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Research Article

Common Fixed Point Results for a Pair of Multivalued Mappings in Complex-Valued *b*-Metric Spaces

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Several fixed point results for the existence of common fixed points of multivalued contractive mappings have been established in complex-valued metric space. In this paper, we study the existence of common fixed points for a pair of multivalued contractive mappings satisfying some rational inequalities in the framework of complex-valued *b*-metric spaces. The contractive condition used in this paper generalizes many contractive conditions used by other authors in the literature. Employing our results, we check the existence solution to the Riemann-Liouville equation.

1. Introduction

Fixed point theory is a well-researched area of mathematics; in particular, results concerning fixed points of contractive type mappings are found useful for determining the existence and uniqueness of solutions of various mathematical models. In this field, Banach [1] introduced the notion of contraction mapping in a complete metric space and gave a fixed point result for finding the fixed point of the contraction mapping. Later in 1969, Kannan [2] gave another contractive type mapping that demonstrated the fixed point theorem. However, in the Kannan contraction result, the continuity property required for the result of Banach was shown to be not necessary. Other authors have also studied several contraction mappings with differing properties (see, for instance, Chaterjea [3]). Since then, the theory of fixed points has been developed regarding results on finding fixed points of self and nonself mappings which are single-valued in a metric space.

Moreover, the study of fixed points for multivalued type contractive mappings was pioneered by Nadler [4] and further studied by Markin [5]. Since then, many researchers have generalized and extended various fixed point results from single-valued contractive mappings to multivalued contractive type mappings. For more literature concerning such extensions and generalizations, see, for instance, [6–12] and other references therein.

On the other hand, the axiomatic development of metric spaces was started by M. Fréchet, a French mathematician in the year 1906. The importance of metric spaces in the natural growth of functional analysis is huge. Several authors have drawn inspirations from the impact of this natural idea to mathematics and functional analysis in particular. Therefore, there have been several generalizations of this notion in the forms of rectangular metric spaces, semimetric spaces, quasimetric spaces, quasisemimetric spaces, *D*-metric spaces, cone metric spaces, and more recently the graphical rectangular *b*-metric spaces. We refer the reader to the following references for surveys on these generalizations [1, 13–18].

One of these generalizations in the last decade is that of Azam et al. [19, 20]. They introduced the notion of complex-valued metric spaces, and some fixed point theorems for mappings with some rational inequalities were established. The central and core idea is to define rational expressions which are not well posed in the cone metric spaces, and thus, such results of analysis cannot be extended to cone metric spaces but to complex-valued metric spaces. Complex-valued metric spaces find interesting applications in many branches of mathematics such as algebraic geometry and number theory as well as in field of studies such as physics, thermodynamics, and electrical engineering.

Furthermore, the idea of b-metric was introduced in 1989 by Bakhtin [21]. Based on this presentation, Rao et al. [22] introduced the concept of fixed point theorems on complex-valued b-metric spaces which is a natural generalization of the complex-valued metric spaces. The relationship between the complex-valued b-metric and cone metric space is well known. Inspired by [22], many authors have proven the existence of fixed points of different mappings satisfying rational inequalities in the framework of complex-valued b-metric spaces (see [19, 23] for more details).

In 2016, Singh et al. [24] introduced a contractive type mapping satisfying some rational inequalities. They obtained the existence of common fixed point for a pair of single-valued mappings satisfying more general contraction conditions in the framework of complex-valued metric spaces. Since then, fixed point theory has been the center of extensive research for many authors (see, e.g., [25, 26]).

Our motivation in this work is in twofolds: we extend the results of [19, 24, 27] from complex-valued metric spaces to complex-valued b-metric spaces; we also generalize the contraction mappings used therein by introducing a multivalued contraction mapping satisfying a general condition in complex-valued b-metric spaces. Our result also improves and strengthens the results of [20, 22, 28, 29] and many other related results in the literature.

2. Preliminaries

In this section, we give some basic definitions and results which will be useful in establishing our main result. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Also, we define partial order $\prec$ and $\preceq$ on $\mathbb{C}$ as follows:

(i) $z_1 \prec z_2$ if and only if $\mathrm{Re}\,(z_1) < \mathrm{Re}\,(z_2)$ and $\mathrm{Im}\,(z_1) < \mathrm{Im}\,(z_2)$

(ii) $z_1 \preceq z_2$ if and only if $\mathrm{Re}\,(z_1) \leq \mathrm{Re}\,(z_2)$ and $\mathrm{Im}\,(z_1) \leq \mathrm{Im}\,(z_2)$

Definition 1. Let X be a nonempty set and $\tau \geq 1$ be a real number. A function $d_c : X \times X \rightarrow \mathbb{C}$ is called complex-valued b-metric, if for all $x, y, z \in X$, the following conditions hold:

(i) $0 \preceq d_c(x, y)$ and $d_c(x, y) = 0$, if and only if $x = y$

(ii) $d_c(x, y) = d_c(y, x)$

(iii) $d(x, y) \preceq \tau(d_c(x, z) + d_c(z, y))$.

