G}_{\mathcal{H}}(\gamma)$, and $\mathcal{T}\mathcal{G}_{\mathcal{H}}(\gamma)$, were studied, respectively, by Ahuja and Jahangiri [4] and Rosy et al. [5].

Let $a_i\in\mathbb{C}$, $((a_i/A_i)\neq 0,\ -1,\ -2,\ \cdots;\ i=1,\ 2,\ \cdots,\ p)$ and $((b_i/B_i)\neq 0,\ -1,\ -2,\ \cdots; i=1,\ 2,\ \cdots,\ q)$, for $A_i>0\ (i=1,\cdots,p)$ and $B_i>0\ (i=1,\cdots,q)$ with

$$1+\sum_{i=1}^{q}B_i-\sum_{i=1}^{p}A_i\geq 0.\tag{6}$$

Wright's generalized hypergeometric functions [6] is defined by

$${}_p\Psi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_i,B_i)_{1,q}\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+nA_i)\ z^n}{\prod_{i=1}^{q}\Gamma(b_i+nB_i)\ n!},\tag{7}$$

which is analytic for suitable bounded values of $|z|$ (see also [7, 8]). The generalized Mittag-Leffler, Bessel-Maitland, and generalized hypergeometric functions are some of the important special cases of Wright's generalized hypergeometric functions, and for their details, one may refer to [8].

For $A_i>0\ (i=1,\cdots,p), B_i>0, b_i>0\ (i=1,\cdots,q)$ with $1+\sum_{i=1}^{q}B_i-\sum_{i=1}^{p}A_i\geq 0$ and $C_i>0\ (i=1,\cdots,r), D_i>0, d_i>0\ (i=1,\cdots,s)$ with $1+\sum_{i=1}^{s}D_i-\sum_{i=1}^{r}C_i\geq 0$, we define Wright's generalized hypergeometric functions:

$$\begin{aligned}{}_p\Psi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_i,B_i)_{1,q}\end{matrix};z\right]&=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+nA_i)\ z^n}{\prod_{i=1}^{q}\Gamma(b_i+nB_i)\ n!},\\ {}_r\Psi_s\left[\begin{matrix}(c_i,C_i)_{1,r}\\(d_i,D_i)_{1,s}\end{matrix};z\right]&=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\Gamma(c_i+nC_i)\ z^n}{\prod_{i=1}^{s}\Gamma(d_i+nD_i)\ n!},\end{aligned}\tag{8}$$

with

$$\frac{\prod_{i=1}^{r}\Gamma(|c_i|+nC_i)/\Gamma|c_i|}{\prod_{i=1}^{s}\Gamma(d_i+nD_i)/\Gamma(d_i)}<1.\tag{9}$$

We consider a harmonic univalent function

$$W(z)=H(z)+\overline{G(z)}\in\mathcal{S}_{\mathcal{H}},\tag{10}$$

where

$$\begin{aligned}H(z)&=z\frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{i=1}^{p}\Gamma(a_i)_p}\Psi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_i,B_i)_{1,q}\end{matrix};z\right]=z+\sum_{n=2}^{\infty}\theta_n\ z^n,\\ G(z)&=\sigma\ z\ \frac{\prod_{i=1}^{s}\Gamma(d_i)}{\prod_{i=1}^{r}\Gamma(c_i)_r}\Psi_s\left[\begin{matrix}(c_i,C_i)_{1,r}\\(d_i,D_i)_{1,s}\end{matrix};z\right]\\&=\sigma\sum_{n=1}^{\infty}\zeta_n\ z^n,\quad |\sigma|<1,\end{aligned}\tag{11}$$

and θ_n and ζ_n are given by

$$\begin{aligned}\theta_n&=\frac{\prod_{i=1}^{p}\Gamma(a_i+(n-1)A_i)/\Gamma(a_i)}{\prod_{i=1}^{q}(\Gamma(b_i+(n-1)B_i)/\Gamma(b_i))(n-1)!},\\ \zeta_n&=\frac{\prod_{i=1}^{r}\Gamma(c_i+(n-1)C_i)/\Gamma(c_i)}{\prod_{i=1}^{s}(\Gamma(d_i+(n-1)D_i)/\Gamma(d_i))(n-1)!}.\end{aligned}\tag{12}$$

