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Research Article

An Equivalence between the Limit Smoothness and the Rate of Convergence for a General Contraction Operator Family

Maxim J. Goldberg [1] **and Seonja Kim** [2]

[1] Theoretical and Applied Science, Ramapo College of New Jersey, Mahwah, NJ 07430, USA
[2] Mathematics Department, Middlesex College, Edison, NJ 08818, USA

Correspondence should be addressed to Maxim J. Goldberg; mgoldber@ramapo.edu

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Let X be a topological space equipped with a complete positive σ-finite measure and T a subset of the reals with 0 as an accumulation point. Let $a_t(x, y)$ be a nonnegative measurable function on $X \times X$ which integrates to 1 in each variable. For a function $f \in L_2(X)$ and $t \in T$, define $A_t f(x) \equiv \int a_t(x, y) f(y)\, dy$. We assume that $A_t f$ converges to f in L_2, as $t \longrightarrow 0$ in T. For example, A_t is a diffusion semigroup (with $T = [0,\infty)$). For W a finite measure space and $w \in W$, select real-valued $h_w \in L_2(X)$, defined everywhere, with $\|h_w\|_{L_2(X)} \leq 1$. Define the distance D by $D(x, y) \equiv \|h_w(x) - h_w(y)\|_{L_2(W)}$. Our main result is an equivalence between the smoothness of an $L_2(X)$ function f (as measured by an L_2-Lipschitz condition involving $a_t(\cdot, \cdot)$ and the distance D) and the rate of convergence of $A_t f$ to f.

1. Introduction

One of the questions that arise in harmonic analysis is the connection between the smoothness of a given function and the rate of approximation by members of a specified family of functions. An important example is the relationship between the smoothness of a function and the speed of convergence of its diffused version to itself, in the limit as time goes to zero. As mentioned in the Introduction of [1], for the Euclidean setting and the heat kernel, see for example [2, 3].

In a more general setting, for a diffusion semigroup $\{T_t f\}_{t\geq 0}$ on a topological space X with a positive σ-finite measure given, for $t > 0$, by an integral kernel operator: $T_t f(x) \equiv \int_X \rho_t(x, y) f(y)\, dy$, Coifman and Leeb in [1, 4] introduce a family of multiscale diffusion distances and establish quantitative results about the equivalence of a bounded function f being Lipschitz and the rate of convergence of $T_t f$ to f, as $t \rightarrow 0^+$. The respective authors of [5–7] consider different aspects of the connection between the smoothness of a function and the rate of convergence of its diffused versions to itself.

As mentioned in, for instance, the Introductions of [5–7], the interest in diffusion semigroups is natural since they play an important role in analysis, both theoretical and applied. Diffusion semigroups include the heat semigroup and, more generally, as discussed in, e.g., [8], arise from considering large classes of elliptic second-order (partial) differential operators on domains in Euclidean space or on manifolds.

For examples of theoretical results involving diffusion semigroups, the interested reader may refer to Chavel [9], Cowling [10], Stein [8], Sturm [11], and Wu [12]. Some applications of diffusion semigroups to dimensionality reduction, embedding, clustering, data representation, manifold parametrization, and multiscale analysis of complex structures can be found in, e.g., [13–22]. Various definitions and procedures for efficient computation of natural diffusion distances can be found in, e.g., [1, 4, 23, 24].

In the present work, we consider a more general family than a diffusion semigroup. For T a subset of the reals having 0 as an accumulation point, for $t \in T$, let $a_t(x, y)$ be a nonnegative measurable function on $X \times X$ which integrates to 1 in each variable. For a function $f \in L_2(X)$ and $t \in T$, define $A_t f$

$(x) = \int_X a_t(x,y)f(y)\,dy$. We assume that for every $f \in L_2(X)$, $\|A_t f - f\|_{L_2} \longrightarrow 0$, as $t \longrightarrow 0, t \in T$. No assumption is made that the family A_t is symmetric or is a semigroup nor is anything assumed about T other than that T has 0 as an accumulation point.

For a finite measure space W, selecting $h_w \in L_2(X)$ for every $w \in W$, we define a distance between points $x, y \in X$ by $D(x,y) \triangleq \|h_w(x) - h_w(y)\|_{L_2(W)}$. We next introduce an L_2 version of being Lipschitz (relative to $\{A_t\}$) using this distance D. Our main result is that a function $f \in L_2(X)$ is L_2-Lipschitz if and only if we have an estimate of the rate of convergence of $\langle f, f - A_t f\rangle$ to 0, namely, $(0 \le)\langle f, f - A_t f\rangle \le c g_D(t)$, where $g_D(t) = 2\int_W \langle h_w, h_w - A_t h_w\rangle\,dw \longrightarrow 0$, as $t \longrightarrow 0, t \in T$.