Then, a set X satisfying such metric d_c written in pair as (X, d_c) is called a complex-valued b-metric space.

Example 2 (see [23]). Let $X = [0, 1]$, define a mapping $d_c : X \times X \rightarrow \mathbb{C}$ by $d_c = |x - y|^2 + i|x - y|^2$ for all $x, y \in X$. Then, (X, d_c) is a complex-valued b-metric space with $\tau = 2$.

Definition 3 (see [19]). *Let (X, d_c) be a complex-valued b-metric space, and then, a point $x \in X$ is*

(i) an interior point of a set $A \subseteq X$, if there exists $0 \prec r \in \mathbb{C}$ such that $N(x, r) = \{y \in X : d_c(x, y) \prec r\} \subseteq A$

(ii) the limit of a set $A \subseteq X$, if for every $0 \prec r \in \mathbb{C}$, $N(x, r) \cap (Dx) \neq \varnothing$

(iii) $A \subseteq X$ which is called an open set if every element A is an interior point A

Definition 4 (see [22]). *Let $\{x_n\}$ be a sequence in a complex-valued b-metric space (X, d_c) and $x \in X$, then*

(i) x is the limit point of $\{x_n\}$ if for every $c \in \mathbb{C}$ with $0 \prec c$, there exits $n_0 \in \mathbb{Q}$ such that $d_c(x_n, x) \prec c$ for all $n > n_0$, we write $\lim_{n\rightarrow\infty} x_n = x$

(ii) if for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $n_0 \in \mathbb{Q}$ such that $d_c(x_n, x_{n+m}) \prec c$ for all $n > n_0$ and $n, m \in \mathbb{Q}$, then $\{x_n\}$ is a cauchy sequence in (X, d_c)

(iii) (X, d_c) is complete if every cauchy sequence is convergent in (X, d_c)

Lemma 5 (see [28]). *Let (X, d_c) be a complex-valued b-metric space and $\{x_n\}$ be a sequence in (X, d_c). Then, $\{x_n\}$ converges to $x \in X$ if and only if $|d_c(x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$.*

Lemma 6 (see [28]). *Let (X, d_c) be a complex-valued b-metric space and $\{x_n\}$ be a sequence in (X, d_c). Then, $\{x_n\}$ is a Cauchy sequence if and only if $|d_c(x_n, x_{n+m})| \rightarrow 0$ as $n \rightarrow \infty$.*

We denote by $CB(X)$ the family of closed and bounded subsets of the set X.

Definition 7 (see [19]). *Let (X, d_c) be a complex-valued b-metric space and $\{x_n\}$ be a sequence in (X, d_c). Denote $s(u) = \{z \in \mathbb{C} : u \preceq z\}$ and*

$$s(x, B) = \bigcup_{b\in B} s(d(a, b)) = \bigcup_{b\in B} \{z \in \mathbb{C} : d(a, b) \preceq z\}, \quad (1)$$

for $a \in X$ and $B \in CB(X)$. For $A, B \in CB(X)$, we denote

$$s(A, B) = \left(\bigcap_{a\in A} s(a, B)\right) \cap \left(\bigcap_{b\in B} s(b, A)\right). \quad (2)$$

Remark 8. Let (X, d_c) be a complex-valued b-metric space with $\tau = 1$. If $\mathbb{C} = \mathbb{R}$, then (X, d_c) is a metric space. Moreover, for $A, B \in CB(X)$, $H(A, B) = \inf s(A, B)$ is the Hausdorff metric induced by d_c.

Definition 9 (see [22]). *Let (X, d_c) be a complex-valued b-metric space.*

(i) Let $T : X \to CB(X)$ be a multivalued mapping. For $x \in X$ and $A \in CB(X)$, define

$$W_x(A) = \{d_c(x, a) \colon a \in A\}. \quad (3)$$

Thus, for $x, y \in X$,

$$W_x(Ty) = \{d_c(x, u) \colon u \in Ty\}. \quad (4)$$

(ii) A mapping $F : X \to 2^{\mathbb{C}}$ is said to be bounded from below if for each $x \in X$ there exists $z_x \in \mathbb{C}$ such that $z_x \precsim v$ for all $v \in F_x$

(iii) For a multivalued mapping $B : X \to CB(X)$, we say that B has a lower bound property on X, d_c, if for any $x \in X$ the mapping $F_x : X \to 2^{\mathbb{C}}$ defined by $F_x(v) = W_x(F_v)$ is bounded below. This implies that for $x, v \in X$, there exists an element $l_x(Bv) \in \mathbb{C}$ such that $l_x(Bv) \precsim a$ for all $a \in W_x(Bv)$, where $l_x(Bv)$ is the lower bound of B associated with (x, v)

(iv) The multivalued mapping $B : X \to CB(X)$ is said to have the greatest lower bound property (g.l.b. property) on (X, d_c) if a greatest lower bound $W_x(Bv)$ exists in $\mathbb{C}$ for every $x, v \in X$. We denote by $d_c(x, Bv)$ the g.l.b. of $W_x(Bv)$. That is,

$$d(x, Bv) = \inf \{d(x, u) \colon u \in Bv\} \quad (5)$$

Definition 10 (see [23]). *Let (X, d_c) be a complex-valued b-metric space and $S, T : X \to CB(X)$ be multivalued mappings.*

(i) A point $x \in X$ is called a fixed point T if $x \in Tx$

(ii) A point $x \in X$ is called a common fixed point of S and T if $x \in Sx$ and $x \in Tx$

3. Main Result

In this section, we state and prove our main findings in the sequel.

