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Research Article

Best Lag Window for Spectrum Estimation of Law Order MA Process

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In this article, we investigate spectrum estimation of law order moving average (MA) process. The main tool is the lag window which is one of the important components of the consistent form to estimate spectral density function (SDF). We show, based on a computer simulation, that the Blackman window is the best lag window to estimate the SDF of MA(1) and MA(2) at the most values of parameters β_i and series sizes n, except for a special case when $\beta = -1$ and $n \geq 40$ in MA(1). In addition, the Hanning–Poisson window appears as the best to estimate the SDF of MA(2) when $\beta_1 = \beta_2 = -0.5$ and $n \geq 40$.

1. Introduction

A set x_t of numerical data (observations) made sequentially in time t is called time series [1]. There are some important processes of a time series: autoregressive, moving average, and autoregressive-moving average processes.

Spectral analysis can be defined as a process that assigns power versus frequency. One of the time series analysis techniques is spectral analysis. The object of spectral analysis is to estimate and study the spectrum of the time series processes for the phenomena of physics and engineering [2].

The spectrum estimation methods can be classified into parametric and nonparametric methods [3]. The consistent estimate of spectral density function $\widehat{f}(\omega)$ is the most important nonparametric spectral analysis method, which depends on lag window $\lambda_T(v)$ and truncation point T [4].

Window functions are used in the estimation of power spectra and bispectra in order to ensure the consistency of the periodogram and the Fourier-type bispectrum estimation methods. A three-dimensional optimum bias lag window is introduced in the estimation of the 4th-order cumulant spectrum, also called trispectrum, which is estimated from the three-dimensional Fourier transform of the 4th order cumulants [5].

Zhongsheng et al. [6] suggested that using windows is one important way to improve bispectrum estimation and also an appropriate window function can be used to reduce variance and suppress noise, but it was noticed that sidelodes in a spectrum of window functions can be ended up in spectrum leak. Thus, one urgent problem which needed to be solved for the application of bispectrum was how to find one appropriate window. He combined a new lag window with Hanning–Poisson window without sidelodes, which is used for nonparametric bispectrum estimation instead of rectangle window. When the spatial location area increases becoming extremely large, it is very difficult [7], or not possible, to evaluate the covariance matrix determined by the set of location distance even for gridded stationary Gaussian process. To alleviate the numerical challenges, he did construct a nonparametric estimator called periodogram of spatial version to represent the sample property in the frequency domain because periodogram requires less computational operation by fast Fourier transform algorithm. Under some regularity conditions on the process, he investigated the asymptotic unbiasedness property of the periodogram as estimator of the spectral density function and achieved the convergence rate.

The basic concepts given in Sections 2–5 present white noise, moving average process of order q and their properties, spectral density function (SDF) on general and SDF of MA(q), and the consistent estimate of SDF, and some important lag windows are reviewed. Section 6 presents a simulation for comparison between the SDF and the consistent estimate of SDF.

2. White Noise

A purely random process ϵ_t, $t \in \mathbb{Z}^+$ is called white noise (Gaussian noise) if it consists of a sequence of uncorrelated independent identically distributed (i.i.d) random variables [3], with mean $\mu_\epsilon = 0$, variance $\operatorname{var}(\epsilon_t) = \sigma_\epsilon^2$, and the autocovariance function

$$R(v) = \operatorname{cov}(\epsilon_t, \epsilon_{t+v}) = \begin{cases} \sigma_\epsilon^2, & v = 0, \\ 0, & v \neq 0. \end{cases} \quad (1)$$

In addition, the autocorrelation function $\rho_v$is

$$\rho_v = \frac{R(v)}{R(0)} = \frac{E(x_t x_{t+v})}{\sigma_x^2} = \begin{cases} 1, & |v| \leq q, \\ 0, & |v| > q. \end{cases} \quad (2)$$

3. Moving Average Process

A stochastic process x_t, $t \in \mathbb{Z}^+$ is called moving average process of order q and denoted by $\text{MA}(q)$. This is given by

$$x_t = \sum_{i=0}^{q} \beta_i \epsilon_{t-i}, \quad (3)$$

where ϵ_t is the white noise with mean zero and covariance σ_ϵ^2 and β_i. $\beta_i = 0$, $i > q$ is the coefficient of the process. The statistical properties of $\text{MA}(q)$ is

$$\mu_x = E(x_t) = \sum_{i=0}^{q} \beta_i E(\epsilon_{t-i}) = 0, \quad (4)$$

$$\sigma_x^2 = \operatorname{var}(x_t) = \sum_{i=0}^{q} \beta_i^2 \operatorname{var}(\epsilon_{t-i}) = \sigma_\epsilon^2 \left(\sum_{i=0}^{q} \beta_i^2 \right), \quad (5)$$

such that ϵ_t be the uncorrelated random process. As a result, the autocovariance function $R(v)$ cuts off after a point x_t, $t > q$, and that implies $\operatorname{cov}(x_t, x_{t+v})$, $|v| > q$ and $R(v)$ as

$$R(v) = E(x_t x_{t+v}) = \begin{cases} \sigma_\epsilon^2 \left(\sum_{i=0}^{q} \beta_i \beta_{i+v} \right), & |v| \leq q, \\ 0, & |v| > q. \end{cases} \quad (6)$$

The autocorrelation function

$$\rho_v = \frac{R(v)}{R(0)} = \frac{E(x_t x_{t+v})}{\sigma_x^2} = \begin{cases} \dfrac{\sigma_\epsilon^2 \left(\sum_{i=0}^{q} \beta_i \beta_{i+v} \right)}{\sigma_x^2}, & |v| \leq q, \\ 0, & |v| > q. \end{cases} \quad (7)$$

Note that μ_x, σ_x^2, $R(v)$, and ρ_v are constants, the finite does not depend on time t for any finite order q. Thus, the moving average process $\text{MA}(q)$ of finite order q is a stationary process [2]. As a special case, MA(1),

$$x_t = \beta_0 \epsilon_t + \beta_1 \epsilon_{t-1}. \quad (8)$$

And $\mu_x = 0$, $\sigma_x^2 = \operatorname{var}(x_t) = \sigma_\epsilon^2 (\beta_0^2 + \beta_1^2)$, and the autocovariance and autocorrelation functions are given by

$$R(v) = \begin{cases} \sigma_\epsilon^2 (\beta_0 \beta_v + \beta_1 \beta_{1+v}), & |v| \leq 1, \\ 0, & |v| > 1, \end{cases}$$
$$\rho_v = \begin{cases} \dfrac{\sigma_\epsilon^2 (\beta_0 \beta_v + \beta_1 \beta_{1+v})}{\sigma_x^2}, & |v| \leq 1, \\ 0, & |v| > 1. \end{cases} \quad (9)$$