From (12), we have for $n\in\mathbb{N}=\{1,\ 2,\ \cdots\}$

$$\begin{aligned}|\theta_n|&\leq\frac{\prod_{i=1}^{p}\Gamma(|a_i|+(n-1)A_i)/\Gamma(|a_i|)}{\prod_{i=1}^{q}(\Gamma(b_i+(n-1)B_i)/\Gamma(b_i))(n-1)!}=\nu_n,\\ |\zeta_n|&\leq\frac{\prod_{i=1}^{r}\Gamma(|c_i|+(n-1)C_i)/\Gamma(c_i)}{\prod_{i=1}^{s}(\Gamma(d_i+(n-1)D_i)/\Gamma(d_i))(n-1)!}=\eta_n.\end{aligned}\tag{13}$$

For some fixed value of $j\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and for

$$\prod_{i=1}^{q}B_i^{B_i}\geq\prod_{i=1}^{p}A_i^{A_i},\qquad \prod_{i=1}^{s}D_i^{D_i}\geq\prod_{i=1}^{r}C_i^{C_i},\tag{14}$$

we denote

$$\begin{aligned}{}_p\Psi_q\left[\begin{matrix}(|a_i|+jA_i,A_i)_{1,p}\\(|b_i|+jB_i,B_i)_{1,q}\end{matrix};1\right]&={}_p\Psi_q^j,\\ {}_r\Psi_s\left[\begin{matrix}(|c_i|+jC_i,C_i)_{1,r}\\(|d_i|+jD_i,D_i)_{1,s}\end{matrix};1\right]&={}_r\Psi_s^j,\end{aligned}\tag{15}$$

provided that

$$\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} |a_i| + \frac{p-q}{2} > \frac{1}{2} + j,$$
$$\sum_{i=1}^{s} d_i - \sum_{i=1}^{r} |c_i| + \frac{r-s}{2} > \frac{1}{2} + j. \quad (16)$$

Making use of (13) and (15), we have

$$\sum_{n=1+j}^{\infty} (n-j)_j \, \nu_n = \frac{\prod_{i=1}^{q} \Gamma(b_i)}{\prod_{i=1}^{p} \Gamma(|a_i|)_p} \Psi_q^j,$$
$$\sum_{n=1+j}^{\infty} (n-j)_j \, \eta_n = \frac{\prod_{i=1}^{s} \Gamma(d_i)}{\prod_{i=1}^{r} \Gamma(|c_i|)_r} \Psi_s^j, \quad (17)$$

provided that (16) holds true.

The convolution of two functions $f(z)$ of the form (1) and $F(z)$ of the form

$$F(z) = z + \sum_{n=2}^{\infty} H_n z^n + \sum_{n=1}^{\infty} \overline{G_n z^n}, \quad (18)$$

is given by

$$(f * F)(z) = f(z) * F(z) = z + \sum_{n=2}^{\infty} h_n H_n z^n + \sum_{n=1}^{\infty} \overline{g_n G_n z^n}. \quad (19)$$

Now, we introduce a convolution operator $\Omega(p, q, r, s)$ as

$$\Omega(p, q, r, s) f(z) = f(z) * W(z) = h(z) * H(z) + \overline{g(z) * G(z)}, \quad (20)$$

where $f = h + \overline{g}$ and $W(z) = H(z) + \overline{G(z)}$ given by (1) and (10), respectively. Hence

$$\Omega(p,q,r,s) f(z) = z + \sum_{n=2}^{\infty} \theta_n h_n z^n + \sum_{n=1}^{\infty} \overline{\zeta_n g_n z^n}. \quad (21)$$