Theorem 11. *Let (X, d_c) be a complete complex-valued b-metric space and let $S, T : X \to CB(X)$ be multivalued mappings with g.l.b. property. Let $\lambda, \mu, \gamma, \delta : X \times X \times X \to [0, 1)$ be mappings such that $\forall x, y \in X$,*

$$\lambda(x_{2n}, y, a) \le \lambda(x_0, y, a), \quad \lambda(x, y_{2n}, a) \le \lambda(x, y_0, a), \quad (6)$$

$$\mu(x_{2n}, y, a) \le \mu(x_0, y, a), \quad \mu(x, y_{2n}, a) \le \mu(x, y_0, a), \quad (7)$$

$$\gamma(x_{2n}, y, a) \le \gamma(x_0, y, a), \quad \gamma(x, y_{2n}, a) \le \gamma(x, y_0, a), \quad (8)$$

$$\delta(x_{2n}, y, a) \le \delta(x_0, y, a), \quad \delta(x, y_{2n}, a) \le \delta(x, y_0, a), \quad (9)$$

$$\begin{aligned} &\lambda(x, y, a) d_c(x, y) + \mu(x, y, a) \frac{d_c(x, Sx) d_c(y, Ty)}{1 + d_c(x, y)} \\ &+ \gamma(x, y, a) \frac{d_c(y, Sx) d_c(x, Ty)}{1 + d_c(x, y)} \\ &+ \delta(x, y, a) \left(\frac{d_c(x, Sx) d_c(x, Ty) + d_c(y, Ty) d_c(y, Sx)}{1 + d_c(x, Ty) + d_c(y, Sx)} \right) \\ &\cdot \in s(Sx, Ty), \end{aligned} \quad (10)$$

$$\tau\lambda(x, y, a) + \mu(x, y, a) + \gamma(x, y, a) + \delta(x, y, a) < 1. \quad (11)$$

Then, S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X, then $Tx_0 \neq \varnothing$. Pick $x_1 \in Tx_0$, set $x = x_0$ and $y = x_1$ in (6). Then, we get

$$\begin{aligned} &\lambda(x_0, x_1, a) d_c(x_0, x_1) + \mu(x_0, x_1, a) \frac{d_c(x_0, Sx_0) d_c(x_1, Tx_1)}{1 + d_c(x_0, x_1)} \\ &+ \gamma(x_0, x_1, a) \frac{d_c(x_1, Sx_0) d_c(x_0, Tx_1)}{1 + d_c(x_0, x_1)} \\ &+ \delta(x_0, x_1, a) \left(\frac{d_c(x_0, Sx_0) d_c(x_0, Tx_1) + d_c(x_1, Tx_1) d_c(x_1, Sx_0)}{1 + d_c(x_0, Tx_1) + d_c(x_1, Sx_0)} \right) \\ &\cdot \in s(Sx_0, Tx_1). \end{aligned} \quad (12)$$

This implies

$$\begin{aligned} &\lambda(x_0, x_1, a) d_c(x_0, x_1) + \mu(x_0, x_1, a) \frac{d_c(x_0, Sx_0) d_c(x_1, Tx_1)}{1 + d_c(x_0, x_1)} \\ &+ \gamma(x_0, x_1, a) \frac{d_c(x_1, Sx_0) d_c(x_0, Tx_1)}{1 + d_c(x_0, x_1)} \\ &+ \delta(x_0, x_1, a) \left(\frac{d_c(x_0, Sx_0) d_c(x_0, Tx_1) + d_c(x_1, Tx_1) d_c(x_1, Sx_0)}{1 + d_c(x_0, Tx_1) + d_c(x_1, Sx_0)} \right) \\ &\cdot \in \bigcap_{p \in Sx_0} s(p, Tx_1), \end{aligned}$$

$$\begin{aligned} &\lambda(x_0, x_1, a) d_c(x_0, x_1) + \mu(x_0, x_1, a) \frac{d_c(x_0, Sx_0) d_c(x_1, Tx_1)}{1 + d_c(x_0, x_1)} \\ &+ \gamma(x_0, x_1, a) \frac{d_c(x_1, Sx_0) d_c(x_0, Tx_1)}{1 + d_c(x_0, x_1)} \delta(x_0, x_1, a) \\ &\cdot \left(\frac{d_c(x_0, Sx_0) d_c(x_0, Tx_1) + d_c(x_1, Tx_1) d_c(x_1, Sx_0)}{1 + d_c(x_0, Tx_1) + d_c(x_1, Sx_0)} \right) \\ &\cdot \in s(p, Tx_1), \forall p \in Sx_0. \end{aligned} \quad (13)$$