Our paper is organized as follows. Following a notation and assumptions section (Section 2), we state the main definitions, provide some examples, and establish our results in Section 3. The paper ends with the Conclusions and Acknowledgments sections.

2. Notation and Assumptions

Let X be a topological space equipped with a complete positive σ-finite measure. The measure on X will be denoted by dx and dy. W is a *finite* measure space, with measure denoted by dw. We assume all spaces involved are such that Fubini's theorem holds on any product of these spaces; e.g., the spaces are σ-finite. All functions are assumed to be real-valued and measurable on the respective spaces; in particular, functions of several variables are assumed to be measurable on the appropriate product spaces.

T will denote a subset of the reals, with 0 as an accumulation point. From now on, $t \longrightarrow 0$ will mean $t \longrightarrow 0, t \in T$. For every $t \in T$, let $a_t(x,y)$ be a nonnegative measurable function on $X \times X$ with the property that $\int_X a_t(x,y)\,dy = \int_X a_t(x,y)\,dx = 1$. For $t \in T$ and a function $f \in L_2(X)$, define $A_t : L_2 \longrightarrow L_2$ by $A_t f(x) = \int_X a_t(x,y)f(y)\,dy$. We assume that for every $f \in L_2(X)$, $\|A_t f - f\|_{L_2} \longrightarrow 0$, as $t \longrightarrow 0$.

No assumption is made that the family A_t is symmetric or is a semigroup nor is anything assumed about T other than that T has 0 as an accumulation point.

Note that A_t is indeed bounded on L_2 with norms not exceeding one, since

$$\begin{aligned}|\langle g, A_t f\rangle| &\le \iint a_t(x,y)|f(x)||g(y)|dxdy \\ &\le \left(\iint a_t(x,y)f^2(x)dxdy\right)^{1/2}\left(\iint a_t(x,y)g^2(y)dxdy\right)^{1/2} \\ &= \left(\left(\int a_t(x,y)dy\right)f^2(x)dx\right)^{1/2} \\ &\quad \left(\left(\int a_t(x,y)\,dx\right)g^2(y)dy\right)^{1/2} = \|f\|_{L_2}\|g\|_{L_2}.\end{aligned} \tag{1}$$

In particular, $\langle f, f - A_t f\rangle \ge 0$.

We will define a family Δ of symmetric distances on $X \times X$ satisfying the triangle inequality with the following properties for every $D \in \Delta$:

(i) $g_D(t) \equiv \iint a_t(x,y)D^2(x,y)\,dxdy < \infty$ for every $t \in T$

(ii) $g_D(t) \longrightarrow 0$, as $t \longrightarrow 0$

3. Main Definitions and Results

We start by describing the family Δ of symmetric distances on $X \times X$.

Definition 1. Select a finite measure space W. For each $w \in W$, select $h_w \in L_2(X)$, defined everywhere, with $\|h_w\|_{L_2(X)} \le 1$. Note that some h_w may be chosen to be identically 0. *Then, the distance $D \in \Delta$ is given by*

$$D(x,y) \equiv \|h_w(x) - h_w(y)\|_{L_2(W)} = \left(\int_W (h_w(x) - h_w(y))^2\,dw\right)^{1/2}. \tag{2}$$

Clearly, D is symmetric and satisfies the triangle inequality (the latter fact follows from the triangle inequality for $L_2(W)$).

Before looking at some examples of such distances, we define our L_2-Lipschitz condition.

Definition 2. For $D \in \Delta$, we say that $f \in L_2(X)$ is L_2-Lipschitz (relative to $\{A_t\}$) if

$$\iint a_t(x,y)(f(x) - f(y))^2\,dxdy \le c\iint a_t(x,y)D^2(x,y)dxdy, \tag{3}$$

for every $t \in T$.