The application of the special functions on the geometric function theory always attracts researchers with various kinds of special functions, for example, hypergeometric functions [9–11], confluent hypergeometric functions [12], generalized hypergeometric functions [6, 13], Bessel functions [14], generalized Bessel functions [15–17], Wright functions [18–21], Fox-Wright functions [6, 22], and Mittag-Leffler functions [23] that have rich applications in analytic and harmonic univalent functions. By using special functions, some researchers introduce operators, for example, Carlson-Shaffer operator [24], Hohlov operator [25], and Dziok-Srivastava operator [26, 27], and obtain interesting results. Motivated with the work of [20], we obtain some inclusion relation between the classes $\mathcal{G}_{\mathcal{H}}(\gamma)$, $K_{\mathcal{H}}^0$, $\mathcal{S}_{\mathcal{H}}^{*,0}$, $\mathcal{C}_{\mathcal{H}}^0$, and $\mathcal{N}_{\mathcal{H}}(\beta)$ by applying the convolution operator Ω.

2. Main Results

In order to establish our main results, we shall require the following lemmas.

Lemma 1 [1]. *If $f = h + \overline{g} \in K_{\mathcal{H}}^0$, where h and g are given by (5) with $g_1 = 0$, then*

$$|h_n| \le \frac{n+1}{2},$$
$$|g_n| \le \frac{n-1}{2}. \quad (22)$$

Lemma 2 [1]. *Let $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^{*,0}$ or $C_{\mathcal{H}}^0$, where h and g are given by (1) with $g_1 = 0$. Then*

$$|h_n| \le \frac{(2n+1)(n+1)}{6},$$
$$|g_n| \le \frac{(2n-1)(n-1)}{6}. \quad (23)$$

Lemma 3 [5]. *Let $f = h + \overline{g}$ be given by (5). If $0 \le \gamma < 1$ and*

$$\sum_{n=2}^{\infty} (2n-1-\gamma)|h_n| + \sum_{n=1}^{\infty} (2n+1+\gamma)|g_n| \le 1-\gamma, \quad (24)$$

then f is a sense-preserving Goodman-Rønning-type harmonic univalent function in $\mathbb{U}$ and $f \in \mathcal{G}_{\mathcal{H}}(\gamma)$.

Remark 4. In [5], it is also shown that $f = h + \overline{g}$ given by (5) is in the family $\mathcal{T}\mathcal{G}_{\mathcal{H}}(\gamma)$, if and only if the coefficient condition (24) holds. Moreover, if $f \in \mathcal{T}\mathcal{G}_{\mathcal{H}}(\gamma)$, then

$$|h_n| = \frac{1-\gamma}{2n-1-\gamma}, \quad n \ge 2,$$
$$|g_n| = \frac{1-\gamma}{2n+1+\gamma}, \quad n \ge 1. \quad (25)$$

Theorem 5. *Let $\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} |a_i| + ((p-q)/2) > 5/2$ and $\sum_{i=1}^{s} d_i - \sum_{i=1}^{r} |c_i| + ((r-s)/2) > 5/2$, and if the inequality*

$$\frac{\prod_{i=1}^{q} \Gamma(b_i)}{\prod_{i=1}^{p} \Gamma(|a_i|)} \left\{ 2\,{}_p\Psi_q^2 + (7-\gamma)\,{}_p\Psi_q^1 + 2(1-\gamma)\left({}_p\Psi_q^0 - 1\right) \right\} + |\sigma| \frac{\prod_{i=1}^{s} \Gamma(d_i)}{\prod_{i=1}^{r} \Gamma(|c_i|)} \left\{ 2\,{}_r\Psi_s^2 + (5+\gamma)\,{}_r\Psi_s^1 \right\} \le 2(1-\gamma), \quad (26)$$

holds, then $\Omega(K_{\mathcal{H}}^0) \subset \mathcal{G}_{\mathcal{H}}(\gamma)$.