So, MA(2) is defined as

$$x_t = \beta_0 \epsilon_t + \beta_1 \epsilon_{t-1} + \beta_2 \epsilon_{t-2}. \quad (10)$$

And, the expected value $\mu_x = E(\beta_0 \epsilon_t + \beta_1 \epsilon_{t-1} + \beta_2 \epsilon_{t-2}) = 0$, and the variance is

$$\sigma_x^2 = \sigma_\epsilon^2 \left(\beta_0^2 + \beta_1^2 + \beta_2^2 \right). \quad (11)$$

It is clear that $\sigma_x^2 = R(v = 0)$, and the autocovariance and autocorrelation functions are given by

$$R(v) = \begin{cases} \sigma_\epsilon^2 (\beta_0 \beta_v + \beta_1 \beta_{1+v} + \beta_2 \beta_{2+v}), & |v| \leq 1, \\ 0, & |v| > 1, \end{cases}$$
$$\rho_v = \begin{cases} \dfrac{\sigma_\epsilon^2 (\beta_0 \beta_v + \beta_1 \beta_{1+v} + \beta_2 \beta_{2+v})}{\sigma_x^2}, & |v| \leq 1, \\ 0b, & |v| > 1. \end{cases} \quad (12)$$

4. Spectral Density Function

If x_t, $t \in \mathbb{Z}$ is a discrete stochastic process with autocorrelation function ρ_v [3,8], a spectral density function (SDF) $f(\omega)$ is defined as a Fourier transform of autocorrelation function ρ_v and is given as

$$f(\omega) = \frac{1}{2\pi} \sum_{v=-\infty}^{\infty} \rho_v e^{-j\omega v}, \quad (13)$$

where $j = \sqrt{-1}$. The formula is rewritten as

$$f(\omega) = \frac{1}{2\pi} \left[\sum_{v=-\infty}^{-1} \rho_v e^{-j\omega v} + \rho_0 + \sum_{v=1}^{\infty} \rho_v e^{-j\omega v} \right],$$
$$f(\omega) = \frac{1}{2\pi} \left[\sum_{v=1}^{\infty} \rho_{-v} e^{j\omega v} + \rho_0 + \sum_{v=1}^{\infty} \rho_v e^{-j\omega v} \right]. \quad (14)$$

Since autocorrelation is an even function [9], it implies $\rho_v = \rho_{-v}$ and $\rho_0 = 1$. Thus,

$$f(\omega) = \frac{1}{2\pi} \left[1 + \sum_{v=1}^{\infty} \rho_v \left(e^{2\pi j\omega v} + e^{-2\pi j\omega v} \right) \right]. \quad (15)$$

Hence,

$$f(\omega)=\frac{1}{2\pi}\left[1+2\sum_{v=1}^{\infty}\rho_v\cos(\omega v)\right]. \quad (16)$$

4.1. SDF of MA (q). Let x_t be the moving average process defined in (3) with autocovariance function $R(v)$ and autocorrelation function ρ_v. The spectral density function $f(\omega)$ defined in (13) is given as

$$\begin{aligned}f(\omega)&=\frac{1}{2\pi}\sum_{v=-\infty}^{\infty}\rho_v e^{-j\omega v}\\&=\frac{1}{2\pi}\left[\sum_{v=-\infty}^{-(q+1)}\rho_v e^{-j\omega v}+\sum_{v=-q}^{q}\rho_v e^{-j\omega v}+\sum_{v=q+1}^{\infty}\rho_v e^{-j\omega v}\right].\end{aligned} \quad (17)$$

From (7),

$$\rho_v=\begin{cases}\dfrac{\sigma_\epsilon^2\left(\sum_{i=0}^{q}\beta_i\beta_{i+v}\right)}{\sigma_x^2}, & |v|\le q,\\ 0, & |v|>q.\end{cases} \quad (18)$$

Then,

$$f(\omega)=\frac{1}{2\pi}\left[\sum_{v=-q}^{-1}\rho_v e^{-j\omega v}+\rho_v e^{-j\omega v}\Big|_{v=0}+\sum_{v=0}^{q}\rho_v e^{-j\omega v}\right]. \quad (19)$$

Since $\rho_v=\rho_{-v}$ and $\rho_v=1$,

$$\begin{aligned}f(\omega)&=\frac{1}{2\pi}\left[1+\sum_{v=1}^{q}\rho_v\left(e^{j\omega v}+e^{-j\omega v}\right)\right]\\&=\frac{1}{2\pi}\left[1+2\sum_{v=1}^{q}\rho_v\cos(\omega v)\right].\end{aligned} \quad (20)$$

Hence,

$$f(\omega)=\frac{1}{2\pi}\left[1+\sum_{v=1}^{q}\frac{\sigma_\epsilon^2\left(\sum_{i=0}^{q}\beta_i\beta_{i+v}\right)}{\sigma_x^2}\cos\omega v\right]. \quad (21)$$

As a special case, MA (1), the spectral density function will be

$$f(\omega)=\frac{1}{2\pi}\left[1+\frac{\sigma_\epsilon^2(\beta_0\beta_1)}{\sigma_x^2}\cos\omega v\right]. \quad (22)$$

And spectral density function of MA (2) is given by

$$f(\omega)=\frac{1}{2\pi}\left[1+\sum_{v=1}^{2}\frac{\sigma_\epsilon^2\left(\beta_0\beta_v+\beta_1\beta_{1+v}+\beta_2\beta_{2+v}\right)}{\sigma_x^2}\cos\omega v\right]. \quad (23)$$

5. The Consistent Estimate of SDF

Let $X_t, t\in\mathbb{Z}$ be a real-valued, weakly stationary, discrete stochastic process (time series) with zero mean and autocovariance function R_v with lag v and autocorrelation function ρ_v [3]. The consistent estimate of R_v and ρ_v are

$$\widehat{R}_v=\frac{1}{n}\sum_{t=1}^{n-|v|}X_tX_{t+|v|},\quad v<n, \quad (24)$$

$$\widehat{\rho}_v=\frac{\widehat{R}_v}{\widehat{R}_0}=\frac{\sum_{t=1}^{n-|v|}X_tX_{t+|v|}}{\sum_{t=1}^{n}x_t^2}. \quad (25)$$

If X_t is a stochastic process of size n, then the consistent form to estimate the spectral density function is [2]

$$\widehat{f}(\omega)=\frac{1}{2\pi}\sum_{v=-T+1}^{T-1}\widehat{\rho}_v\lambda_T(v)\cos(v\omega),\quad -\pi\le\omega\le\pi, \quad (26)$$

where T is the truncation point $0\le T\le n$ and $\lambda_T(v)$ is the lag window, which weighting the autocorrelation function.

