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Research Article

New Fixed Point Theorems for θ-ϕ-Contraction on Rectangular b-Metric Spaces

Abdelkarim Kari,[1] Mohamed Rossafi ,[2] El Miloudi Marhrani,[1] and Mohamed Aamri[1]

[1] *Laboratory of Algebra, Analysis and Applications Faculty of Sciences Ben M'Sik, Hassan II University, Casablanca, Morocco*
[2] *Department of Mathematics, University of Ibn Tofail, 133 Kenitra, Morocco*

Correspondence should be addressed to Mohamed Rossafi; rossafimohamed@gmail.com

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The Banach contraction principle is the most celebrated fixed point theorem and has been generalized in various directions. In this paper, inspired by the concept of θ-ϕ-contraction in metric spaces, introduced by Zheng et al., we present the notion of θ-ϕ-contraction in b-rectangular metric spaces and study the existence and uniqueness of a fixed point for the mappings in this space. Our results improve many existing results.

1. Introduction

The Banach contraction principle is a fundamental result in fixed point theory [1]. Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle (see [2–4]).

Many generalizations of the concept of metric spaces have been defined, and some fixed point theorems were proven in these spaces. In particular, b-metric spaces were introduced by Bakhtin [5] and Czerwik [6] as a generalization of metric spaces. Many mathematicians worked on this interesting space. For more, the reader can refer to [7–10].

In 2000, generalized metric spaces were introduced by Branciari [11], in such a way that triangle inequality is replaced by the quadrilateral inequality $d(x, y) \le d(x, z) + d(z, u) + d(u, y)$ for all pairwise distinct points x, y, z, and u. Any metric space is a generalized metric space, but in general, generalized metric space might not be a metric space. Various fixed point results were established on such spaces (see [12–17] and references therein).

Recently, George et al. [10] announced the notion of b-rectangular metric space; many authors initiated and studied many existing fixed point theorems in such spaces (see [18–23]).

Very recently, Zheng et al. [24] introduced a new concept of θ-ϕ-contractions and established some fixed point results for such mappings in complete metric spaces and generalized the results of Brower and Kannan. For more works related to theta-contractions, see [25–27].

In this paper, we introduce a new notion of generalized θ-ϕ-contractions and establish some fixed point results for such mappings in complete b-rectangular metric spaces. The results presented in the paper extend the corresponding results of Kannan [3] and Reich [4] on b-rectangular metric spaces. Also, we derive some useful corollaries of these results.

2. Preliminaries

Definition 1 (see [7]). Let X be a nonempty set and $s \ge 1$ be a given real number and let $d : X \times X \to [0,+\infty[$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y: (1) $d(x, y) = 0$, if only if $x = y$; (2) $d(x, y) = d(y, x)$; and (3) $d(x, y) \le s[d(x, u) + d(u, v) + d(v, y)]$ (b − rectangular inequality).

Then (X, d) is called a b-rectangular metric space.

Example 2 (see [19]). Let $X = A \cup B$, where $A = \{1/n : n \in \{2, 3, 4, 5, 6, 7\}\}$ and $B = [1, 2]$. Define $d : X \times X \to [0, +\infty[$ as follows:

$$\begin{cases} d(x,y) = d(y,x) \quad \text{for all } x, y \in X, \\ d(x,y) = 0 \Leftrightarrow y = x, \end{cases}$$
$$\begin{cases} d\left(\frac{1}{2},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) = d\left(\frac{1}{6},\frac{1}{7}\right) = 0,05, \\ d\left(\frac{1}{2},\frac{1}{4}\right) = d\left(\frac{1}{3},\frac{1}{7}\right) = d\left(\frac{1}{5},\frac{1}{6}\right) = 0,08, \\ d\left(\frac{1}{2},\frac{1}{6}\right) = d\left(\frac{1}{3},\frac{1}{4}\right) = d\left(\frac{1}{5},\frac{1}{7}\right) = 0,4, \\ d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{3},\frac{1}{6}\right) = d\left(\frac{1}{4},\frac{1}{7}\right) = 0,24, \\ d\left(\frac{1}{2},\frac{1}{7}\right) = d\left(\frac{1}{3},\frac{1}{5}\right) = d\left(\frac{1}{4},\frac{1}{6}\right) = 0,15, \\ d(x,y) = (|x-y|)^2 \text{ otherwise.} \end{cases} \tag{1}$$

Then (X, d) is a b-rectangular metric space with coefficient $s = 3$.

Lemma 3 (see [20]). *Let (X, d) be a b-rectangular metric space.*

(a) Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, with $x \neq y, x_n \neq x$, and $y_n \neq y$ for all $n \in \mathbb{N}$. Then, we have $(1/s)d(x,y) \leq \lim_{n\to\infty} \inf\ d(x_n, y_n) \leq \lim_{n\to\infty} \sup\ d(x_n, y_n) \leq sd(x,y)$

(b) if $y \in X$ and $\{x_n\}$ is a Cauchy sequence in X with $x_n \neq x_m$ for any $m, n \in \mathbb{N}, m \neq n$, converging to $x \neq y$, then $(1/s)d(x,y) \leq \lim_{n\to\infty} \inf\ d(x_n, y) \leq \lim_{n\to\infty} \sup\ d(x_n, y) \leq sd(x,y)$, for all $x \in X$

Lemma 4. *Let (X, d) be a b-rectangular metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2}$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that

$$\begin{aligned} \varepsilon &\leq \lim_{k\to\infty} \inf\ d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq \lim_{k\to\infty} \sup\ d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq s\varepsilon, \\ \varepsilon &\leq \lim_{k\to\infty} \inf\ d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq \lim_{k\to\infty} \sup\ d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq s\varepsilon, \\ \varepsilon &\leq \lim_{k\to\infty} \inf\ d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leq \lim_{k\to\infty} \sup\ d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leq s\varepsilon, \\ \frac{\varepsilon}{s} &\leq \lim_{k\to\infty} \inf\ d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq \lim_{k\to\infty} \sup\ d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq s^2\varepsilon. \end{aligned} \tag{3}$$

Proof. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that

$$m(k) > n(k) > k, \varepsilon \leq d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \text{and } d\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right) < \varepsilon, \tag{4}$$

for all positive integers k. By the b-rectangular inequality, we have

$$\begin{aligned} \varepsilon \leq d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\leq s\left[d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right)\right. \\ &\quad \left. + d\left(x_{m_{(k)+1}}, x_{m_{(k)-1}}\right) + d\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right)\right]. \end{aligned} \tag{5}$$