Since $x_1 \in Sx_0$, we get

$$\begin{aligned}&\lambda(x_0,x_1,a)d_c(x_0,x_1)+\mu(x_0,x_1,a)\frac{d_c(x_0,Sx_0)d_c(x_1,Tx_1)}{1+d_c(x_0,x_1)}\\&\quad+\gamma(x_0,x_1,a)\frac{d_c(x_1,Sx_0)d_c(x_0,Tx_1)}{1+d_c(x_0,x_1)}\\&\quad+\delta(x_0,x_1,a)\left(\frac{d_c(x_0,Sx_0)d_c(x_0,Tx_1)+d_c(x_1,Tx_1)d_c(x_1,Sx_0)}{1+d_c(x_0,Tx_1)+d_c(x_1,Sx_0)}\right)\\&\quad\cdot\in s(x_1,Tx_1),\end{aligned}$$

$$\begin{aligned}&\lambda(x_0,x_1,a)d_c(x_0,x_1)+\mu(x_0,x_1,a)\frac{d_c(x_0,Sx_0)d_c(x_1,Tx_1)}{1+d_c(x_0,x_1)}\\&\quad+\gamma(x_0,x_1,a)\frac{d_c(x_1,Sx_0)d_c(x_0,Tx_1)}{1+d_c(x_0,x_1)}\\&\quad+\delta(x_0,x_1,a)\left(\frac{d_c(x_0,Sx_0)d_c(x_0,Tx_1)+d_c(x_1,Tx_1)d_c(x_1,Sx_0)}{1+d_c(x_0,Tx_1)+d_c(x_1,Sx_0)}\right)\\&\quad\cdot\in\bigcup_{q\in Tx_1}s(d_c(x_1,q)).\end{aligned} \tag{14}$$

Thus, there exists some $x_2 \in Tx_1$ such that

$$\begin{aligned}&\lambda(x_0,x_1,a)d_c(x_0,x_1)+\mu(x_0,x_1,a)\frac{d_c(x_0,Sx_0)d_c(x_1,Tx_1)}{1+d_c(x_0,x_1)}\\&\quad+\gamma(x_0,x_1,a)\frac{d_c(x_1,Sx_0)d_c(x_0,Tx_1)}{1+d_c(x_0,x_1)}\\&\quad+\delta(x_0,x_1,a)\left(\frac{d_c(x_0,Sx_0)d_c(x_0,Tx_1)+d_c(x_1,Tx_1)d_c(x_1,Sx_0)}{1+d_c(x_0,Tx_1)+d_c(x_1,Sx_0)}\right)\\&\quad\cdot\in\bigcup_{q\in Tx_1}s(d_c(x_1,x_2)).\end{aligned} \tag{15}$$

By using the g.l.b. property of S and T, we obtain that

$$\begin{aligned}d_c(x_1,x_2)&\preceq\lambda(x_0,x_1,a)d_c(x_0,x_1)\\&\quad+\mu(x_0,x_1,a)\frac{d_c(x_0,x_1)d_c(x_1,x_2)}{1+d_c(x_0,x_1)}\\&\quad+\gamma(x_0,x_1,a)\frac{d_c(x_1,x_1)d_c(x_0,x_2)}{1+d_c(x_0,x_1)}\\&\quad+\delta(x_0,x_1,a)\left(\frac{d_c(x_0,x_1)d_c(x_0,x_2)+d_c(x_1,x_2)d_c(x_1,x_1)}{1+d_c(x_0,x_2)+d_c(x_1,x_1)}\right),\end{aligned} \tag{16}$$

which implies that

$$\begin{aligned}|d_c(x_1,x_2)|&\le\lambda(x_0,x_1,a)|d_c(x_0,x_1)|\\&\quad+\mu(x_0,x_1,a)\frac{|d_c(x_0,x_1)||d_c(x_1,x_2)|}{|1+d_c(x_0,x_1)|}\\&\quad+\delta(x_0,x_1,a)\frac{|d_c(x_0,x_1)||d_c(x_0,x_2)|}{|1+d_c(x_0,x_2)|},\\|d_c(x_1,x_2)|&=\lambda(x_0,x_1,a)|d_c(x_0,x_1)|\\&\quad+\mu(x_0,x_1,a)\frac{|d_c(x_0,x_1)|}{|1+d_c(x_0,x_1)|}|d_c(x_1,x_2)|\\&\quad+\delta(x_0,x_1,a)\frac{|d_c(x_0,x_2)|}{|1+d_c(x_0,x_2)|}|d_c(x_0,x_1)|.\end{aligned} \tag{17}$$

That is,

$$\begin{aligned}|d_c(x_1,x_2)|&\le\lambda(x_0,x_1,a)|d_c(x_0,x_1)|+\mu(x_0,x_1,a)|d_c(x_1,x_2)|\\&\quad+\delta(x_0,x_1,a)|d_c(x_0,x_1)|,\end{aligned} \tag{18}$$

from which we get

$$|d_c(x_1,x_2)|\le\frac{(\lambda(x_0,x_1,a)+\delta(x_0,x_1,a))}{1-\mu(x_0,x_1,a)}|d_c(x_0,x_1)|. \tag{19}$$