The consistent estimate of SDF depends on two important sides, select an appropriate value of a truncation point T and an appropriate lag window $\lambda_T(v)$.

There are a lot of lag windows suggested by researchers [3, 6, 10–12]. Table 1 contains the most important of lag windows as shown in previous papers.

6. The Empirical Aspect

A simulation experiment is applied to achieve our goal by using Matlab software according to the following assumptions:

(1) Generate MA (1) process, $x_t=\epsilon_t+\beta\epsilon_{t-1}$ and MA (2), $x_t=\epsilon_t+\beta_1\epsilon_{t-1}+\beta_2\epsilon_{t-2}$, where the white noise ϵ_t with $\mu_\epsilon=0$ and $\sigma_\epsilon^2=1$. Empirically, the initial white noise $\epsilon_0=\mu_\epsilon$ for MA (1) and $\epsilon_0=\mu_\epsilon$ and $\epsilon_1=\mu_\epsilon$ for MA (2). The parameter $\beta_0=1$ for all processes, with different values of the parameters β, β_1, and β_2 given in Tables 2 and 3.

(2) The different values of series sizes $n=10$, 100, 500, 1000, and 10000.

(3) The run size value of simulation $k=1000$.

(4) The appropriate value of a truncation point T was calculated according to the closing window algorithm.

(5) The values of ω are $[-\pi{:}\ (0.0251){:}\ \pi]$ where the number of values is $L=250$, and $\widehat{\rho}_v$ is defined in equation (25), and the lag windows $\lambda_T(v)$ are defined in Table 1.

(6) The spectral density function of moving average process $f(\omega_i)$ of MA (1) process is

$$f(\omega)=\frac{1+2\beta\cos\omega+\beta^2}{2\pi\left(1+\beta^2\right)},\qquad -\pi\le\omega\le\pi, \quad (27)$$

Table 1: Lag windows.

Windows	The form of $\lambda_T(v)$	Notes
Rectangular	1	For all $v \le T$, $\lambda_T(v) = 0$ if $v > T$
Triangular	$1 - v/T$	$v \le T$
Hanning	$\alpha - \alpha \cos(\pi v/T)$	$v \le T$, $\alpha = 0.5$
Hamming	$\alpha_1 - \alpha_2 \cos(\pi v/T)$	$v \le T$, $\alpha_1 = 0.54$, $\alpha_2 = 0.46$
Blackman	$0.42 - 0.5 \cos(\pi v/T) + 0.8 \cos(2\pi v/T)$	$v \le T$
Blackman–Harris	$\alpha_1 - \alpha_2 \cos(\pi v/T) + \alpha_3 \cos(2\pi v/T) + \alpha_4 \cos(3\pi v/T)$	$v \le T$, $\alpha_1 = 0.402$, $\alpha_2 = 0.498$, $\alpha_3 = 0.98$, $\alpha_4 = 0.001$
Flat top	$\alpha_1 - \alpha_2 \cos(\pi v/T) + \alpha_3 \cos(2\pi v/T) + \alpha_4 \cos(3\pi v/T) + \alpha_5 \cos(4\pi v/T)$	$v \le T$, $\alpha_1 = 0.21557895$, $\alpha_2 = 0.41663158$, $\alpha_3 = 0.277263158$, $\alpha_4 = 0.083578947$, $\alpha_5 = 0.006947368$
Exponential	$(0.1)^{v/T}$	$v \le T$
Gaussian	$e^{-1/2(\alpha v/T)^2}$	$v \le T$, $\alpha = 2.5$
Welch(Riesz)	$1 - (v/T)^2$	$v \le T$
Cosine	$\sin(\pi v/T)$	$v \le T$
Parzen	$\begin{cases} 1 - 6(\mathrm{v/T})^2(1 - \lvert \mathrm{v/T} \rvert), & v \le 1/2T, \\ 2(1 - \lvert v/T \rvert)^3, & v \ge 1/2T, \end{cases}$	$v \le T$
Bohman	$[1 - v/T]\cos \pi v/T + 1/\pi \sin \pi v/T$	$v \le T$
Poisson	$e^{-\alpha v/T}$	$e = 2.71828$, $\alpha = 2$, $v \le T$
Hanning–Poisson	$0.5 * [1 + \cos(\pi v/T)\exp(-\alpha v/T)]$	$\alpha = 2$, $v \le T$
Cauchy	$1/(1 + [\alpha * v/T]^2)$	$v \le T$
Bartlett–Hanning	$0.62 - 0.48\lvert v/T - 0.5 \rvert - 0.38 \cos(\pi v/T)$	$v \le T$

Table 2: Values of the parameter of MA(1).

	Different values of parameter β		
The value of β	−1.5, −1, −0.5	0, 0.5	1, 1.5, 2

Table 3: Values of the parameters of MA(2).

β_1, β_2	Different values of parameters							
β_1	1	−1	0.5	−0.5	0.5	0.5	1	−1.5
β_2	−1	1	0.5	−0.5	−0.5	1	−0.5	1

Table 4: MSE values of MA(1) spectrum estimation with size ($n = 10$).