Proof. Let $f = h + \overline{g} \in K_{\mathcal{H}}^0$, where h and g are given by (1) with $g_1 = 0$. We have to prove that $\Omega(f) \in \mathcal{G}_{\mathcal{H}}(\gamma)$, where $\Omega(f)$ is defined by (21). To prove $\Omega(f) \in \mathcal{G}_{\mathcal{H}}(\gamma)$, in view of Lemma 3, it is sufficient to prove that $P_1 \leq 1-\gamma$, where

$$P_1 = \sum_{n=2}^{\infty} (2n-1-\gamma)\,|\theta_n h_n| + \sum_{n=2}^{\infty} (2n+1+\gamma)\,|\zeta_n g_n|. \quad (27)$$

By using Lemma 1,

$$\begin{aligned} P_1 &\leq \sum_{n=2}^{\infty} (n+1)(2n-1-\gamma)|\theta_n| + \sum_{n=2}^{\infty} (n-1)(2n+1+\gamma)|\zeta_n| \\ &= \frac{1}{2}\left[\sum_{n=2}^{\infty} \{2(n-1)(n-2) + (7-\gamma)(n-1) + 2(1-\gamma)\}\nu_n\right] \\ &\quad + \frac{|\sigma|}{2}\left[\sum_{n=2}^{\infty} \{2(n-2) + (5+\gamma)\}\eta_n\right] \\ &= \frac{1}{2}\left[\frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{i=1}^{p}\Gamma(|a_i|)}\left\{2 + (7-\gamma)\,{}_p\Psi_q^1 + 2\,(1-\gamma)\left({}_p\Psi_q^0 - 1\right)\right\}\right. \\ &\quad \left. + |\sigma|\,\frac{\prod_{i=1}^{s}\Gamma(d_i)}{\prod_{i=1}^{r}\Gamma(|c_i|)}\,\left\{2_r\Psi_s^2 + (5+\gamma)_r\Psi_s^1\right\}\right] \leq 1-\gamma, \end{aligned} \quad (28)$$

by the given hypothesis. This completes the proof of Theorem 5.

The result is sharp for the function

$$L(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+1}{2}\right) z^n - \sum_{n=2}^{\infty} \left(\frac{n-1}{2}\right) \bar{z}^n. \quad (29)$$

Theorem 6. *Let* $\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} |a_i| + ((p-q)/2) > 7/2$ *and* $\sum_{i=1}^{s} d_i - \sum_{i=1}^{r} |c_i| + ((r-s)/2) > 7/2$, *and if the inequality*

$$\begin{aligned} &\frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{i=1}^{p}\Gamma(|a_i|)}\,\left\{4_p\Psi_q^3 + (28-2\gamma)_p\Psi_q^2 + (39-9\gamma)_p\Psi_q^1\right. \\ &\quad \left. + 6\,(1-\gamma)\,\left({}_p\Psi_q^0 - 1\right)\right\} + |\sigma|\,\frac{\prod_{i=1}^{s}\Gamma(d_i)}{\prod_{i=1}^{r}\Gamma(|c_i|)} \\ &\quad \cdot \left\{4_r\Psi_s^3 + 2\,(10+\gamma)_r\Psi_s^2 + 3\,(5+\gamma)_r\Psi_s^1\right\} \leq 6\,(1-\gamma), \end{aligned} \quad (30)$$

holds, then $\Omega(\mathcal{S}_{\mathcal{H}}^{*,0}) \subset \mathcal{G}_{\mathcal{H}}(\gamma)$ *and* $\Omega(\mathcal{C}_{\mathcal{H}}^{0}) \subset \mathcal{G}_{\mathcal{H}}(\gamma)$.