Now let us consider some examples of distances $D \in \Delta$. For the first one, let X be a bounded subset of $\mathbb{R}^n$ having (some) finite measure dx. Let $W = \{1, 2, \cdots, n\}$, with dw indicating unit masses assigned at each point of W. For $k = 1, \cdots, n$, let $h_k(x) = cx_i$, for $x = (x_1, \cdots, x_n) \in X$, where c is a suitable constant to ensure that $\|h_k\|_{L^2} \le 1$. Then, $D_1(x,y) \equiv \left(\int_W (h_w(x) - h_w(y))^2\,dw\right)^{1/2} = \left(c^2\sum_{k=1}^n (x_k - y_k)^2\right)^{1/2}$, a multiple of the Euclidean distance on X.

For our second example, let X be a finite measure space with measure dx. Let $W = (0,1) \times X$, with $dw = s^{\alpha-1}dsdx$, where $\alpha > 0$. For $w = (s,u) \in W$, let $h_w(x) = \sqrt{a_s(x,u)}$. Clearly, $\|h_w\|_{L_2} = \sqrt{\int a_s(x,u)\,dx} = 1$. Then,

$$\begin{aligned}D_2^2(x,y) &\equiv \int_X\int_0^1\left(\sqrt{a_s(x,u)} - \sqrt{a_s(y,u)}\right)^2 s^{\alpha-1}dsdu \\ &= \int_0^1\left\|\sqrt{a_s(x,\cdot)} - \sqrt{a_s(y,\cdot)}\right\|_{L_2}^2 s^{\alpha-1}ds,\end{aligned} \tag{4}$$

an analog of the distance considered by Coifman and Leeb in [1, 4] for a semigroup. Note that $\sqrt{a_s(x,u)}$ and $\sqrt{a_s(y,u)}$ are normalized in L_2 with respect to u as well.

To gain some understanding of this distance D_2 (although we will use the case of $\mathbb{R}^n$ with Lebesgue measure, not a finite measure space), let us calculate the distance $D_2 = (\int_0^1 \|\sqrt{a_s(x,\cdot)} - \sqrt{a_s(y,\cdot)}\|_{L_2}^2 s^{\alpha-1} ds)^{1/2}$ for the basic case when $X = \mathbb{R}^n$ with dx Lebesgue measure and $\{A_t\}$ is the heat flow semigroup. (While the derivation right after the statement of Proposition 2.6 in [5] by Coifman and Goldberg has a calculation of this distance, we present a more detailed computation here.)

We easily see that $\int_X (\sqrt{a_s(x,u)} - \sqrt{a_s(y,u)})^2 du = 2 - 2e^{-|x-y|^2/(16s)}$, where $|x-y|$ is the Euclidean distance between the points x and y. Thus, $D_2^2(x,y) = 2\int_0^1 (1 - e^{-|x-y|^2/(16s)}) s^{\alpha-1} ds$.

If $|x-y| \geq 1$, $e^{-|x-y|^2/(16s)}$ is bounded away from 1 for $0 < s < 1$, so $D_2^2(x,y) \sim \int_0^1 s^{\alpha-1} ds = c$.

If $|x-y| < 1$, write

$$\int_0^1 \left(1 - e^{-|x-y|^2/(16s)}\right) s^{\alpha-1} ds = \int_0^{|x-y|^2} \left(1 - e^{-|x-y|^2/(16s)}\right) s^{\alpha-1} ds + \int_{|x-y|^2}^1 \left(1 - e^{-|x-y|^2/(16s)}\right) s^{\alpha-1} ds. \tag{5}$$

For the first summand, observe that $\int_0^{|x-y|^2} (1 - e^{-|x-y|^2/(16s)}) s^{\alpha-1} ds \sim \int_0^{|x-y|^2} s^{\alpha-1} ds = c|x-y|^{2\alpha}$. For the second summand, an easy calculation shows that, for $\alpha \neq 1$,

$$\int_{|x-y|^2}^1 \left(1 - e^{-|x-y|^2/(16s)}\right) s^{\alpha-1} ds \sim \int_{|x-y|^2}^1 \frac{|x-y|^2}{s} s^{\alpha-1} ds = c\frac{1}{\alpha-1}\left(|x-y|^2 - |x-y|^{2\alpha}\right) \sim \begin{cases} |x-y|^{2\alpha}, & 0 < \alpha < 1. \\ |x-y|^2, & \alpha > 1. \end{cases} \tag{6}$$