Windows	Different values of parameter β							
	$\beta = -1.5$	$\beta = -1$	$\beta = -0.5$	$\beta = 0$	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$	$\beta = 2$
Rectangular	6.317333361	6.316064139	6.319364383	6.332573978	6.345797365	6.349105366	6.347832956	6.345797365
Triangular	6.311369594	6.310767548	6.31289241	6.321968603	6.329353747	6.331318921	6.330993772	6.330034865
Hanning	6.32083044	6.318996447	6.323517904	6.341436557	6.360466439	6.36544573	6.363049374	6.360179286
Hamming	6.314117829	6.313367766	6.315545492	6.324429262	6.332322646	6.33409771	6.333856336	6.332585922
Blackman	6.302260759	6.303272395	6.303558226	6.308619677	6.312332078	6.31249273	6.312505741	6.312483751
Blackman–Harris	6.318938009	6.317435825	6.32159714	6.338771007	6.356981779	6.361658436	6.35932391	6.35667816
Flat top	6.317741146	6.316691593	6.320307479	6.336122551	6.352782751	6.356984761	6.354866599	6.35249844
Exponential	6.320794905	6.318896804	6.323176328	6.339687335	6.358287302	6.362522503	6.360466436	6.357282728
Gaussian	6.312937309	6.312397469	6.314383952	6.322872665	6.329995016	6.331686557	6.331498156	6.33048736
Welch(Riesz)	6.312989401	6.312497347	6.31450491	6.323271286	6.329490736	6.331504406	6.331276246	6.330651116
Cosine	6.317333361	6.316064139	6.319364383	6.332573978	6.345797365	6.349105366	6.347832956	6.345797365
Parzen	6.321912789	6.319469585	6.323215615	6.336148995	6.354453504	6.356559699	6.355270565	6.350618256
Bohman	6.311609354	6.31128131	6.313085632	6.321097799	6.327607439	6.329162515	6.329065939	6.328263199
Poisson	6.310203987	6.309449206	6.311849782	6.321901239	6.331036428	6.333238834	6.332666646	6.331360452
Hanning–Poisson	6.308901111	6.308291579	6.310471547	6.319777571	6.32816172	6.330066936	6.329647765	6.328471139
Cauchy	6.316534346	6.315404723	6.318452614	6.330768767	6.342655521	6.345688248	6.344625789	6.342838347
Bartlett–Hanning	6.319845036	6.318180978	6.322379624	6.339105847	6.356579046	6.361152556	6.35903439	6.356414425
Min-MSE	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman

Table 5: MSE values of MA(1) spectrum estimation with size ($n = 100$).

Windows	Different values of parameter β							
	$\beta = -1.5$	$\beta = -1$	$\beta = -0.5$	$\beta = 0$	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$	$\beta = 2$
Rectangular	6.317333361	6.316064139	6.319364383	6.332573978	6.345797365	6.349105366	6.347832956	6.345797365
Triangular	6.316005269	6.315502259	6.316731003	6.321849529	6.326985356	6.327757368	6.326900064	6.32579965
Hanning	6.31806519	6.316167753	6.32110453	6.341162052	6.360347991	6.365372888	6.364252615	6.360647922
Hamming	6.316660153	6.315968816	6.317763862	6.324682905	6.33243793	6.334173767	6.332762578	6.332165359
Blackman	6.315362421	6.315718152	6.314932087	6.310802524	6.308365932	6.307273013	6.306340167	6.306147794
Blackman–Harris	6.317848489	6.316128283	6.320633953	6.33882696	6.356290341	6.360849096	6.359792368	6.356255191
Flat top	6.317638588	6.316102605	6.320181441	6.336478466	6.352199786	6.356363117	6.355264552	6.351894477
Exponential	6.318221266	6.31637155	6.321318053	6.340398962	6.360502172	6.36618394	6.364286426	6.36219272
Gaussian	6.31645154	6.315884494	6.317326823	6.323030172	6.329212062	6.330404138	6.329199826	6.328365722
Welch(Riesz)	6.316230368	6.315713366	6.316921366	6.32245267	6.327205341	6.327606375	6.326989738	6.325262877
Cosine	6.317333361	6.316064139	6.319364383	6.332573978	6.345797365	6.349105366	6.347832956	6.345797365
Parzen	6.319052677	6.317233201	6.322761625	6.339863334	6.366401923	6.374981893	6.370220836	6.373119197
Bohman	6.316253855	6.315808587	6.316897146	6.321355062	6.325958709	6.326704444	6.325615407	6.324696778
Poisson	6.315823722	6.315261275	6.31666959	6.322265743	6.328176627	6.329372137	6.328423389	6.327531799
Hanning–Poisson	6.315661537	6.31523451	6.316291145	6.320398054	6.324977482	6.325848083	6.324882348	6.324291094
Cauchy	6.317136032	6.31600932	6.318921608	6.330651747	6.342351301	6.345144417	6.343959584	6.342050158
Bartlett–Hanning	6.317852285	6.316125822	6.320611356	6.338842477	6.356317622	6.36082472	6.359718723	6.356437047
Min-MSE	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman

Table 6: MSE values of MA(1) spectrum estimation with size ($n = 500$).

Windows	Different values of parameter β							
	$\beta = -1.5$	$\beta = -1$	$\beta = -0.5$	$\beta = 0$	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$	$\beta = 2$
Rectangular	6.317333361	6.316064139	6.319364383	6.332573978	6.345797365	6.349105366	6.347832956	6.345797365
Triangular	6.316377185	6.315952991	6.317159619	6.321833153	6.326693206	6.326801285	6.326774568	6.325874867
Hanning	6.317998059	6.316081423	6.32096733	6.340993146	6.360522531	6.365614361	6.363381808	6.361253929
Hamming	6.3167219	6.316048238	6.317890033	6.324838119	6.332280487	6.333954984	6.333562764	6.331611969
Blackman	6.315612827	6.316016966	6.31507097	6.311104415	6.307897643	6.306975267	6.306457259	6.30597069
Blackman–Harris	6.317810644	6.316076264	6.32049217	6.338673171	6.356415512	6.36107786	6.358807307	6.356916986
Flat top	6.31761655	6.31607157	6.320014315	6.336284248	6.352314562	6.35658088	6.354268933	6.352515183
Exponential	6.318063049	6.316124968	6.321060167	6.340735216	6.360493164	6.367239265	6.364810109	6.361301206
Gaussian	6.316557743	6.316032352	6.317501811	6.323066838	6.329037777	6.329918169	6.329657358	6.328171358
Welch(Riesz)	6.316420872	6.315995647	6.31722662	6.322204748	6.32712537	6.3263227	6.326546164	6.326204822
Cosine	6.317333361	6.316064139	6.319364383	6.332573978	6.345797365	6.349105366	6.347832956	6.345797365
Parzen	6.318577978	6.316312267	6.321970931	6.342973131	6.365195001	6.378587971	6.3755627	6.367369776
Bohman	6.316393708	6.316018758	6.317117179	6.321211233	6.32579264	6.326101485	6.325881926	6.324644039
Poisson	6.316384516	6.315902859	6.317233642	6.322333304	6.327747006	6.328624662	6.328294354	6.327060016
Hanning–Poisson	6.316239399	6.31590009	6.316876052	6.320464524	6.324529924	6.32515849	6.324897286	6.323651196
Cauchy	6.317167665	6.316053652	6.318972459	6.330640244	6.342321031	6.344935342	6.343939081	6.342161726
Bartlett–Hanning	6.317810152	6.316073852	6.320517205	6.338685687	6.356453679	6.360945471	6.358981262	6.356983283
Min-MSE	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman

Table 7: MSE values of MA(1) spectrum estimation with size ($n = 1000$).