Proof. Let $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}^{*,0}$ (or $\mathcal{C}_{\mathcal{H}}^{0}$), where h and g are given by (1) with $g_1 = 0$; we need to prove that $\Omega(f) \in \mathcal{G}_{\mathcal{H}}(\gamma)$, where $\Omega(f)$ is defined by (21). In view of Lemma 3, it is sufficient to prove that $P_1 \leq 1-\gamma$, where P_1 is given by (27). Now using Lemma 2, we have

$$\begin{aligned} P_1 &\leq \frac{1}{6}\left[\sum_{n=2}^{\infty} (n+1)(2n+1)(2n-1-\gamma)|\theta_n|\right. \\ &\quad \left. + |\sigma|\sum_{n=2}^{\infty} (n-1)(2n-1)(2n+1+\gamma)|\zeta_n|\right] \\ &= \frac{1}{6}\left[\sum_{n=2}^{\infty} \{4(n-1)(n-2)(n-3) + (28-2\gamma)(n-1)(n-2)\right. \\ &\quad \left. + (39-9\gamma)(n-1) + 6(1-\gamma)\}\,\nu_n\right] \\ &\quad + \frac{|\sigma|}{6}\left[\sum_{n=2}^{\infty} \{4(n-1)(n-2)(n-3)\right. \\ &\quad \left. + (20+2\gamma)(n-1)(n-2) + (15+3\gamma)(n-1)\}\eta_n\right] \\ &= \frac{1}{6}\left[\frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{i=1}^{p}\Gamma(|a_i|)}\left\{4_p\Psi_q^3 + (28-2\gamma)_p\Psi_q^2 + (39-9\gamma)_p\Psi_q^1\right.\right. \\ &\quad \left.\left. + 6(1-\gamma)\left({}_p\Psi_q^0 - 1\right)\right\}\right] \\ &\quad + \frac{|\sigma|}{6}\left[\frac{\prod_{i=1}^{s}\Gamma(d_i)}{\prod_{i=1}^{r}\Gamma(|c_i|)}\left\{4_r\Psi_s^3 + 2(10+\gamma)_r\Psi_s^2 + 3(5+\gamma)_r\Psi_s^1\right\}\right] \\ &\leq 1-\gamma, \end{aligned} \quad (31)$$

by the given hypothesis. Thus, the proof of Theorem 6 is established.

The result is sharp for the function

$$f(z) = H(z) + \overline{G(z)}, \quad (32)$$

where

$$\begin{aligned} H(z) &= \frac{z - (1/2)z^2 + (1/6)z^3}{(1-z)^3}, \\ G(z) &= \frac{(1/2)z^2 + (1/6)z^3}{(1-z)^3}. \end{aligned} \quad (33)$$

In our next theorem, we establish connections between $\mathcal{T}\mathcal{G}_{\mathcal{H}}(\gamma)$ and $\mathcal{G}_{\mathcal{H}}(\gamma)$.

Theorem 7. *Let* $\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} |a_i| + ((p-q)/2) > 1/2$ *and* $\sum_{i=1}^{s} d_i - \sum_{i=1}^{r} |c_i| + ((r-s)/2) > 1/2$, *and if the inequality*

$$\frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{i=1}^{p}\Gamma(|a_i|)}\,\left({}_p\Psi_q^0 - 1\right) + |\sigma|\,\frac{\prod_{i=1}^{s}\Gamma(d_i)}{\prod_{i=1}^{r}\Gamma(|c_i|)_r}\,\Psi_s^0 \leq 1, \quad (34)$$

holds, then $\Omega(\mathcal{T}\mathcal{G}_{\mathcal{H}}(\gamma)) \subseteq \mathcal{G}_{\mathcal{H}}(\gamma)$.

Proof. Let $f = h + \bar{g} \in \mathcal{TG}_{\mathcal{H}}(\gamma)$ be given by (1). We have to prove that $P_2 \le 1 - \gamma$, where

$$P_2 = \sum_{n=2}^{\infty} (2n - 1 - \gamma) |\theta_n h_n| + |\sigma| \sum_{n=1}^{\infty} (2n + 1 + \gamma) |\zeta_n g_n|. \tag{35}$$

Now, using Remark 4, we have

$$\begin{aligned} P_2 &\le (1-\gamma) \sum_{n=2}^{\infty} \nu_n + (1-\gamma)\sigma \sum_{n=1}^{\infty} \eta_n = (1-\gamma) \\ &\cdot \left(\frac{\prod_{i=1}^{q} \Gamma(b_i)}{\prod_{i=1}^{p} \Gamma(|a_i|)} \left({}_p\Psi_q^0 - 1 \right) + |\sigma| \frac{\prod_{i=1}^{s} \Gamma(d_i)}{\prod_{i=1}^{r} \Gamma(|c_i|)_r} \Psi_s^0 \right) \\ &\le 1 - \gamma, \end{aligned} \tag{36}$$

by the given hypothesis. This completes the proof of Theorem 7.