Let $\rho=(\lambda(x_0,x_1,a)+\delta(x_0,x_1,a))/(1-\mu(x_0,x_1,a))$. Clearly, $\rho<1$; then, we can inductively define a sequence $\{x_n\}\in X$ such that $|d_c(x_n,x_{n+m})|\le\rho^n|d_c(x_0,x_1)|$, for $n=0,1,\cdots,x_{2n+1}\in Sx_{2n}$, and $x_{2n+2}\in Tx_{2n+1}$. Now, for $m>n$ and the fact that X is complex-valued b-metric space, we get

$$\begin{aligned}|d_c(x_n,x_m)|&\le\tau|d_c(x_n,x_{n+1})|+\tau^2|d_c(x_{n+1},x_{n+2})|+\cdots\\&\quad+\tau^m|d_c(x_{m-1},x_m)|\le\tau\rho^n d_c(x_0,x_1)\\&\quad+\tau^2\rho^{n+1}d_c(x_0,x_1)+\cdots+\tau^m\rho^{m-1}d_c(x_0,x_1)\\&\le\tau\rho^n\left(1+\tau\rho+(\tau\rho)^2+\cdots+(\tau\rho)^{m-1}\right)d_c(x_0,x_1).\end{aligned} \tag{20}$$

That is,

$$|d_c(x_n,x_m)|\le\left[\frac{\tau\rho^n}{1-\tau\rho}\right]d_c(x_0,x_1). \tag{21}$$

Letting $m,n\to\infty$ in (21), we get $|d_c(x_n,x_m)|\to 0$. This implies by Lemma 6 that $\{x_n\}$ is a cauchy sequence in X. Since X is complete, there exists $u\in X$ such that $x_n\to u$ as $n\to\infty$.

Next, we show that $u\in Su$ and $u\in Tu$. Again, from (6), we have

$$\begin{aligned}&\lambda(x_{2n},u,a)d_c(x_{2n},u)+\mu(x_{2n},u,a)\frac{d_c(x_{2n},Sx_{2n})d_c(u,Tu)}{1+d_c(x_{2n},u)}\\&\quad+\gamma(x_{2n},u,a)\frac{d_c(u,Sx_{2n})d_c(x_{2n},Tu)}{1+d_c(x_{2n},u)}\\&\quad+\delta(x_{2n},u,a)\left(\frac{d_c(x_{2n},Sx_{2n})d_c(x_{2n},Tu)+d_c(u,Tu)d_c(u,Sx_{2n})}{1+d_c(x_{2n},Tu)+d_c(u,Sx_{2n})}\right)\\&\quad\cdot\in s(Sx_{2n},Tu).\end{aligned} \tag{22}$$

This implies

$$\lambda(x_{2n},u,a)d_c(x_{2n},u)+\mu(x_{2n},u,a)\frac{d_c(x_{2n},Sx_{2n})d_c(u,Tu)}{1+d_c(x_{2n},u)}$$
$$+\gamma(x_{2n},u,a)\frac{d_c(u,Sx_{2n})d_c(x_{2n},Tu)}{1+d_c(x_{2n},u)}$$
$$+\delta(x_{2n},u,a)\left(\frac{d_c(x_{2n},Sx_{2n})d_c(x_{2n},Tu)+d_c(u,Tu)d_c(u,Sx_{2n})}{1+d_c(x_{2n},Tu)+d_c(u,Sx_{2n})}\right)$$
$$\cdot\in\bigcap_{p\in Sx}s(p,Tu),$$

$$\lambda(x_{2n},u,a)d_c(x_{2n},u)+\mu(x_{2n},u,a)\frac{d_c(x_{2n},Sx_{2n})d_c(u,Tu)}{1+d_c(x_{2n},u)}$$
$$+\gamma(x_{2n},u,a)\frac{d_c(u,Sx_{2n})d_c(x_{2n},Tu)}{1+d_c(x_{2n},u)}$$
$$+\delta(x_{2n},u,a)\left(\frac{d_c(x_{2n},Sx_{2n})d_c(x_{2n},Tu)+d_c(u,Tu)d_c(u,Sx_{2n})}{1+d_c(x_{2n},Tu)+d_c(u,Sx_{2n})}\right)$$
$$\cdot\in s(p,Tu),\forall p\in Sx_{2n}. \tag{23}$$

Now, since $x_{2n+1}\in Sx_{2n}$, we have

$$\lambda(x_{2n},u,a)d_c(x_{2n},u)+\mu(x_{2n},u,a)\frac{d_c(x_{2n},Sx_{2n})d_c(u,Tu)}{1+d_c(x_{2n},u)}$$
$$+\gamma(x_{2n},u,a)\frac{d_c(u,Sx_{2n})d_c(x_{2n},Tu)}{1+d_c(x_{2n},u)}$$
$$+\delta(x_{2n},u,a)\left(\frac{d_c(x_{2n},Sx_{2n})d_c(x_{2n},Tu)+d_c(u,Tu)d_c(u,Sx_{2n})}{1+d_c(x_{2n},Tu)+d_c(u,Sx_{2n})}\right)$$
$$\cdot\in s(x_{2n+1},Tu). \tag{24}$$