Taking the upper and lower limits as $k \to \infty$ in (5) and using (2) (4), we obtain

$$\varepsilon \leq \lim_{k\to\infty} \inf\ d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq \lim_{k\to\infty} \sup\ d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq s\varepsilon. \tag{6}$$

Using the b-rectangular inequality again, we have

$$\begin{aligned} \varepsilon \leq d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) &\leq s\left[d\left(x_{n_{(k)}}, x_{m_{(k)-1}}\right)\right. \\ &\quad \left. + d\left(x_{m_{(k)-1}}, x_{m_{(k)}}\right) + d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right)\right]. \end{aligned} \tag{7}$$

Taking the upper and lower limits as $k \to \infty$ in (7) and using (2) and (4), we obtain

$$\varepsilon \leq \lim_{k\to\infty} \inf\ d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq \lim_{k\to\infty} \sup\ d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq s\varepsilon. \tag{8}$$

Using the b-rectangular inequality again, we have

$$\begin{aligned} \varepsilon \leq d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) &\leq s\left[d\left(x_{m_{(k)}}, x_{m_{(k)-1}}\right)\right. \\ &\quad \left. + d\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right) + d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right)\right]. \end{aligned} \tag{9}$$

Taking the upper and lower limits as $k \to \infty$ in (9) and using (2) and (4), we obtain

$$\varepsilon \leq \lim_{k\to\infty} \inf\ d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leq \lim_{k\to\infty} \sup\ d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leq s\varepsilon. \tag{10}$$

Using the b-rectangular inequality again, we have

$$d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq s\left[d\left(x_{m_{(k)+1}}, x_{m_{(k)}}\right) + d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) + d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right)\right], \tag{11}$$

$$\begin{aligned} \varepsilon \leq d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\leq s\left[d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right)\right. \\ &\quad \left. + d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) + d\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right)\right]. \end{aligned} \tag{12}$$

Taking the upper and lower limits as $k \to \infty$ in (11) and (12) and using (2) (6), we obtain

$$\frac{\varepsilon}{s} \le \lim_{k\to\infty} \inf d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \le \lim_{k\to\infty} \sup d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \le s^2\varepsilon. \quad (13)$$

The following definition was given by Ding et al. in [13].

Definition 5 (see [13]). Let Θ be the family of all functions $\theta :]0,+\infty[\to]1,+\infty[$ such that $[(\theta_1)]\theta$ is increasing; $[(\theta_2)]$ for each sequence $(x_n) \subset]0,+\infty[$; $\lim_{n\to 0} x_n = 0$ if and only if $\lim_{n\to\infty} \theta(x_n) = 1$; and $[(\theta_3)]\theta$ is continuous.

In [21] Radenovic et al. presented the concept of θ-ϕ -contractions on metric spaces.

Definition 6 (see [21]). Let Φ be the family of all functions $\phi : [1,+\infty[\to [1,+\infty[$, such that $[(\phi_1)]\phi$ is nondecreasing; $[(\phi_2)]$ for each $t \in]1,+\infty[$, $\lim_{n\to\infty}\phi^n(t) = 1$; and $[(\phi_3)]\phi$ is continuous.

Lemma 7 (see [21]). *If $\phi \in \Phi$, then $\phi(1) = 1$, and $\phi(t) < t$ for all $t \in]1,\infty[$.*

Definition 8 (see [21]). Let (X, d) be a metric space and $T : X \to X$ be a mapping.

T is said to be a θ-ϕ-contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \theta[d(Tx, Ty)] \le \phi(\theta[N(x, y)]), \quad (14)$$

where

$$N(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty)\}. \quad (15)$$

In [27], Zheng et al. proved the following nice result.

Theorem 9 (see [21]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a θ-ϕ -contraction. Then, T has a unique fixed point.*

3. Main Result

In this paper, using the idea introduced by Zheng et al., we present the concept θ-ϕ-contraction in b-rectangular metric spaces, and we prove some fixed point results for such spaces.

Definition 10. Let (X, d) be a b-rectangular metric space with parameter $s > 1$ space and $T : X \to X$ be a mapping.

(1) T is said to be a θ-contraction if there exist $\theta \in \Theta$ and $r \in]0, 1[$ such that

$$d(Tx, Ty) > 0 \Rightarrow \theta\left[s^2 d(Tx, Ty)\right] \le (\theta[M(x, y)])^r, \quad (16)$$

where

$$M(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}. \quad (17)$$

(2) T is said to be a θ-ϕ-contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$d(Tx, Ty) > 0 \Rightarrow \theta\left[s^2 d(Tx, Ty)\right] \le \phi[\theta(M(x, y))], \quad (18)$$

where

$$M(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}. \quad (19)$$

(3) T is said to be a θ-ϕ-Kannan-type contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that $d(Tx, Ty) > 0$, we have

$$\theta\left[s^2 d(Tx, Ty)\right] \le \varphi\left[\theta\left(\frac{d(x, Tx) + d(y, Ty)}{2}\right)\right]. \quad (20)$$

(4) T is said to be a θ-ϕ-Reich-type contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that $d(Tx, Ty) > 0$, we have

$$\theta\left[s^2 d(Tx, Ty)\right] \le \phi\left[\theta\left(\frac{d(x, y) + d(x, Tx) + d(y, Ty)}{3}\right)\right]. \quad (21)$$

Theorem 11. *Let (X, d) be a complete b-rectangular metric space and let $T : X \to X$ be an θ -contraction, i.e., there exist $\theta \in \Theta$ and $r \in]0, 1[$ such that for any $x, y \in X$, we have*

$$d(Tx, Ty) > 0 \Rightarrow \theta\left[s^2 d(Tx, Ty)\right] \le (\theta[M(x, y)])^r. \quad (22)$$

Then, T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X and define a sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0, \quad (23)$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then the proof is finished.

We can suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$.