The result is sharp for the function

$$f(z) = z - \sum_{n=2}^{\infty} \left(\frac{1-\gamma}{2n-1-\gamma} \right) |x_n| z^n + \sum_{n=1}^{\infty} \left(\frac{1-\gamma}{2n+1+\gamma} \right) |y_n| \bar{z}^n, \tag{37}$$

where

$$\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1. \tag{38}$$

3. Some Consequences of the Main Results

If we let $p = q = r = s = 1$ and $a_1 = A_1 = c_1 = C_1 = 1$ in (10), then $W(z)$ reduces to a harmonic univalent function $E(z)$ involving the following generalized Mittag-Leffler functions as

$$E(z) = z\Gamma(b_1) E^{1,1}_{b_1,B_1}[z] + \sigma z \Gamma(d_1) \overline{E}^{1,1}_{d_1,D_1}[z], \tag{39}$$

where

$$\begin{aligned} E^{1,1}_{b_1,B_1}[z] &= {}_1\Psi_1 \left[\begin{matrix} (1,1) \\ (b_1, B_1) \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(b_1 + nB_1)}, \\ E^{1,1}_{d_1,D_1}[z] &= {}_1\Psi_1 \left[\begin{matrix} (1,1) \\ (d_1, D_1) \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(d_1 + nD_1)}. \end{aligned} \tag{40}$$

With these specializations, the convolution operator $\Omega(p, q, r, s)$ reduces to the operator $\Phi(b_1 ; B_1 ; d_1 ; D_1)$, which is defined as

$$\begin{aligned} \Phi(b_1 ; B_1 ; d_1 ; D_1) f(z) &= f(z) * E(z) = h(z) * z\Gamma(b_1) E^{1,1}_{b_1,B_1}[z] \\ &+ \sigma \overline{g(z) * z\Gamma(d_1) E^{1,1}_{d_1,D_1}[z]}. \end{aligned} \tag{41}$$

For these specific values of $p = q = r = s = 1$ and $a_1 = A_1 = c_1 = C_1 = 1$, Theorems 5–7 yield the following results.

Corollary 8. *If the inequality*

$$\begin{aligned} &\Gamma(b_1) \left\{ 2E^{3,1}_{b_1+2B_1,B_1}(1) + (7-\gamma) E^{2,1}_{b_1+B_1,B_1}(1) + 2(1-\gamma) \left(E^{1,1}_{b_1,B_1} - 1 \right) \right\} \\ &+ |\sigma| \, \Gamma(d_1) \left\{ 2\, E^{3,1}_{d_1+2D_1,D_1}(1) + (5+\gamma)\, E^{2,1}_{d_1+D_1,D_1}(1) \right\} \le 2\,(1-\gamma), \end{aligned} \tag{42}$$

holds, then $\Phi(K^0_{\mathcal{H}}) \subset \mathcal{G}_{\mathcal{H}}(\gamma)$.

Corollary 9. *If the inequality*

$$\begin{aligned} &\Gamma(b_1) \left\{ 4E^{4,1}_{b_1+3B_1,B_1}(1) + (28-2\gamma) E^{3,1}_{b_1+2B_1,B_1}(1) \right. \\ &\left. + (39-9\gamma) E^{2,1}_{b_1+B_1,B_1}(1) + 2(1-\gamma) \left(E^{1,1}_{b_1,B_1} - 1 \right) \right\} \\ &+ |\sigma| \, \Gamma(d_1) \left\{ 4E^{4,1}_{d_1+3D_1,D_1}(1) + 2(10+\gamma)\, E^{3,1}_{d_1+2D_1,D_1}(1) \right. \\ &\left. + 3(5+\gamma)\, E^{2,1}_{d_1+D_1,D_1}(1) \right\} \le 6\,(1-\gamma), \end{aligned} \tag{43}$$

holds, then $\Phi(\mathcal{S}^{*,0}_{\mathcal{H}}) \subset \mathcal{G}_{\mathcal{H}}(\gamma)$ *and* $\Phi(C^0_{\mathcal{H}}) \subset \mathcal{G}_{\mathcal{H}}(\gamma)$.