From which we obtain that

$$\lambda(x_{2n},u,a)d_c(x_{2n},u)+\mu(x_{2n},u,a)\frac{d_c(x_{2n},Sx_{2n})d_c(u,Tu)}{1+d_c(x_{2n},u)}$$
$$+\gamma(x_{2n},u,a)\frac{d_c(u,Sx_{2n})d_c(x_{2n},Tu)}{1+d_c(x_{2n},u)}$$
$$+\delta(x_{2n},u,a)\left(\frac{d_c(x_{2n},Sx_{2n})d_c(x_{2n},Tu)+d_c(u,Tu)d_c(u,Sx_{2n})}{1+d_c(x_{2n},Tu)+d_c(u,Sx_{2n})}\right)$$
$$\cdot\in s(x_{2n+1},Tu)=\bigcup_{q\in Tu}s(d(x_{2n+1},q)). \tag{25}$$

This implies that there exists some $u_n\in Tu$ such that

$$\lambda(x_{2n},u,a)d_c(x_{2n},u)+\mu(x_{2n},u,a)\frac{d_c(x_{2n},Sx_{2n})d_c(u,Tu)}{1+d_c(x_{2n},u)}$$
$$+\gamma(x_{2n},u,a)\frac{d_c(u,Sx_{2n})d_c(x_{2n},Tu)}{1+d_c(x_{2n},u)}$$
$$+\delta(x_{2n},u,a)\left(\frac{d_c(x_{2n},Sx_{2n})d_c(x_{2n},Tu)+d_c(u,Tu)d_c(u,Sx_{2n})}{1+d_c(x_{2n},Tu)+d_c(u,Sx_{2n})}\right)$$
$$\cdot s(d(x_{2n+1},u_n)). \tag{26}$$

That is,

$$d_c(x_{2n+1},u_n)\preceq\lambda(x_{2n},u,a)d_c(x_{2n},u)$$
$$+\mu(x_{2n},u,a)\frac{d_c(x_{2n},Sx_{2n})d_c(u,Tu)}{1+d_c(x_{2_n},u)}$$
$$+\gamma(x_{2n},u,a)\frac{d_c(u,Sx_{2n})d_c(x_{2n},Tu)}{1+d_c(x_{2_n},u)}$$
$$+\delta(x_{2n},u,a)\left(\frac{d_c(x_{2_n},Sx_{2n})d_c(x_{2n},Tu)+d_c(u,Tu)d_c(u,Sx_{2n})}{1+d_c(x_{2n},Tu)+d_c(u,Sx_{2n})}\right). \tag{27}$$

By using the g.l.b. property of S and T, we get

$$d_c(x_{2n+1},u_n)\preceq\lambda(x_{2n},u,a)d_c(x_{2n},u)$$
$$+\mu(x_{2n},u,a)\frac{d_c(x_{2n},x_{2n+1})d_c(u,u_n)}{1+d_c(x_{2n},u_n)}$$
$$+\gamma(x_{2n},u,a)\frac{d_c(u,x_{2n+1})d_c(x_{2n},u_n)}{1+d_c(x_{2n},u)}$$
$$+\delta(x_{2n},u,a)\left(\frac{d_c(x_{2n},x_{2n+1})d_c(x_{2n},u_n)+d_c(u,u_n)d_c(u,x_{2n+1})}{1+d_c(x_{2n},u_n)+d_c(u,x_{2n+1})}\right). \tag{28}$$

From $d_c(u,u_n)\preceq\tau(d_c(u,x_{2n+1})+d_c(x_{2n+1},u_n))$, we have

$$d_c(u,u_n)\preceq\tau d_c(u,x_{2n+1})+\tau\lambda(x_{2n},u,a)d_c(x_{2n},u)$$
$$+\tau\mu(x_{2n},u,a)\frac{d_c(x_{2n},x_{2n+1})d_c(u,u_n)}{1+d_c(x_{2n},u)}$$
$$+\tau\gamma(x_{2n},u,a)\frac{d_c(u,x_{2n+1})d_c(x_{2n},u_n)}{1+d_c(x_{2n},u)}$$
$$+\tau\delta(x_{2n},u,a)\left(\frac{d_c(x_{2n},x_{2n+1})d_c(x_{2n},u_n)+d_c(u,u_n)d_c(u,x_{2n+1})}{1+d_c(x_{2n},u_n)+d_c(u,x_{2n+1})}\right)$$
$$\preceq\tau d_c(u,x_{2n+1})+\tau\lambda(x_0,u,a)d_c(x_{2n},u)$$
$$+\tau\mu(x_0,u,a)\frac{d_c(x_{2n},x_{2n+1})d_c(u,u_n)}{1+d_c(x_{2n},u)}$$
$$+\tau\gamma(x_0,u,a)\frac{d_c(u,x_{2n+1})d_c(x_{2n},u_n)}{1+d_c(x_{2n},u)}$$
$$+\tau\delta(x_0,u,a)\left(\frac{d_c(x_{2n},x_{2n+1})d_c(x_{2n},u_n)+d_c(u,u_n)d_c(u,x_{2n+1})}{1+d_c(x_{2n},u_n)+d_c(u,x_{2n+1})}\right). \tag{29}$$