Substituting $x = x_{n-1}$ and $y = x_n$, from (22), for all $n \in \mathbb{N}$, we have

$$\theta[d(x_n, x_{n+1})] \leq \theta\left[s^2 d(x_n, x_{n+1})\right] \leq \left[\theta(M(x_{n-1}, x_n))\right]^r, \quad \forall n \in \mathbb{N}, \tag{24}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+1}\right) \\ &= \max\left(d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}. \end{aligned} \tag{25}$$

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, by (24), we have

$$\theta(d(x_n, x_{n+1})) \leq (\theta(d(x_n, x_{n+1})))^r < \theta(d(x_n, x_{n+1})), \tag{26}$$

which is a contradiction. Hence, $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Thus,

$$\theta(d(x_n, x_{n+1})) \leq (\theta(d(x_{n-1}, x_n)))^r. \tag{27}$$

Repeating this step, we conclude that

$$\begin{aligned} \theta(d(x_n, x_{n+1})) &\leq (\theta(d(x_{n-1}, x_n)))^r \\ &\leq (\theta(d(x_{n-2}, x_{n-1})))^{r^2} \\ &\leq \cdots \leq \theta(d(x_0, x_1))^{r^n}. \end{aligned} \tag{28}$$

From (27) and using (θ_1), we get

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{29}$$

Therefore, $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is a monotone strictly decreasing sequence of nonnegative real numbers. Consequently, there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) \quad = \alpha. \tag{30}$$

Now, we claim that $\alpha = 0$. Arguing by contradiction, we assume that $\alpha > 0$. Since $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence, then we have

$$d(x_n, x_{n+1}) \geq \alpha \quad \forall n \in \mathbb{N}. \tag{31}$$

By property of θ, we get

$$1 < \theta(\alpha) \leq \theta(d(x_0, x_1))^{r^n}. \tag{32}$$

By letting $n \to \infty$ in inequality (32), we obtain

$$1 < \theta(\alpha) \leq 1. \tag{33}$$

It is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{34}$$

Next, we shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{35}$$

We assume that $x_n \neq x_m$ for every $n, m \in \mathbb{N}$, $n \neq m$. Indeed, suppose that $x_n = x_m$ for some $n = m + k$ with $k > 0$ and using (29), we have

$$d(x_m, x_{m+1}) = d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{36}$$

Continuing this process, we can that

$$d(x_m, x_{n+1}) = d(x_n, x_{n+1}) < d(x_m, x_{m+1}). \tag{37}$$

It is a contradiction. Therefore, $d(x_n, x_m) > 0$ for every $n, m \in \mathbb{N}$, $n \neq m$.

Applying (22) with $x = x_{n-1}$ and $y = x_{n+1}$, we have

$$\begin{aligned} \theta[d(x_n, x_{n+2})] &= \theta[d(Tx_{n-1}, Tx_{n+1})] \\ &\leq \theta\left[s^2 d(Tx_{n-1}, Tx_{n+1})\right] \\ &\leq [\theta(M(x_{n-1}, x_{n+1}))]^r, \end{aligned} \tag{38}$$

where

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_n)\} \\ &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\}. \end{aligned} \tag{39}$$

So, we get

$$\theta(d(x_n, x_{n+2})) \leq [\theta(\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1})\})]^r. \tag{40}$$

Take $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$. Thus, by (40), one can write

$$\theta(a_n) \leq [\theta(\max(a_{n-1}, b_{n-1}))]^r. \tag{41}$$

By (θ_1), we get

$$a_n < \max\{a_{n-1}, b_{n-1}\}. \tag{42}$$

By (36), we have

$$b_n \leq b_{n-1} \leq \max\{a_{n-1}, b_{n-1}\}. \tag{43}$$

It implies that

$$\max\{a_n, b_n\} \leq \max\{a_{n-1}, b_{n-1}\}, \quad \forall n \in \mathbb{N}. \tag{44}$$

Therefore, the sequence $\max\{a_{n-1}, b_{n-1}\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence of real numbers. Thus, there exists $\lambda \geq 0$ such that

$$\lim_{n\to\infty} \max\{a_n, b_n\} = \lambda. \quad (45)$$

By (34) assume that $\lambda > 0$, we have

$$\lambda = \lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \sup\max\{a_n, b_n\} = \lim_{n\to\infty} \max\{a_n, b_n\}. \quad (46)$$

Taking the lim $\sup_n \to \infty$ in (40) and using the property of θ, we obtain

$$\theta\left(\lim_{n\to\infty} \sup a_n\right) \le \theta\left(\lim_{n\to\infty} \max\{a_{n-1}, b_{n-1}\}\right)^r < \theta\left(\lim_{n\to\infty} \max\{a_{n-1}, b_{n-1}\}\right). \quad (47)$$

Therefore,

$$\theta(\lambda) < \theta(\lambda). \quad (48)$$

By (θ_1), we get

$$\lambda < \lambda. \quad (49)$$

It is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \quad (50)$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., $\lim_{n\to\infty} d(x_n, x_m) = 0$, for all $n, m \in \mathbb{N}$. Suppose to the contrary. By Lemma 4 Then, there is an ε>0 such that for an integer k there exists two sequences $\{n_{(k)}\}$ and $\{m_{(k)}\}$ such that [i)] $\varepsilon \le \lim_{k\to\infty} \inf d(x_{m_{(k)}}, x_{n_{(k)}}) \le \lim_{k\to\infty} \sup d(x_{m_{(k)}}, x_{n_{(k)}}) \le s\varepsilon$, [ii)] $\varepsilon \le \lim_{k\to\infty} \inf d(x_{n_{(k)}}, x_{m_{(k)+1}}) \le \lim_{k\to\infty} \sup d(x_{n_{(k)}}, x_{m_{(k)+1}}) \le s\varepsilon$, [iii)] $\varepsilon \le \lim_{k\to\infty} \inf d(x_{m_{(k)}}, x_{n_{(k)+1}}) \le \lim_{k\to\infty} \sup d(x_{m_{(k)}}, x_{n_{(k)+1}}) \le s\varepsilon$, and [vi)] $\varepsilon/s \le \lim_{k\to\infty} \inf d(x_{m_{(k)+1}}, x_{n_{(k)+1}}) \le \lim_{k\to\infty} \sup d(x_{m_{(k)+1}}, x_{n_{(k)+1}}) \le s^2\varepsilon$.