Windows	Different values of parameter β							
	$\beta = -1.5$	$\beta = -1$	$\beta = -0.5$	$\beta = 0$	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$	$\beta = 2$
Rectangular	6.317333361	6.316064139	6.319364383	6.332573978	6.345797365	6.349105366	6.347832956	6.345797365
Triangular	6.316450439	6.316010799	6.317116444	6.320669904	6.326647222	6.327854445	6.326523968	6.326018604
Hanning	6.317952234	6.316071633	6.320986333	6.341457537	6.360453565	6.364877187	6.364266231	6.360568627
Hamming	6.316764056	6.316057244	6.317872553	6.324411712	6.332343093	6.334629512	6.332750509	6.332238824
Blackman	6.315724711	6.316043998	6.314941086	6.309316698	6.307721597	6.307680769	6.307010748	6.307337395
Blackman–Harris	6.317776777	6.316069431	6.320493511	6.338898231	6.356290763	6.360347095	6.359908255	6.356366975
Flat top	6.317596093	6.316067342	6.320008079	6.33625915	6.352063455	6.35591282	6.355375232	6.352147618
Exponential	6.318033755	6.316091775	6.321147607	6.342219386	6.360665727	6.366052121	6.364741973	6.36159135
Gaussian	6.316604308	6.316050059	6.317449541	6.322230584	6.329024221	6.330798748	6.329079574	6.328691584
Welch(Riesz)	6.316455978	6.316033153	6.317132955	6.320433157	6.32697944	6.327870934	6.326603021	6.325936882
Cosine	6.317333361	6.316064139	6.319364383	6.332573978	6.345797365	6.349105366	6.347832956	6.345797365
Parzen	6.318549726	6.316178874	6.322383809	6.347943676	6.366544801	6.376414843	6.37358532	6.370508696
Bohman	6.316450929	6.316043932	6.317023713	6.32014857	6.325690506	6.327020438	6.325281865	6.325242589
Poisson	6.316489926	6.315984895	6.317228693	6.321646101	6.327736255	6.329177908	6.327932997	6.327379837
Hanning–Poisson	6.31635782	6.315983781	6.316859	6.319750927	6.324514567	6.325702868	6.324375171	6.324183754
Cauchy	6.317174388	6.316059423	6.318959102	6.330351343	6.342309448	6.345216009	6.343919346	6.342150534
Bartlett–Hanning	6.317775594	6.316068281	6.320526597	6.338945371	6.356382878	6.360441317	6.359704803	6.356411672
Min-MSE	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman

and the spectral density function $f(\omega_i)$ of MA(2) is given by

$$f(\omega) = \frac{\left(1 + \beta_1^2 + \beta_2^2\right) + 2\left(\beta_1 + \beta_1\beta_2\right)\cos\omega + 2\beta_2\cos\omega}{2\pi\left(1 + \beta_1^2 + \beta_2^2\right)}, \quad -\pi \le \omega \le \pi. \tag{28}$$

(7) The criterion used to evaluate the windows performance was the mean square error (MSE) calculated with the following formula:

$$\text{MSE} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{L}\left(f(\omega_i) - \widehat{f}_j(\omega_i)\right)^2}{kL}, \tag{29}$$

where k and L were defined in (3) and (5), respectively, and $\widehat{f}_j(\omega_i)$ is the consistent estimate of the SDF formula in (26).

Table 8: MSE values of MA(1) spectrum estimation with size ($n = 10000$).

Windows	Different values of parameter β							
	$\beta = -1.5$	$\beta = -1$	$\beta = -0.5$	$\beta = 0$	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$	$\beta = 2$
Rectangular	6.317333361	6.316064139	6.319364383	6.332573978	6.345797365	6.349105366	6.347832956	6.345797365
Triangular	6.316479573	6.31605864	6.317209379	6.321468334	6.325557514	6.326554021	6.325547978	6.325765604
Hanning	6.317973878	6.316064925	6.321014701	6.341128648	6.361206169	6.365776987	6.363952022	6.360915895
Hamming	6.316744145	6.316063416	6.31784647	6.324713492	6.331653522	6.333806476	6.333038184	6.331920613
Blackman	6.315702546	6.316062088	6.315077573	6.310816877	6.305092725	6.305663759	6.305800605	6.308807089
Blackman–Harris	6.317798694	6.316064702	6.320549488	6.338791963	6.356691063	6.360995072	6.359337896	6.357012486
Flat top	6.317620052	6.316064479	6.320083994	6.336445166	6.352033339	6.356039601	6.354631114	6.35296128
Exponential	6.318011853	6.316067131	6.321018293	6.341142875	6.361633925	6.367540154	6.366462249	6.361999646
Gaussian	6.316593257	6.316062641	6.317479185	6.322877081	6.328090783	6.329596079	6.328766781	6.32845892
Welch(Riesz)	6.316481157	6.31606079	6.317261818	6.321657877	6.325736418	6.326149294	6.324618196	6.32548475
Cosine	6.317333361	6.316064139	6.319364383	6.332573978	6.345797365	6.349105366	6.347832956	6.345797365
Parzen	6.318413973	6.316076684	6.321580321	6.34373885	6.368067039	6.381168828	6.381470473	6.371855429
Bohman	6.316438799	6.316061985	6.31710822	6.321099464	6.32463446	6.32540767	6.324748977	6.32506562
Poisson	6.31653195	6.316056095	6.317320073	6.322133862	6.326857319	6.328248328	6.327603009	6.32730352
Hanning–Poisson	6.316391338	6.31605599	6.316955154	6.320308685	6.323512583	6.32461211	6.324213674	6.324156156
Cauchy	6.317177571	6.316063631	6.31897445	6.33054192	6.342080247	6.344927928	6.343628113	6.342071815
Bartlett–Hanning	6.317794749	6.316064567	6.320558417	6.338764354	6.356901826	6.361046061	6.359323892	6.356678845
Min-MSE	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman

Table 9: MSE values of MA(2) spectrum estimation with size ($n = 10$).