Combining with the estimate for the first summand, and with the case $|x-y| \geq 1$, we obtain that for $0 < \alpha < 1$,

$$D_2(x,y) \sim \begin{cases} |x-y|^\alpha, & |x-y| < 1, \\ 1, & |x-y| \geq 1, \end{cases} \tag{7}$$

while for $\alpha > 1$,

$$D_2(x,y) \sim \begin{cases} |x-y|, & |x-y| < 1. \\ 1, & |x-y| \geq 1. \end{cases} \tag{8}$$

Our third example is a variation of our second example above. As in the second example, let X be a finite measure space with measure dx and let $W = (0,1) \times X$, with $dw = s^{\alpha-1} ds dx$, where $\alpha > 0$. For $w = (s,u) \in W$, let $h_w(x) = a_s(x,u)/\|a_s(\cdot,u)\|_{L_2}$. Clearly, $\|h_w\|_{L_2} = 1$.

Then, our new distance is given by

$$D_3^2(x,y) = \int_X \int_0^1 \left(\frac{a_s(x,u)}{\|a_s(\cdot,u)\|_{L_2}} - \frac{a_s(y,u)}{\|a_s(\cdot,u)\|_{L_2}}\right)^2 s^{\alpha-1} ds du. \tag{9}$$

(For a related example, see Section 4 of [23] and the very last example in Section 2 of [5].)

Let us specialize to the case of a symmetric diffusion semigroup with the following additional requirement: $a_s(z,z)$ is constant over $z \in X$ (but varies with s). Let $a_s(\cdot,\cdot)$ denote the value of $a_s(z,z)$ for every $z \in X$. Under these assumptions, using the semigroup property, we easily obtain

$$D_3^2(x,y) = \int_0^1 \left(2 - 2\frac{a_{2s}(x,y)}{a_{2s}(\cdot,\cdot)}\right) s^{\alpha-1} ds. \tag{10}$$

In the very special subcase of $X = \mathbb{R}^n$ equipped with Lebesgue measure and $\{A_t\}$ the heat flow semigroup,

$$2 - 2\frac{a_{2s}(x,y)}{a_{2s}(\cdot,\cdot)} = 2 - 2e^{-|x-y|^2/(8s)}, \tag{11}$$

and we thus obtain the same estimates for $D_3(x,y)$ as for $D_2(x,y)$ above.

We now return to the general development. The following simple result is the key tautology to prove our Theorem 6.

Proposition 3. *For $f \in L_2(X)$, $\iint a_t(x,y)(f(x) - f(y))^2 dx dy = 2\langle f, f - A_t f\rangle$.*

Proof. Using Fubini's theorem and the assumption that $a_t(x,y)$ integrates to 1 in each variable, we see that

$$\begin{aligned} \iint a_t(x,y)(f(x)-f(y))^2 dxdy &= \iint a_t(x,y)\left(f^2(x) + f^2(y) - 2f(x)f(y)\right)^2 dxdy \\ &= \iint a_t(x,y) f^2(x) dxdy + \iint a_t(x,y) f^2(y) dxdy \\ &\quad - 2\int f(x)\left(\int a_t(x,y) f(y) dy\right) dx \\ &= 2\|f\|_{L_2(X)}^2 - 2\langle f, A_t f\rangle = 2\langle f, f\rangle - 2\langle f, A_t f\rangle \\ &= 2\langle f, f - A_t f\rangle. \end{aligned} \tag{12}$$

For $D \in \Delta$, letting $g_D(t) \equiv \iint a_t(x,y) D^2(x,y) dx dy$, we obtain the following result.

Proposition 4. $g_D(t) = 2\int_W \langle h_w, h_w - A_t h_w\rangle dw$.

Proof. Using the definition of $D(x,y)$, Fubini's theorem, and Proposition 3, we observe that

$$\iint a_t(x,y) D^2(x,y)dxdy = \int_W \left(\iint a_t(x,y)(h_w(x) - h_w(y))^2 dxdy\right) dw = 2\int_W \langle h_w, h_w - A_t h_w\rangle dw. \tag{13}$$

Corollary 5. *$g_D(t) < \infty$ for every $t \in T$ and $g_D(t) \longrightarrow 0$ as $t \longrightarrow 0$.*

Proof. Since $|\langle h_w, h_w - A_t h_w\rangle| \le 2$ and $\int_W dw < \infty$, the result that $g_D(t) < \infty$, for every $t \in T$, follows from Proposition 4. From one of our initial assumptions that for every $f \in L_2(X)$, $\|A_t f - f\|_{L_2} \longrightarrow 0$, as $t \longrightarrow 0$, we obtain that $\langle h_w, h_w - A_t h_w\rangle \longrightarrow 0$, as $t \longrightarrow 0$, for every w. Hence, $g_D(t) \longrightarrow 0$, as $t \longrightarrow 0$, by the dominated convergence theorem.