Now, using (i), (ii), and (34), we conclude that

$$\lim_{k\to\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \lim_{k\to\infty} \max\left\{d\left(x_{m_{(k)}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)}}, x_{m(k)+1}\right), d \cdot \left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right)\right\} \le s\varepsilon. \quad (51)$$

Now, applying (22) with $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we obtain

$$\theta\left[s^2 d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right] \le \left[\theta\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]^r. \quad (52)$$

Letting $k \to \infty$ the above inequality and using (θ_3), (51) and (iv), we obtain

$$\theta\left(\frac{\varepsilon}{s} s^2\right) = \theta(\varepsilon s) \le \theta\left(s^2 \lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \le \left[\theta\left(\lim_{k\to\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]^r. \quad (53)$$

Therefore,

$$\theta(s\varepsilon) \le [\theta(s\varepsilon)]^r < \theta(s\varepsilon). \quad (54)$$

Since θ is increasing, we get

$$s\varepsilon < s\varepsilon, \quad (55)$$

which is a contradiction. Then,

$$\lim_{n,m\to\infty} d(x_m, x_n) = 0. \quad (56)$$

Hence, $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d), there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0. \quad (57)$$

Now, we show that $d(Tz, z) = 0$; arguing by contradiction, we assume that

$$d(Tz, z) > 0. \quad (58)$$

Since $x_n \to z$ as $n \to \infty$ for all $n \in \mathbf{N}$, then from Lemma 3, we conclude that

$$\frac{1}{s} d(z, Tz) \le \lim_{n\to\infty} \sup d(Tx_n, Tz) \le sd(z, Tz). \quad (59)$$

Now, applying (22) with $x = x_n$ and $y = z$, we have

$$\theta\left(s^2 d(Tx_n, Tz)\right) \le [\theta(M(x_n, z))]^r, \quad \forall n \in \mathbb{N}, \quad (60)$$

where

$$M(x_n, z) = \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(z, Tx_n)\}. \quad (61)$$

Therefore,

$$\theta\left(s^2 d(Tx_n, Tz)\right) \le [\theta(\max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(z, Tx_n)\})]^r. \quad (62)$$

By letting $n \to \infty$ in inequality (62), using (59) and θ_3, we obtain

$$\theta\left[s^2 \frac{1}{s} d(z, Tz)\right] = \theta[sd(z, Tz)] \le \theta\left[s^2 \lim_{n\to\infty} d(Tx_n, Tz)\right] \le [\theta(d(z, Tz))]^r < \theta(d(z, Tz)). \quad (63)$$

By (θ_1), we get

$$sd(z, Tz) < d(z, Tz), \quad (64)$$

which implies that

$$d(z, Tz)(s-1) < 0 \Rightarrow s < 1, \quad (65)$$

which is a contradiction. Hence, $Tz = z$.

Uniqueness: now, suppose that $z, u \in X$ are two fixed points of T such that $u \neq z$. Therefore, we have

$$d(z, u) = d(Tz, Tu) > 0. \quad (66)$$

Applying (22) with $x = z$ and $y = u$, we have

$$\theta(d(z, u)) = \theta(d(Tu, Tz)) \leq \theta(s^2 d(Tu, Tz)) \leq [\theta(M(z, u))]^r, \quad (67)$$

where

$$M(z, u) = \max\{d(z, u), d(z, Tz), d(u, Tu), d(u, Tz)\} = d(z, u). \quad (68)$$

Therefore, we have

$$\theta(d(z, u)) \leq [\theta(d(z, u))]^r < \theta(d(z, u)), \quad (69)$$

which implies that

$$d(z, u) < d(z, u), \quad (70)$$

which is a contradiction. Therefore, $u = z$.

Corollary 12. *Let (X, d) be a complete b-rectangular metric space and $T : X \to X$ be the given mapping. Suppose that there exist $\theta \in \Theta$ and $k \in]0, 1[$ such that for any $x, y \in X$, we have*

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^2 d(Tx, Ty)] \leq [\theta(d(x, y))]^k. \quad (71)$$

Then, T has a unique fixed point.

Example 13. Let $X = A \cup B$, where $A = \{0, (1/2), (1/3), (1/4)\}$ and $B = [1, 2]$.

Define $d : X \times X \to [0, +\infty[$ as follows:

$$\begin{cases} d(x, y) = d(y, x) & \text{for all } x, y \in X, \\ d(x, y) = 0 \Leftrightarrow y = x, \end{cases}$$
$$\begin{cases} d\left(0, \frac{1}{2}\right) = d\left(\frac{1}{2}, \frac{1}{3}\right) = 0,16, \\ d\left(0, \frac{1}{3}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = 0,04, \\ d\left(0, \frac{1}{4}\right) = d\left(\frac{1}{2}, \frac{1}{4}\right) = 0,25, \\ d(x, y) = (|x - y|)^2 \text{ otherwise.} \end{cases} \quad (72)$$

Then, (X, d) is a b-rectangular metric space with coefficient $s = 3$. However, we have the following: (1) (X, d) is not a metric space, as $d(0, (1/4)) = 0.25 > 0.08 = d(0, (1/3)) + d((1/3), (1/4))$. (2) (X, d) is not a rectangular metric space, as $d((1/2), (1/4)) = 0.25 > 0.24 = d((1/2), 0) + d(0, (1/3)) + d((1/3), (1/4))$.

Define mapping $T : X \to X$ by

$$T(x) = \begin{cases} \frac{1}{4} & \text{if } x \in [1, 2], \\ \frac{1}{3} & \text{if } x \in A. \end{cases} \quad (73)$$

Evidently, $T(x) \in X$. Let $\theta(t) = e^{\sqrt{t}}, r = 8/9$. It is obvious that $\theta \in \Theta$ and $r \in]0, 1[$.

Consider the following possibilities:

(1) $x \in [1, 2]$, $y \in A$. Then,

$$T(x) = \frac{1}{4}, T(y) = \frac{1}{3}, d(Tx, Ty) = 0.04. \quad (74)$$

On the other hand,

$$\theta[s^2 d(Tx, Ty)] = e^{0.6},$$

$$\begin{aligned} M(x, y) &= \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\} \\ &\geq d(x, Tx) = \left(\left|x - \frac{1}{4}\right|\right)^2 \geq \left(\left|1 - \frac{1}{4}\right|\right)^2 = \left(\frac{3}{4}\right)^2. \end{aligned} \quad (75)$$

Hence,

$$\left[\theta\left(\frac{3}{4}\right)^2\right]^{8/9} = [e^{2/3}] \leq [\theta(d(x, Tx))]^{8/9} \leq [\theta(M(x, y))]^{8/9}. \quad (76)$$

On the other hand,

$$e^{0.6} - e^{2/3} \leq 0, \quad (77)$$

which implies that

$$\begin{aligned} &\theta(s^2 d(Tx, Ty) \leq \phi[\theta(d(x, Tx))]^{8/9} \\ &\leq [\theta(\max\{d(x, y), d(x, Tx), d(y, Ty)\}, d(y, Tx))]^{8/9}. \end{aligned} \quad (78)$$

(2) If $x, y \in [1, 2]$ or $x, y \in A$. Then,

$$T(x) = T(y) = \frac{1}{4} \text{ or } T(x) = T(y) = \frac{1}{3}, \text{then } d(Tx, Ty) = 0, \quad (79)$$

which implies that

$$\theta\left(s^2 d(Tx, Ty) \leq [\theta(\max\{d(x,y), d(x,Tx), d(y,Ty)\}, d(y,Tx))]^{8/9}\right. \quad (80)$$

Hence, condition (22) is satisfied. Therefore, T has a unique fixed point $z = 1/3$.