Therefore, we get

$$|d_c(u,u_n)|\le\tau|d_c(u,x_{2n+1})|+\tau\lambda(x_0,u,a)|d_c(x_{2n},u)|$$
$$+\tau\mu(x_0,u,a)\frac{|d_c(x_{2n},x_{2n+1})||d_c(u,u_n)|}{1+|d_c(x_{2n},u)|}$$
$$+\tau\gamma(x_0,u,a)\frac{|d_c(u,x_{2n+1})||d_c(x_{2n},u_n)|}{1+|d_c(x_{2n},u)|}+\tau\delta(x_0,u,a)$$
$$\cdot\left(\frac{|d_c(x_{2n},x_{2n+1})||d_c(x_{2n},u_n)|+|d_c(u,u_n)||d_c(u,x_{2n+1})|}{1+|d_c(x_{2n},u_n)|+|d_c(u,x_{2n+1})|}\right). \tag{30}$$

By letting $n\to\infty$ in the above inequality, we get $|d_c(u,u_n)|\to 0$ as $n\to\infty$. By Lemma 5, we get that $\lim_{n\to\infty}u_n=u$. Since Tu is closed, we have $u\in Tu$. Proceeding in a similar fashion, it follows easily that $u \in Su$. Therefore, S and T have a common fixed point.

Corollary 12. *Let (X, d_c) be a complete complex-valued b-metric space and let $S, T : X \to CB(X)$ be multivalued mappings with g.l.b. property. Let $\lambda, \mu, \gamma, \delta : X \times X \to [0, 1)$ be mappings such that $\forall x, y \in X$:*

$$\lambda(x_{2n}, y) \le \lambda(x_0, y), \quad \lambda(x, y_{2n}) \le \lambda(x, y_0),$$

$$\mu(x_{2n}, y) \le \mu(x_0, y), \quad \mu(x, y_{2n}) \le \mu(x, y_0),$$

$$\gamma(x_{2n}, y) \le \gamma(x_0, y), \quad \gamma(x, y_{2n}) \le \gamma(x, y_0),$$

$$\delta(x_{2n}, y) \le \delta(x_0, y), \quad \delta(x, y_{2n}) \le \delta(x, y_0),$$

$$\begin{aligned} &\lambda(x, y) d_c(x, y) + \mu(x, y) \frac{d_c(x, Sx) d_c(y, Ty)}{1 + d_c(x, y)} \\ &\quad + \gamma(x, y) \frac{d_c(y, Sx) d_c(x, Ty)}{1 + d_c(x, y)} \\ &\quad + \delta(x, y) \left(\frac{d_c(x, Sx) d_c(x, Ty) + d_c(y, Ty) d_c(y, Sx)}{1 + d_c(x, Ty) + d_c(y, Sx)} \right) \\ &\quad \cdot \in s(Sx, Ty), \end{aligned}$$

$$\tau\lambda(x, y) + \mu(x, y) + \gamma(x, y) + \delta(x, y) < 1. \quad (31)$$

Then, S and T have a unique common fixed point.

Proof. The proof follows by setting $\lambda(x, y, a) = \lambda(x, y)$, $\mu(x, y, a) = \mu(x, y)$, $\gamma(x, y, a) = \gamma(x, y)$, and $\delta(x, y, a) = \delta(x, y)$ in the proof of Theorem 11.

By setting $\mu = \gamma = 0$ in Theorem 11, we have the following corollary.

Corollary 13. *Let (X, d_c) be a complete complex valued b-metric space and let $S, T : X \to CB(X)$ be multivalued mappings with g.l.b. property. Let $\lambda, \delta : X \times X \times X \to [0, 1)$ be mappings such that $\forall x, y \in X$,*

$$\lambda(x_{2n}, y, a) \le \lambda(x_0, y, a), \quad \lambda(x, y_{2n}, a) \le \lambda(x, y_0, a),$$

$$\delta(x_{2n}, y, a) \le \delta(x_0, y, a), \quad \delta(x, y_{2n}, a) \le \delta(x, y_0, a),$$

$$\begin{aligned} &\lambda(x, y, a) d_c(x, y) \\ &\quad + \delta(x, y, a) \left(\frac{d_c(x, Sx) d_c(x, Ty) + d_c(y, Ty) d_c(y, Sx)}{1 + d_c(x, Ty) + d_c(y, Sx)} \right) \\ &\quad \cdot \in s(Sx, Ty), \end{aligned}$$

$$\tau\lambda(x, y, a) + \delta(x, y, a) < 1. \quad (32)$$

Then, S and T have a unique common fixed point.