Corollary 10. *If the inequality*

$$\Gamma(b_1) \left\{ \left(E^{1,1}_{b_1,B_1} - 1 \right) \right\} + |\sigma| \, \Gamma(d_1) \left(E^{1,1}_{d_1,D_1} \right) \le 1, \tag{44}$$

holds, then $\Phi(\mathcal{TG}_{\mathcal{H}}(\gamma)) \subset \mathcal{G}_{\mathcal{H}}(\gamma)$.

Remark 11. If we put $p = q = r = s = 1$, $a_1 = c_1 = 1$, $A_1 = C_1 = 0$, and $\sigma = 1$, then

$$\begin{aligned} W(z) = z &+ \sum_{n=2}^{\infty} \frac{\Gamma(b_1)}{\Gamma(b_1 + B_1(n-1))(n-1)!} z^n \\ &+ \sum_{n=1}^{\infty} \frac{\Gamma(\bar{d}_1)}{\Gamma(d_1 + D_1(n1))(n1)!} \bar{z}^n, \end{aligned} \tag{45}$$

and results of Theorems 5–7 reduce to corresponding results of Maharana and Sahoo [28].

Remark 12. If we put $p = r = 2$, $q = s = 1$, $A_1 = A_2 = B_1 = C_1 = C_2 = D_1 = 1$, and $\sigma = 1$, then

$$W(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} (a_2)_{n-1}}{(b_1)_{n-1} (n-1)!} z^n + \sum_{n=2}^{\infty} \frac{(c_1)_{n1} (\bar{c}_2)_{n1}}{(d_1)_{n1} (n1)!} \bar{z}^n, \tag{46}$$

and results of Theorems 5–7 reduce to corresponding results of Porwal and Dixit [11].

Data Availability

No data is required.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors equally worked on the results, and they read and approved the final manuscript.

References

[1] J. Clunie and T. Sheil-Small, "Harmonic univalent functions," *Annales Academiae Scientiarum Fennicae Series A I Mathematica*, vol. 9, pp. 3–25, 1984.

[2] P. Duren, *Harmonic Mappings in the Plane*, Cambridge Tracts in Mathematics, 156, Cambridge University Press, Cambridge, 2004.

[3] O. P. Ahuja, "Planar harmonic univalent and related mappings," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 6, no. 4, 2005.

[4] O. P. Ahuja and J. M. Jahangiri, "Noshiro-type harmonic univalent functions," *Scientiae Mathematicae Japonicae*, vol. 56, no. 2, pp. 293–299, 2002.

[5] T. Rosy, B. A. Stephen, K. G. Subramanian, and J. M. Jahangiri, "Goodman-Rønning-type harmonic univalent functions," *Kyungpook Mathematical Journal*, vol. 41, no. 1, 2001.

[6] E. M. Wright, "The asymptotic expansion of the generalized hypergeometric function," *Proceedings of the London Mathematical Society*, vol. s2-46, no. 1, pp. 389–408, 1940.

[7] H. M. Srivastava, "Some Fox-Wright generalized hypergeometric functions and associated families of convolution operators," *Applicable Analysis and Discrete Mathematics*, vol. 1, no. 1, pp. 56–71, 2007.

[8] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Series Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1984.

[9] O. P. Ahuja, "Connections between various subclasses of planar harmonic mappings involving hypergeometric functions," *Applied Mathematics and Computation*, vol. 198, no. 1, pp. 305–316, 2008.