Theorem 14. *Let (X, d) be a complete b-rectangular metric space and $T : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,*

$$d(Tx, Ty) > 0 \Rightarrow \theta\left[s^2 d(Tx, Ty)\right] \leq \phi[\theta(M(x,y))], \quad (81)$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx)\}. \quad (82)$$

Then, T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X and define a sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0, \quad (83)$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then the proof is finished.

We can suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$.

Substituting $x = x_{n-1}$ and $y = x_n$, from (81), for all $n \in \mathbb{N}$, we have

$$\theta[d(x_n, x_{n+1})] \leq \theta\left[s^2 d(x_n, x_{n+1})\right] \leq \phi[\theta(M(x_{n-1}, x_n))], \quad \forall n \in \mathbb{N}. \quad (84)$$

As in the proof of Theorem 11, we conclude that

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \quad (85)$$

If for some n, $M(x_{n-1}, x_n) = \{d(x_n, x_{n+1})\}$, it follows from (84), (θ_1), and using Lemma 7 we get

$$\theta(d(x_n, x_{n+1})) \leq \phi(\theta(d(x_n, x_{n+1}))). \quad (86)$$

It implies that

$$d(x_n, x_{n+1}) < d(x_n, x_{n+1}), \quad (87)$$

which is a contradiction. Hence, $M(x_{n-1}, x_n) = \{d(x_{n-1}, x_n)\}$.

Therefore,

$$\theta(d(x_n, x_{n+1})) \leq \phi(\theta(d(x_{n-1}, x_n))) < \theta(d(x_{n-1}, x_n)). \quad (88)$$

Since θ is increasing, so

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}). \quad (89)$$

Therefore, $\{d(x_{n+1}, x_n)\}_{n\in\mathbb{N}}$ is a monotone strictly decreasing sequence of nonnegative real numbers. Consequently, there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \alpha. \quad (90)$$

Now, we claim that $\alpha = 0$. Arguing by contradiction, we assume that $\alpha > 0$. Since $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence, we have

$$d(x_n, x_{n+1}) \geq \alpha \quad \forall n \in \mathbb{N}. \quad (91)$$

Thus, we have

$$\begin{aligned} 1 < \theta(\alpha) \leq \theta(d(x_n, x_{n+1})) &\leq \phi[\theta(d(x_{n-1}, x_n))] \\ &\leq \cdots \leq \phi^n[\theta(d(x_0, x_1))]. \end{aligned} \quad (92)$$

By letting $n \to \infty$ in inequality (92), using (Φ_2), we obtain

$$1 < \theta(\alpha) \leq 1. \quad (93)$$

It is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \quad (94)$$

Next, we shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \quad (95)$$

We assume that $x_n \neq x_m$ for every $n, m \in \mathbb{N}, n \neq m$. Indeed, suppose that $x_n = x_m$ for some $n = m + k$ with $k > 0$, so we have $x_{n+1} = Tx_n = Tx_m = x_{m+1}$.

By (89), we get

$$d(x_m, x_{m+1}) = d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \quad (96)$$

Continuing this process, we can that

$$d(x_m, x_{m+1}) < d(x_m, x_{m+1}). \quad (97)$$

It is a contradiction. Therefore,

$$d(x_n, x_m) > 0 \text{ for every } n, m \in \mathbb{N}, n \neq m. \quad (98)$$

Applying (81) with $x = x_{n-1}$ and $y = x_{n+1}$, we have

$$\theta[d(x_n, x_{n+2})] \leq \theta\left[s^2 d(x_n, x_{n+2})\right] \leq \phi[\theta(M(x_{n-1}, x_{n+1}))], \quad (99)$$

where

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1})\}. \end{aligned} \quad (100)$$

Therefore,

$$\theta((x_n, x_{n+2})) \leq \phi[\theta(\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1})\})], \tag{101}$$

which implies that

$$d(x_n, x_{n+2}) < \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1})\}. \tag{102}$$

Take $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$. By (102), we have

$$a_n < \max\{a_{n-1}, b_{n-1}\}. \tag{103}$$

Again by (89), we get

$$b_n \leq b_{n-1} \leq \max\{a_{n-1}, b_{n-1}\}. \tag{104}$$

Therefore,

$$\max\{a_n, b_n\} \leq \max\{a_{n-1}, b_{n-1}\}, \quad \forall n \in \mathbb{N}. \tag{105}$$

Then, the sequence $\{\max\{a_n, b_n\}\}_n$ is monotone nonincreasing, so it converges to some $\beta \geq 0$ such that

$$\lim_{n\to\infty} \max\{a_n, b_n\} = \beta. \tag{106}$$

By (94) assume that$\beta > 0$, we have

$$\beta = \lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \sup\max\{a_n, b_n\} = \lim_{n\to\infty} \max\{a_n, b_n\}. \tag{107}$$