Windows	Different values of parameters β_1, β_2							
	$\beta_1 = 1$, $\beta_2 = -1$	$\beta_1 = -1$, $\beta_2 = 1$	$\beta_1 = 0.5$, $\beta_2 = 0.5$	$\beta_1 = -0.5$, $\beta_2 = -0.5$	$\beta_1 = 0.5$, $\beta_2 = -0.5$	$\beta_1 = 0.5$, $\beta_2 = 1$	$\beta_1 = 1$, $\beta_2 = -0.5$	$\beta_1 = -1.5$, $\beta_2 = 1$
Rectangular	6.321565024	6.321565024	6.360138264	6.316064139	6.316064139	6.361978011	6.332573978	6.317034709
Triangular	6.312005203	6.316707374	6.341600802	6.308265322	6.308265322	6.343799682	6.319994439	6.313156769
Hanning	6.329331965	6.323962003	6.378387396	6.321932771	6.321932771	6.379542556	6.343610193	6.318259519
Hamming	6.314426416	6.319360693	6.343378579	6.310669467	6.310669467	6.345847113	6.322433753	6.31590812
Blackman	6.29540239	6.314673225	6.326490025	6.293526586	6.293526586	6.329993335	6.302467223	6.309973629
Blackman–Harris	6.326158188	6.323331333	6.375330581	6.319121344	6.319121344	6.376639217	6.340221321	6.317448053
Flat top	6.323428374	6.323339845	6.370649079	6.317007048	6.317007048	6.371981296	6.336830045	6.31758825
Exponential	6.327426077	6.324458741	6.373850374	6.320445609	6.320445609	6.375765692	6.341101042	6.319028806
Gaussian	6.31279587	6.318682021	6.341512875	6.309263029	6.309263029	6.343876486	6.320566376	6.315181164
Welch(Riesz)	6.313917191	6.317934726	6.341881423	6.310367315	6.310367315	6.343773555	6.321380659	6.314643892
Cosine	6.321565024	6.321565024	6.360138264	6.316064139	6.316064139	6.361978011	6.332573978	6.317034709
Parzen	6.323495637	6.326494597	6.362753447	6.317361895	6.317361895	6.36755149	6.335711083	6.322006755
Bohman	6.310834358	6.31799328	6.33935597	6.307546938	6.307546938	6.341615062	6.31841308	6.314405216
Poisson	6.311294317	6.31569628	6.343823934	6.307123183	6.307123183	6.346107879	6.320193056	6.311748163
Hanning–Poisson	6.309034828	6.314967826	6.34063026	6.30517467	6.30517467	6.343068984	6.317560801	6.311049725
Cauchy	6.320072257	6.320909745	6.356521406	6.314953792	6.314953792	6.358409071	6.330385723	6.316629309
Bartlett–Hanning	6.327290745	6.323273715	6.373709134	6.320385895	6.320385895	6.374995983	6.340729357	6.317872708
Min-MSE	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman

7. Results

(1) For the first order of moving average process MA (1), when we study the different values of parameter β as shown in Table 2 and size n of the process, we get the results given in Tables 4–8.

(2) For the second order of moving average process MA (2), when we study the different values of parameters β_1, β_2 and size of process n, we get the results given in Tables 9–13.

8. Conclusion

(1) In MA (1) with the different parameters and series sizes, the best lag window which gives the minimum mean square error (MSE) between the SDF $f(\omega)$ and the consistent estimate of SDF $\widehat{f}(\omega)$, and the results in Tables 4–8 can be summarized in Table 14.

(2) The results in Tables 9–13 shows that, in MA (2), with the following different parameters and series sizes, the best lag window which gives the minimum mean

Table 10: MSE values of MA(2) spectrum estimation with size ($n = 100$).

Windows	Different values of parameters β_1, β_2							
	$\beta_1 = 1$, $\beta_2 = -1$	$\beta_1 = -1$, $\beta_2 = 1$	$\beta_1 = 0.5$, $\beta_2 = 0.5$	$\beta_1 = -0.5$, $\beta_2 = -0.5$	$\beta_1 = 0.5$, $\beta_2 = -0.5$	$\beta_1 = 0.5$, $\beta_2 = 1$	$\beta_1 = 1$, $\beta_2 = -0.5$	$\beta_1 = -1.5$, $\beta_2 = 1$
Rectangular	6.321565024	6.321565024	6.360138264	6.316064139	6.327068304	6.361978011	6.332573978	6.317034709
Triangular	6.317117423	6.31777068	6.333128008	6.315160112	6.31939826	6.333007724	6.321457949	6.316118821
Hanning	6.324557448	6.324401497	6.382064674	6.316268444	6.332689354	6.385154783	6.340913819	6.317578586
Hamming	6.3188133	6.318956624	6.34002707	6.315876185	6.32190157	6.34072313	6.324910813	6.316534387
Blackman	6.313845839	6.314385509	6.304330089	6.315309193	6.312027333	6.301064093	6.310514309	6.315641361
Blackman–Harris	6.323731576	6.323627073	6.376027133	6.316180189	6.331031122	6.378416063	6.338483312	6.317427316
Flat top	6.322939163	6.322816414	6.369886183	6.316129535	6.32939882	6.371527219	6.336027589	6.317269995
Exponential	6.324861824	6.324570285	6.380687038	6.316526045	6.33277471	6.384549625	6.341098715	6.317662753
Gaussian	6.318063427	6.318296571	6.335634552	6.315744141	6.320580289	6.335698617	6.323003143	6.316391836
Welch(Riesz)	6.317425293	6.317763848	6.333879433	6.315526842	6.319839881	6.333353187	6.321746436	6.316253592
Cosine	6.321565024	6.321565024	6.360138264	6.316064139	6.327068304	6.361978011	6.332573978	6.317034709
Parzen	6.327015561	6.326095813	6.383062251	6.31761612	6.335030668	6.39084977	6.344262279	6.318203885
Bohman	6.317318887	6.317663081	6.33110916	6.315617866	6.319220643	6.33055528	6.321152307	6.316253023
Poisson	6.317049932	6.318023383	6.334833716	6.314767193	6.319506318	6.335092961	6.32198115	6.316019062
Hanning–Poisson	6.316388107	6.317432572	6.330017371	6.314701346	6.318221449	6.329926155	6.320154072	6.315904302
Cauchy	6.320811057	6.320866313	6.355186428	6.315977384	6.325731496	6.356628158	6.330567105	6.316894163
Bartlett–Hanning	6.323712379	6.323606229	6.376095466	6.316194872	6.331127061	6.378773405	6.338590688	6.317423083
Min-MSE	Blackman	Blackman	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman

Table 11: MSE values of MA(2) spectrum estimation with size ($n = 500$).