[10] S. Ponnusamy and F. Rønning, "Starlikeness properties for convolutions involving hypergeometric series," *Annales Universitatis Mariae Curie-Skłodowska Sectio A. Mathematica*, vol. 52, no. 1, pp. 141–155, 1998.

[11] S. Porwal and K. K. Dixit, "An application of hypergeometric functions on harmonic univalent functions," *Bulletin of Mathematical Analysis and Applications*, vol. 2, no. 4, pp. 97–105, 2010.

[12] S. S. Miller and P. T. Mocanu, "Univalence of Gaussian and confluent hypergeometric functions," *Proceedings of the American Mathematical Society*, vol. 110, no. 2, pp. 333–342, 1990.

[13] S. Owa and H. M. Srivastava, "Univalent and starlike generalized hypergeometric functions," *Canadian Journal of Mathematics*, vol. 39, no. 5, pp. 1057–1077, 1987.

[14] B. A. Frasin, "Sufficient conditions for integral operator defined by Bessel functions," *Journal of Mathematical Inequalities*, vol. 4, no. 2, pp. 301–306, 2007.

[15] A. Baricz, *Generalized Bessel Functions of the First Kind*, vol. 1994 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2010.

[16] S. R. Mondal and A. Swaminathan, "Geometric properties of generalized Bessel functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 35, no. 1, pp. 179–194, 2012.

[17] S. Porwal, "Connections between various subclasses of planar harmonic mappings involving generalized Bessel functions," *Thai Journal of Mathematics*, vol. 13, no. 1, pp. 33–42, 2015.

[18] M. K. Aouf and J. Dziok, "Distortion and convolutional theorems for operators of generalized fractional calculus involving Wright function," *Journal of Applied Analysis*, vol. 14, no. 2, pp. 183–192, 2008.

[19] G. Murugusundaramoorthy and R. K. Raina, "On a subclass of harmonic functions associated with Wright's generalized hypergeometric functions," *Hacettepe Journal of Mathematics and Statistics*, vol. 38, no. 2, pp. 129–136, 2009.

[20] R. K. Raina and P. Sharma, "Harmonic univalent functions associated with Wright's generalized hypergeometric functions," *Integral Transforms and Special Functions*, vol. 22, no. 8, pp. 561–572, 2011.

[21] R. K. Raina and T. S. Nahar, "On characterization of certain Wright's generalized hypergeometric functions involving certain subclasses of analytic functions," *Lithuanian Academy of Sciences*, vol. 10, no. 2, pp. 219–230, 1999.

[22] V. B. L. Chaurasia and H. S. Parihar, "Certain sufficiency conditions on Fox-Wright functions," *Demonstratio Mathematica*, vol. 41, no. 4, pp. 813–822, 2008.

[23] A. Attiya, "Some applications of Mittag-Leffler function in the unit disk," *Filomat*, vol. 30, no. 7, pp. 2075–2081, 2016.

[24] B. C. Carlson and D. B. Shaffer, "Starlike and prestarlike hypergeometric functions," *SIAM Journal on Mathematical Analysis*, vol. 15, no. 4, pp. 737–745, 1984.

[25] Y. E. Hohlov, "Convolution operators preserving univalent functions," *Pliska Studia Mathematica Bulgarica*, vol. 10, pp. 87–92, 1989.

[26] J. Dziok and H. M. Srivastava, "Certain subclasses of analytic functions associated with the generalized hypergeometric function," *Integral Transforms and Special Functions*, vol. 14, no. 1, pp. 7–18, 2003.

[27] S. Porwal, K. K. Dixit, V. Kumar, A. L. Pathak, and P. Dixit, "A new subclass of harmonic univalent functions defined by Dziok-Srivastava operator," *Advances in Theoretical and Applied Mathematics*, vol. 5, no. 1, pp. 109–119, 2010.

[28] S. Maharana and S. K. Sahoo, "Inclusion properties of planar harmonic mappings associated with the Wright function," *Complex Variables and Elliptic Equations*, pp. 1–23, 2020.