Taking the $\limsup_{n\to\infty}$ in (101) and using (θ_3), (ϕ_3), and Lemma 7, we obtain

$$\theta\left(\lim_{n\to\infty} \sup a_n\right) \leq \phi\left[\theta\left(\limsup_{n\to\infty} \max\{a_{n-1}, b_{n-1}\}\right)\right], \tag{108}$$

which implies that

$$\theta(\beta) \leq \phi[\theta(\beta)]. \tag{109}$$

Therefore,

$$\beta < \beta, \tag{110}$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{111}$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., $\lim_{n\to\infty} d(x_n, x_m) = 0$, for all $n, m \in \mathbb{N}$. Suppose to the contrary, then there is an $\varepsilon > 0$ such that for an integer k, there exist two sequences $\{n_{(k)}\}$ and $\{m_{(k)}\} m_{(k)} > n_{(k)} > k$, such that [i)] $\varepsilon \leq \lim_{k\to\infty} \inf d(x_{m_{(k)}}, x_{n_{(k)}}) \leq \lim_{k\to\infty} \sup d(x_{m_{(k)}}, x_{n_{(k)}}) \leq s\varepsilon$, [i)] $\varepsilon \leq \lim_{k\to\infty} \inf d(x_{n_{(k)}}, x_{m_{(k)+1}}) \leq \lim_{k\to\infty} \sup d(x_{n_{(k)}}, x_{m_{(k)+1}}) \leq s\varepsilon$, [iii)] $\varepsilon \leq \lim_{k\to\infty} \inf d(x_{m_{(k)}}, x_{n_{(k)+1}}) \leq \lim_{k\to\infty} \sup d(x_{m_{(k)}}, x_{n_{(k)+1}}) \leq s\varepsilon$, and [vi)] $\varepsilon/s \leq \lim_{k\to\infty} \inf d(x_{m_{(k)+1}}, x_{n_{(k)+1}}) \leq \lim_{k\to\infty} \sup d(x_{m_{(k)+1}}, x_{n_{(k)+1}}) \leq s^2\varepsilon$.

Since T is a θ-ϕ-contraction, applying (81) with $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we have

$$\begin{aligned} \theta\left(d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) &\leq \theta\left(s^2 d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \\ &\leq \varphi\left[\theta\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]. \end{aligned} \tag{112}$$

As in the proof of Theorem 11, we have

$$M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \max\left\{d\left(x_{m_{(k)}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right), \left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right)\right\}, \tag{113}$$

$$\lim_{k\to\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq s\varepsilon. \tag{114}$$

By letting $k \to \infty$ in inequality (112) and using (θ_1), (θ_3), (Φ_3), vi), (114) and Lemma 7, we obtain

$$\begin{aligned} \theta\left(s^2 \frac{\varepsilon}{s}\right) = \theta(s\varepsilon) &\leq \theta\left[s^2 \lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right] \\ &\leq \varphi\left[\theta \lim_{k\to\infty}\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]. \end{aligned} \tag{115}$$

Therefore,

$$\theta(s\varepsilon) < \theta(s\varepsilon). \tag{116}$$

It is a contradiction. Therefore,

$$\lim_{n,m\to\infty} d(x_m, x_n) = 0. \tag{117}$$

Hence, $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d), there exists z in X such that

$$\lim_{x\to\infty} d(x_n, z) = 0. \tag{118}$$

Now, we show that $d(Tz, z) = 0$ arguing by contradiction, we assume that

$$d(Tz, z) > 0. \tag{119}$$

As in the proof of Theorem 11, we conclude that

$$\frac{1}{s} d(z, Tz) \leq \lim_{n\to\infty} \sup d(Tx_n, Tz) \leq s d(z, Tz). \tag{120}$$

Since T is a θ-ϕ-contraction, applying (81) with $x = x_n$ and $y = z$, we conclude that

$$\theta\left(s^2 d(Tx_n, Tz)\right) \leq \phi([\theta(M(x_n, z))]), \tag{121}$$

where

$$M(x_n, z) = \max \{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(z, Tx_n)\}. \tag{122}$$

This implies that

$$\theta(s^2 d(Tx_n, Tz)) \le \phi[\theta(\max \{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(z, Tx_n)\})]. \tag{123}$$

By letting $n \to \infty$ in inequality (123) and using (θ_3), (Φ_3), (120) and Lemma 7, we obtain

$$\begin{aligned} \theta\left[s^2 \frac{1}{s} d(z, Tz)\right] &= \theta[sd(z, Tz)] \\ &\le \theta\left[s^2 \lim_{n\to\infty} d(Tx_n, Tz)\right] \\ &\le \phi\left[\left(\theta\left(\lim_{n\to\infty} \max \{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(z, Tx_n)\}\right)\right)\right] \\ &= \phi[\theta(d(z, Tz))] < \theta(d(z, Tz)). \end{aligned} \tag{124}$$

As θ is increasing, then we deduced that

$$d(z, Tz) < sd(z, Tz). \tag{125}$$

Therefore, $s < 1$. It is a contradiction. So, $z = Tz$. Thus, T has a fixed point.

Uniqueness: let $z, u \in$ fix (T) where $z \ne u$. Then, from

$$d(Tz, Tu) > 0. \tag{126}$$

Applying (81) with $x = z$ and $y = u$, we have

$$\theta(d(z, u)) = \theta(d(Tu, Tz)) \le \theta(s^2 d(Tu, Tz)) \le \varphi[\theta(M(z, u))], \tag{127}$$

where

$$M(z, u) = \max \{d(z, u), d(z, Tz), d(u, Tu), d(u, Tz)\} = d(z, u). \tag{128}$$

Therefore, we have

$$\theta(d(z, u)) \le \varphi[\theta(d(z, u))] < \theta(d(z, u)). \tag{129}$$

This implies that $d(z, u) < d(z, u)$. It is a contradiction. Therefore, $u = z$.

Following from Theorem 14, we obtain the fixed point theorems for the θ-ϕ-Kannan-type contraction and the θ-ϕ-Reich-type contraction. The results presented in the paper improve and extend the corresponding results due to the Kannan-type contraction and Reich-type contraction on rectangular b-metric space.

Theorem 15. *Let (X, d) be a complete b-rectangular metric space and $T : X \to X$ be a Kannan-type contraction, then T has a unique fix.*

Proof. Since T is a Kannan-type contraction, then there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$\begin{aligned} \theta[s^2 d(Tx, Ty)] &\le \phi\left[\theta\left(\frac{d(Tx, x) + d(Ty, y)}{2}\right)\right] \\ &\le \phi[\theta(\max \{d(x, Tx), d(y, Ty)\})] \\ &\le \phi[\theta(\max \{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\})]. \end{aligned} \tag{130}$$

Therefore, T is θ-ϕ-contraction. As in the proof of Theorem 14 we conclude that T has a unique fixed point.

Theorem 16. *Let (X, d) be a complete b-rectangular metric space and $T : X \to X$ be a Reich-type contraction. Then, T has a unique fixed point.*

Proof. Since T is a Reich-type contraction, then there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$\begin{aligned} \theta[s^2 d(Tx, Ty)] &\le \phi\left[\theta\left(\frac{d(x, y) + d(Tx, x) + d(Ty, y)}{3}\right)\right] \\ &\le \phi[\theta(\max \{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\})]. \end{aligned} \tag{131}$$

Therefore, T is a θ-ϕ-contraction. As in the proof of Theorem 14 we conclude that T has a unique fixed point.