Windows	Different values of parameters β_1, β_2							
	$\beta_1 = 1$, $\beta_2 = -1$	$\beta_1 = -1$, $\beta_2 = 1$	$\beta_1 = 0.5$, $\beta_2 = 0.5$	$\beta_1 = -0.5$, $\beta_2 = -0.5$	$\beta_1 = 0.5$, $\beta_2 = -0.5$	$\beta_1 = 0.5$, $\beta_2 = 1$	$\beta_1 = 1$, $\beta_2 = -0.5$	$\beta_1 = -1.5$, $\beta_2 = 1$
Rectangular	6.321565024	6.321565024	6.360138264	6.316064139	6.327068304	6.361978011	6.332573978	6.317034709
Triangular	6.317647266	6.3179566	6.331517324	6.31588179	6.319328483	6.332050539	6.321301252	6.316365871
Hanning	6.324250822	6.324265461	6.38206513	6.316091945	6.332703845	6.384872656	6.340870645	6.317506423
Hamming	6.31909505	6.319081614	6.340031293	6.316038557	6.321887866	6.340985719	6.324950371	6.316600764
Blackman	6.31455688	6.314500879	6.302256466	6.315982718	6.312578132	6.301911887	6.310992796	6.315763958
Blackman–Harris	6.32348736	6.323494163	6.375675873	6.316082845	6.331131988	6.378262669	6.338526006	6.317365882
Flat top	6.322718912	6.322716438	6.369293183	6.316075253	6.329517997	6.371648059	6.336213328	6.317227254
Exponential	6.324840245	6.324407841	6.382496376	6.316163699	6.333349526	6.385428999	6.341664876	6.317538431
Gaussian	6.31833174	6.31844159	6.334939146	6.316012361	6.320477191	6.335768283	6.322893236	6.316483275
Welch(Riesz)	6.317505476	6.31797286	6.331959995	6.315952597	6.319175721	6.332387703	6.321159104	6.316401338
Cosine	6.321565024	6.321565024	6.360138264	6.316064139	6.327068304	6.361978011	6.332573978	6.317034709
Parzen	6.328047906	6.325832428	6.392371593	6.316475625	6.337532096	6.395529955	6.347464105	6.317834331
Bohman	6.317608082	6.317812419	6.329753704	6.315989451	6.319144591	6.330648137	6.320906056	6.316364675
Poisson	6.317936785	6.318181261	6.3331846	6.3157986	6.319916772	6.333917098	6.322115039	6.316370761
Hanning–Poisson	6.317367925	6.317591571	6.328278066	6.31579334	6.318733654	6.328904817	6.320328848	6.316263887
Cauchy	6.320839839	6.320904802	6.354921707	6.316047154	6.325630499	6.356491956	6.330489775	6.31691898
Bartlett–Hanning	6.323469349	6.32350886	6.375966284	6.316079627	6.331101469	6.378496766	6.338527326	6.317373725
Min-MSE	Blackman	Blackman	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman

square error (MSE) between the SDF $f(\omega)$ and the consistent estimate of SDF $\widehat{f}(\omega)$ is shown in Table 15.

(3) In MA(1) with series sizes $n < 40$ and any value of parameter β, the best lag window which gives the minimum mean square error (MSE) between the SDF $f(\omega)$ and the consistent estimate of SDF $\widehat{f}(\omega)$ is the Blackman window, as shown in Table 9.

(4) In MA(1) with series sizes $n \geq 40$ and parameter $\beta = -1$ or belongs to neighborhood -1 with radius 0.3, the best lag window which gives the minimum mean square error (MSE) between the SDF $f(\omega)$ and the consistent estimate of SDF $\widehat{f}(\omega)$ is the Hanning–Poisson window, as shown in Table 9.

(5) Blackman window is the best window to estimate the SDF for white noise ϵ_t, where $\mathrm{MA}(1) = \epsilon_t$ when $\beta = 0$, as shown in Table 9.

(6) In MA(2) with series sizes $n < 40$ and any values of parameters β_1, β_2, the best lag window which gives the minimum mean square error (MSE) between the

Table 12: MSE values of MA(2) spectrum estimation with size ($n = 1000$).