Recalling Definition 2, we can now prove the following theorem, which is of interest only due to Corollary 5.

Theorem 6. *For $D \in \Delta$ and $f \in L_2(X)$, f is L_2-Lipschitz if and only if $\langle f, f - A_t f\rangle \le c g_D(t)$, for every $t \in T$.*

Proof. First, suppose that $f \in L_2(X)$ is L_2-Lipschitz. Then, by Proposition 3, we have

$$\langle f, f - A_t f\rangle = \frac{1}{2}\iint a_t(x,y)(f(x) - f(y))^2 dxdy \cdot \le c\iint a_t(x,y) D^2(x,y)dxdy = c g_D(t), \tag{14}$$

for $t \in T$.

Conversely, suppose $\langle f, f - A_t f\rangle \le c g_D(t)$, for every $t \in T$. Then, by Proposition 3 again,

$$\iint a_t(x,y)(f(x) - f(y))^2 dxdy = 2\langle f, f - A_t f\rangle \le c g_D(t) = c\iint a_t(x,y) D^2(x,y)dxdy. \tag{15}$$

Thus, f is L_2-Lipschitz.

It is easy to see that $(1/2)\|f - A_t f\|_{L_2}^2 \le \langle f, f - A_t f\rangle \le \|f\|_{L_2}\|f - A_t f\|_{L_2}$, so Theorem 6 establishes an equivalence between f being L_2-Lipschitz and having an estimate of the speed of convergence of $\|f - A_t f\|_{L_2}$ to 0, as $t \to 0$.

Note that if $\{A_t\}$ is a symmetric semigroup (and $T = [0, \infty)$), then

$$\begin{aligned}\langle f, f - A_t f\rangle &= \langle f, f\rangle - \langle f, A_t f\rangle \\ &= \langle f, f\rangle - \langle A_{t/2} f, A_{t/2} f\rangle \\ &= \|f\|_{L_2}^2 - \|A_{t/2} f\|_{L_2}^2 \\ &= \left(\|f\|_{L_2} - \|A_{t/2} f\|_{L_2}\right)\left(\|f\|_{L_2} + \|A_{t/2} f\|_{L_2}\right),\end{aligned} \tag{16}$$

so $\|f\|_{L_2}(\|f\|_{L_2} - \|A_{t/2} f\|_{L_2}) \le \langle f, f - A_t f\rangle \le 2\|f\|_{L_2}(\|f\|_{L_2} - \|A_{t/2} f\|_{L_2})$. Hence, if $\{A_t\}$ is a symmetric semigroup, $\langle f, f - A_t f\rangle \sim \|f\|_{L_2} - \|A_{t/2} f\|_{L_2}$.

4. Conclusions

For X a topological space equipped with a complete positive σ-finite measure, W a finite measure space, and selecting everywhere-defined real-valued $h_w \in L_2(X)$ for every $w \in W$ with $\|h_w\|_{L_2(X)} \le 1$, we have defined a distance D by $D(x,y) \triangleq \|h_w(x) - h_w(y)\|_{L_2(W)}$.

For T a subset of the reals having 0 as an accumulation point and for $t \in T$, letting $a_t(x,y)$ be a nonnegative measurable function on $X \times X$ which integrates to 1 in each variable, we have considered bounded operators A_t on $L_2(X)$ given by $A_t f(x) = \int_X a_t(x,y) f(y)\, dy$. Assuming that for every $f \in L_2(X)$, $\|A_t f - f\|_{L_2} \longrightarrow 0$, as $t \longrightarrow 0$, $t \in T$, we have shown that $\iint a_t(x,y)(f(x) - f(y))^2\, dxdy \le c\iint a_t(x,y) D^2(x,y)\, dxdy$, for every $t \in T$ if and only if $(0 \le)\langle f, f - A_t f\rangle \le c g_D(t)$, where $g_D(t) = 2\int_W \langle h_w, h_w - A_t h_w\rangle\, dw \longrightarrow 0$, as $t \longrightarrow 0$, $t \in T$.

Data Availability

No data were used to support this study.

Disclosure

Ramapo College had no involvement with the writing or submission for publication of this work.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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