Corollary 17. *Let (X, d) be a complete b-rectangular metric space and $T : X \to X$ be a Kannan type mapping, i.e., there exists $\alpha \in]0, (1/2)[$ such that for all $x, y \in X$,*

$$d(Tx, Ty) > 0 \Rightarrow s^2 d(Tx, Ty) \le \alpha[(d(Tx, x) + d(Ty, y))]. \tag{132}$$

Then, T has a unique fixed point.

Proof. Let $\theta(t) = e^t$ for all $t \in]0, +\infty[$, and $\phi(t) = t^{2\alpha}$ for all t $\in [1,+\infty[$. Clearly $\phi \in \Phi$ and $\theta \in \Theta$. We prove that T is a θ-ϕ-Kannan-type contraction. Indeed,

$$\begin{aligned} \theta(s^2 d(Tx, Ty)) &= e^{s^2 d(Tx,Ty)} \\ &\le e^{\alpha(d(Tx,x)+d(Ty,y))} \\ &= e^{2\alpha\left(\frac{d(Tx,x)+d(Ty,y)}{2}\right)} \\ &= \left[e^{\left(\frac{d(Tx,x)+d(Ty,y)}{2}\right)}\right]^{2\alpha} \\ &= \phi\left[\theta\left(\frac{d(Tx, x) + d(Ty, y)}{2}\right)\right]. \end{aligned} \tag{133}$$

As in the proof of Theorem 15, T has a unique fixed point $x \in X$.

Corollary 18. *Let (X, d) be a complete b-rectangular metric space and $T : X \to X$ be a Reich-type mapping, i.e., there exists $\lambda \in]0, (1/3)[$ such that for all $x, y \in X$,*

$$d(x, y) > 0 \Rightarrow s^2 d(Tx, Ty) \le \lambda[(d(x, y) + d(Tx, x) + d(Ty, y))]. \tag{134}$$

Then, T has a unique fixed point.

Proof. Let $\theta(t) = e^t$ for all $t \in]0, +\infty[$ and $\phi(t) = t^{3\lambda}$ for all $t \in [1, +\infty[$.

We prove that T is a θ-ϕ-Reich-type contraction. Indeed,

$$\begin{aligned}\theta\left(s^2 d(Tx, Ty)\right) &= e^{s^2 d(Tx,Ty)}\\ &\le e^{\lambda(d(x,y)+d(Tx,x)+d(Ty,y))}\\ &= e^{3\lambda\left(\frac{d(x,y)+d(Tx,x)+d(Ty,y)}{3}\right)}\\ &= \phi\left[\theta\left(\frac{d(x, y) + d(Tx, x) + d(Ty, y)}{3}\right)\right].\end{aligned} \tag{135}$$

As in the proof of Theorem 16, T has a unique fixed point $x \in X$.

Corollary 19. *(Theorem 11 Let (X, d) be a complete b-rectangular metric space and $T : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $r \in]0, 1[$ such that for any $x, y \in X$,*

$$d(Tx, Ty) > 0 \Rightarrow \theta\left[s^2 d(Tx, Ty)\right] \le [\theta(M(x, y))]^r, \tag{136}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}. \tag{137}$$

Then, T has a unique fixed point.

Proof. By taking $\phi(t) = t^r$, with $r \in]0, 1[$, obvious $\phi \in \Phi$, then we conclude that T is a θ-ϕ-contraction. As in the proof of Theorem 14, T has a unique fixed point.

Very recently, Kari et al. in [8] proved the result (Theorem 1) on (α, η)-complete rectangular b-metric spaces. In this paper, we prove this result in complete rectangular b-metric spaces.

Corollary 20. *Let $d(X, d)$ be a complete b-rectangular metric space with parameter $s > 1$, and let T be self-mapping on X. If for all $x, y \in X$, we have*

$$\begin{aligned}d(Tx, Ty) > 0 \Rightarrow \theta\left(s^2.d(Tx, Ty)\right)\\ \le \phi[\theta(\beta_1 d(x, y) + \beta_2 d(Tx, x) + \beta_3 d(Ty, y) + \beta_4 d(y, Tx))],\end{aligned} \tag{138}$$

where $\theta \in \Theta$, $\phi \in \Phi$, $\beta_i \ge 0$ for $i \in \{1, 2, 3, 4\}$, $\sum_{i=0}^{i=4}\beta_i \le 1$. Then, T has a unique fixed point.

Proof. We prove that T is a θ-ϕ-contraction. Indeed,

$$\begin{aligned}\theta\left(s^2.d(Tx, Ty)\right) &\le \phi[\theta(\beta_1 d(x, y) + \beta_2 d(Tx, x)\\ &\quad + \beta_3 d(Ty, y) + \beta_4 d(y, Tx))]\\ &\le \phi[\theta(\beta_1 + \beta_2 + \beta_3 + \beta_4)\\ &\quad \cdot (\max\{d(x, y), d(Tx, x), d(Ty, y), d(y, Tx)\})]\\ &\le \phi[\theta(\max\{d(x, y), d(Tx, x), d(Ty, y), d(y, Tx)\})].\end{aligned} \tag{139}$$

As in the proof of Theorem 14, T has a unique fixed point.

Example 21. Let $X = A \cup B$, where $A = \{1/n : n \in \{3, 4, 5, 6\}\}$ and $B = [(1/2), (3/2)]$. Define $d : X \times X \to [0, +\infty[$ as follows:

$$\begin{cases} d(x, y) = d(y, x) & \text{for all } x, y \in X,\\ d(x, y) = 0 \Leftrightarrow y = x, \end{cases}$$
$$\begin{cases} d\left(\frac{1}{3}, \frac{1}{4}\right) = d\left(\frac{1}{4}, \frac{1}{5}\right) = 0, 1,\\ d\left(\frac{1}{3}, \frac{1}{5}\right) = d\left(\frac{1}{4}, \frac{1}{6}\right) = 0, 05,\\ d\left(\frac{1}{3}, \frac{1}{6}\right) = d\left(\frac{1}{5}, \frac{1}{6}\right) = 0, 5,\\ d(x, y) = (|x - y|)^2 \text{ otherwise.} \end{cases} \tag{140}$$