Windows	Different values of parameters β_1, β_2							
	$\beta_1 = 1$ $\beta_2 = -1$	$\beta_1 = -1$ $\beta_2 = 1$	$\beta_1 = 0.5$ $\beta_2 = 0.5$	$\beta_1 = -0.5$ $\beta_2 = -0.5$	$\beta_1 = 0.5$ $\beta_2 = -0.5$	$\beta_1 = 0.5$ $\beta_2 = 1$	$\beta_1 = 1$ $\beta_2 = -0.5$	$\beta_1 = -1.5$ $\beta_2 = 1$
Rectangular	6.321565024	6.321565024	6.360138264	6.316064139	6.327068304	6.361978011	6.332573978	6.317034709
Triangular	6.317820977	6.317924846	6.330318905	6.315974265	6.319357377	6.331935269	6.322220406	6.31635336
Hanning	6.324327739	6.324326234	6.382575142	6.316078585	6.332692347	6.385185403	6.340355037	6.317535755
Hamming	6.319024374	6.319025792	6.339567906	6.316050848	6.321898419	6.340702612	6.325423809	6.316573783
Blackman	6.314648712	6.314272258	6.301303387	6.31602455	6.312563425	6.303054634	6.311731336	6.315768701
Blackman–Harris	6.323590394	6.323528609	6.376110223	6.316074223	6.331115503	6.378782602	6.33805233	6.317400392
Flat top	6.322829702	6.32274256	6.369677188	6.316070147	6.329499258	6.372115746	6.335781648	6.31726237
Exponential	6.324558834	6.324377777	6.38375854	6.316109209	6.333391536	6.385552536	6.340534867	6.317564817
Gaussian	6.318372808	6.318392146	6.334241625	6.316038988	6.320469553	6.335580791	6.323583399	6.316457184
Welch(Riesz)	6.317801343	6.318001235	6.330356383	6.316012613	6.319149002	6.33221923	6.322532917	6.316363813
Cosine	6.321565024	6.321565024	6.360138264	6.316064139	6.327068304	6.361978011	6.332573978	6.317034709
Parzen	6.32641214	6.325521793	6.39535827	6.316237555	6.337953601	6.395432172	6.344184482	6.317829935
Bohman	6.31773952	6.317754417	6.329118219	6.316029039	6.319103227	6.330499792	6.321678537	6.316343762
Poisson	6.318063244	6.31811306	6.332348143	6.315931055	6.319983864	6.33388695	6.322647099	6.316386733
Hanning–Poisson	6.317481915	6.317495665	6.32750183	6.315928748	6.318794459	6.328913444	6.320862533	6.316280627
Cauchy	6.320877617	6.320906541	6.354635902	6.316056093	6.325630084	6.356453375	6.330722247	6.316911431
Bartlett–Hanning	6.323551102	6.323559592	6.376281513	6.316072595	6.331091791	6.378745441	6.338191821	6.317395507
Min-MSE	Blackman	Blackman	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman

Table 13: MSE values of MA(2) spectrum estimation with size ($n = 10000$).

Windows	Different values of parameters β_1, β_2							
	$\beta_1 = 1$ $\beta_2 = -1$	$\beta_1 = -1$ $\beta_2 = 1$	$\beta_1 = 0.5$ $\beta_2 = 0.5$	$\beta_1 = -0.5$ $\beta_2 = -0.5$	$\beta_1 = 0.5$ $\beta_2 = -0.5$	$\beta_1 = 0.5$ $\beta_2 = 1$	$\beta_1 = 1$ $\beta_2 = -0.5$	$\beta_1 = -1.5$ $\beta_2 = 1$
Rectangular	6.321565024	6.321565024	6.360138264	6.316064139	6.327068304	6.361978011	6.332573978	6.317034709
Triangular	6.317855079	6.318048015	6.329728119	6.31605515	6.319269444	6.331734076	6.321390891	6.31638888
Hanning	6.324333631	6.324210633	6.383194947	6.316065401	6.33274672	6.38583431	6.340955416	6.317515983
Hamming	6.319018997	6.319132064	6.338999008	6.316062978	6.321848752	6.34011287	6.324872728	6.316591972
Blackman	6.314602201	6.314606383	6.302553934	6.316060833	6.31262263	6.304153737	6.311101155	6.315776292
Blackman–Harris	6.323590977	6.323453625	6.376995885	6.316065041	6.331187991	6.379739984	6.338630758	6.317378991
Flat top	6.32283238	6.322718592	6.370476692	6.31606468	6.329578574	6.373369557	6.33625461	6.317242264
Exponential	6.324542244	6.324301408	6.38464717	6.316068875	6.333565517	6.385292434	6.341509798	6.317542611
Gaussian	6.318369328	6.31851271	6.333705322	6.316061752	6.320389388	6.335275274	6.322893501	6.316477891
Welch(Riesz)	6.317807718	6.318110539	6.329288881	6.316058835	6.318937909	6.332372703	6.321295763	6.316390903
Cosine	6.321565024	6.321565024	6.360138264	6.316064139	6.327068304	6.361978011	6.332573978	6.317034709
Parzen	6.326203496	6.325611467	6.39781678	6.316083922	6.338511857	6.391717309	6.346256854	6.317781749
Bohman	6.317746178	6.317883746	6.328662769	6.316060713	6.319016136	6.33040712	6.320991031	6.31636812
Poisson	6.318134832	6.318228945	6.332174728	6.316050845	6.320001297	6.33355514	6.322214269	6.316430102
Hanning–Poisson	6.317554127	6.317635162	6.327408815	6.316050674	6.318825202	6.328555408	6.320436569	6.316326227
Cauchy	6.320876447	6.320929331	6.354453735	6.316063333	6.325595081	6.356433889	6.33050151	6.316916254
Bartlett–Hanning	6.323555462	6.323474752	6.376724998	6.316064831	6.331122539	6.379295097	6.338600735	6.31738112
Min-MSE	Blackman	Blackman	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman

Table 14: Best lag window with minimum MSE between $f(\omega)$ and $\hat{f}(\omega)$. of MA(1).

Sample size	Parameters							
	$\beta = -1.5$	$\beta = -1$	$\beta = -0.5$	$\beta = 0$	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$	$\beta = 2$
10	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman
100	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman
500	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman
1000	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman
10000	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman

Table 15: Best lag window with minimum MSE between $f(\omega)$ and $\widehat{f}(\omega)$of.MA (1)

Series size n	Parameters							
	$\beta_1 = 1$ $\beta_2 = -1$	$\beta_1 = -1$ $\beta_2 = 1$	$\beta_1 = 0.5$ $\beta_2 = 0.5$	$\beta_1 = -0.5$ $\beta_2 = -0.5$	$\beta_1 = 0.5$ $\beta_2 = -0.5$	$\beta_1 = 0.5$ $\beta_2 = 1$	$\beta_1 = 1$ $\beta_2 = -0.5$	$\beta_1 = -1.5$ $\beta_2 = 1$
10	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman	Blackman
100	Blackman	Blackman	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman
500	Blackman	Blackman	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman
1000	Blackman	Blackman	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman
10000	Blackman	Blackman	Blackman	Hanning–Poisson	Blackman	Blackman	Blackman	Blackman

SDF $f(\omega)$ and the consistent estimate of SDF $\widehat{f}(\omega)$ is the Blackman window, as shown in Table 15.

(7) In MA (2) with series sizes $n \geq 40$ and parameters $\beta_1 = \beta_2 = -0.5$, the best lag window is the Hanning–Poisson window, or the Blackman window, as shown in Table 15.

Data Availability

The data included in the article were calculated using the MATLAB software and the method of calculation of the data is given in Section 6.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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