Then, (X, d) is a b-rectangular metric space with coefficient $s = 3$. However we have the following: [1)]$(X, d)$is not a metric space, as $d((1/5), (1/6)) = 0.5 > 0.15 = d((1/5), (1/4)) + d((1/4), (1/6))$. [2)] $(X, d)$is not ab-metric space for$s = 3$, as $d((1/5), (1/6)) = 0.5 > 0.45 = 3[d((1/5), (1/4)) + d((1/4), (1/6))]$. [3)] $(X, d)$is not a rectangular metric space, as $d((1/5), (1/6)) = 0.5 > 0.2 = d((1/5), (1/3)) + d((1/3), (1/4)) + d((1/4), (1/6))$. Define mapping $T : X \to X$ by

$$T(x) = \begin{cases} \dfrac{\sqrt{x} + 4}{5} & \text{if } x \in \left[\dfrac{1}{2}, \dfrac{3}{2}\right],\\ 1 & \text{if } x \in A. \end{cases} \tag{141}$$

Then, $T(x) \in [(1/2), (3/2)]$. Let $\theta(t) = \sqrt{t} + 1$, $\phi(t) = (t + 1)/2$. It is obvious that $\theta \in \Theta$ and $\phi \in \Phi$.

Consider the following possibilities:

$$d(Tx, Ty) = \left(\frac{\sqrt{x} - \sqrt{y}}{5}\right)^2. \tag{142}$$

Case 1. $x, y \in [(1/2), (3/2)]$, with $x \neq y$ and assume that $x > y$.

Therefore,

$$\begin{aligned}\theta\Big(s^2 d(Tx, Ty) &= \frac{3}{5}\left(\sqrt{x} - \sqrt{y}\right) + 1,\\ \phi[\theta(d(x, y))] &= \frac{x - y}{2} + 1.\end{aligned} \tag{143}$$

On the other hand,

$$\begin{aligned}\theta(s^2 d(Tx,Ty) - \phi[\theta(d(x,y))] &= \frac{6(\sqrt{x}-\sqrt{y}) - 5(x-y)}{10} \\ &= \frac{1}{10}((\sqrt{x}-\sqrt{y}))[6-5(\sqrt{x}+\sqrt{y})].\end{aligned} \tag{144}$$

Since $x, y \in [(1/2), (3/2)]$, then

$$6 - 5\sqrt{6} \le [6 - 5(\sqrt{x}+\sqrt{y})] \le 6 - \frac{10}{\sqrt{2}} \le 0, \tag{145}$$

which implies that

$$\begin{aligned}\theta(s^2 d(Tx,Ty) &\le \phi[\theta(d(x,y))] \\ &\le \phi[\theta(\max\{d(x,y), d(x,Tx), d(y,Ty)\}, d(y,Tx))].\end{aligned} \tag{146}$$

Case 2. $x \in [(1/2), (3/2)]$, $y \in A$, or $y \in [(1/2), (3/2)]$, $x \in A$.

Therefore, $T(x) = (\sqrt{x}+4)/5$, $T(y) = 1$, then $d(Tx,Ty) = (|(\sqrt{x}-1)/5|)^2$.

In this case, consider two possibilities:

(1) $x \ge 1$: then $\sqrt{x} \ge 1$. Therefore,

$$d(Tx,Ty) = \left(\frac{\sqrt{x}-1}{5}\right)^2. \tag{147}$$

So, we have

$$\theta(s^2 d(Tx,Ty) = \frac{3}{5}(\sqrt{x}-1) + 1,$$

$$\begin{aligned}M(x,y) &= \max\{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx)\} \\ &\ge d(y,Ty) = (1-y)^2 \ge \left(1-\frac{1}{3}\right)^2 = \left(\frac{2}{3}\right)^2,\end{aligned}$$

$$\phi\left[\theta\left(\left(\frac{2}{3}\right)^2\right)\right] = \frac{1}{3} + 1. \tag{148}$$

On the other hand,

$$\begin{aligned}\theta(s^2 d(Tx,Ty) - \phi\left[\theta\left(d\left(1,\frac{1}{3}\right)\right)\right] &= \frac{3}{5}(\sqrt{x}-1) - \frac{1}{3} \\ &= \frac{1}{15}(9\sqrt{x}-14).\end{aligned} \tag{149}$$

Since $x \in [1, (3/2)]$, then

$$\frac{1}{15}(9\sqrt{x}-14) \le 0. \tag{150}$$

This implies that

$$\begin{aligned}\theta(s^2 d(Tx,Ty) &\le \phi[\theta(d(y,Ty))] \\ &\le \phi\left[\theta\left(d\left(1,\frac{1}{3}\right)\right)\right] \\ &\le \phi[\theta(d(y,Ty)] \\ &\le \phi[\theta(\max\{d(x,y), d(x,Tx), d(y,Ty)\}, d(y,Tx))].\end{aligned} \tag{151}$$

(2) $x < 1$: then $\sqrt{x} < 1$. Therefore,

$$d(Tx,Ty) = \left(\left|\frac{1-\sqrt{x}}{5}\right|\right)^2 = \left(\frac{1-\sqrt{x}}{5}\right)^2. \tag{152}$$

So, we have

$$\theta(s^2 d(Tx,Ty) = \frac{3}{5}(1-\sqrt{x}) + 1,$$

$$\begin{aligned}M(x,y) &= \max\{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx)\} \\ &\ge \left(\frac{2}{3}\right)^2,\end{aligned}$$

$$\phi\left[\theta\left(\left(\frac{2}{3}\right)^2\right)\right] = \frac{1}{3} + 1. \tag{153}$$

On the other hand,

$$\theta(s^2 d(Tx,Ty) - \phi\left[\theta\left(d\left(1,\frac{1}{3}\right)\right)\right] = \frac{3}{5}(1-\sqrt{x}) - \frac{1}{3} = \frac{1}{15}(4-9\sqrt{x}). \tag{154}$$

Since $x \in [(1/2), 1]$, then

$$\frac{1}{15}(4-9\sqrt{x}) \le 0. \tag{155}$$

This implies that

$$\begin{aligned}\theta(s^2 d(Tx,Ty) &\le \phi[\theta(d(y,Ty))] \\ &\le \phi[\theta(\max\{d(x,y), d(x,Tx), d(y,Ty)\}, d(y,Tx))].\end{aligned} \tag{156}$$

Hence, condition (81) is satisfied. Therefore, T has a unique fixed point $z = 1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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