Also, by setting $\gamma = \delta = 0$ in Theorem 11, we have the following consequence.

Corollary 14. *Let (X, d_c) be a complete complex-valued b-metric space and let $S, T : X \to CB(X)$ be multivalued mappings with g.l.b. property. Let $\lambda, \mu, \gamma, \delta : X \times X \times X \to [0, 1)$ be mappings such that $\forall x, y \in X$,*

$$\begin{aligned} &\lambda(x_{2n}, y, a) \le \lambda(x_0, y, a), \quad \lambda(x, y_{2n}, a) \le \lambda(x, y_0, a), \\ &\mu(x_{2n}, y, a) \le \mu(x_0, y, a), \quad \mu(x, y_{2n}, a) \le \mu(x, y_0, a). \\ &\lambda(x, y, a) d_c(x, y) + \mu(x, y, a) \frac{d_c(x, Sx) d_c(y, Ty)}{1 + d_c(x, y)} \in s(Sx, Ty), \\ &\tau\lambda(x, y, a) + \mu(x, y, a) < 1. \end{aligned} \quad (33)$$

Then, S and T have a unique common fixed point.

Corollary 15. *Let (X, d_c) be a complete complex-valued b-metric space and let $S, T : X \to CB(X)$ be multivalued mappings with g.l.b property. Let $\lambda, \mu : X \times X \times X \to [0, 1)$ be mappings such that $\forall x, y \in X$,*

$$\lambda(x_{2n}, y, a) \le \lambda(x_0, y, a), \quad \lambda(x, y_{2n}, a) \le \lambda(x, y_0, a),$$

$$\mu(x_{2n}, y, a) \le \mu(x_0, y, a), \quad \mu(x, y_{2n}, a) \le \mu(x, y_0, a),$$

$$\gamma(x_{2n}, y, a) \le \gamma(x_0, y, a), \quad \gamma(x, y_{2n}, a) \le \gamma(x, y_0, a),$$

$$\delta(x_{2n}, y, a) \le \delta(x_0, y, a), \quad \delta(x, y_{2n}, a) \le \delta(x, y_0, a),$$

$$\begin{aligned} &\lambda(x, y, a) d_c(x, y) + \mu(x, y, a) \frac{d_c(x, Sx) d_c(y, Ty)}{1 + d_c(x, y)} \\ &\quad + \gamma(x, y, a) \frac{d_c(y, Sx) d_c(x, Ty)}{1 + d_c(x, y)} \\ &\quad + \delta(x, y, a) \left(\frac{d_c(x, Sx) d_c(x, Ty) + d_c(y, Ty) d_c(y, Sx)}{1 + d_c(x, Ty) + d_c(y, Sx)} \right) \\ &\quad \cdot \in s(Sx, Ty), \end{aligned}$$

$$\tau\lambda(x, y, a) + \mu(x, y, a) + \gamma(x, y, a) + \delta(x, y, a) < 1. \quad (34)$$

Then, S and T have a unique common fixed point.

By putting $S = T$ in our main theorem, we obtain the following corollary.

Corollary 16. *Let (X, d_c) be a complete complex-valued b-metric space and let $T : X \to CB(X)$ be multivalued mappings with g.l.b. property. Let $\lambda, \mu, \gamma, \delta : X \times X \times X \to [0, 1)$ be mappings such that $\forall x, y \in X$,*

(i) $\lambda(x_{2n}, y, a) \le \lambda(x_0, y, a)$, $\mu(x_{2n}, y, a) \le \mu(x_0, y, a)$, $\gamma(x_{2n}, y, a) \le \gamma(x_0, y, a,)$

$$\delta(x_{2n}, y, a) \le \delta(x_0, y, a) \quad (35)$$

(ii) $\lambda(x, y, a) d_c(x, y) + \mu(x, y, a)(d_c(x, Tx) d_c(y, Ty)/1 + d_c(x, y)) + \gamma(x, y, a)(d_c(y, Tx) d_c(x, Ty)/1 + d_c(x, y))$

$$+\delta(x,y,a)(d_c(x,Tx)d_c(x,Ty)+d_c(y,Ty)d_c(y,Tx) /1+d_c(x,Ty)+d_c(y,Tx))\in s(Tx,Ty)$$

(iii) $\tau\lambda(x,y,a)+\mu(x,y,a)+\gamma(x,y,a)+\delta(x,y,a)<1$

Then, T has a unique fixed point in X.

Corollary 17. *Let (X,d_c) be a complete complex-valued b -metric space and $T:X\to X$. Let $\lambda:X\times X\times X\to[0,1)$ be mappings such that $\forall x,y\in X$,*

(i) $\lambda(x,y,a)\le b, b\in\mathbb{R}$

(ii) $d_c(Tx,Ty)\preceq\lambda(x,y,a)d_c(x,y)$

(iii) $\tau\lambda(x,y,a)<1$

Then, T has a unique fixed point in X.

Example 18. Let $X=[0,1]$. Define $d_c:X\times X\to\mathbb{C}$ by $d_c(x,y)=|x-y|e^{i\theta}$, where $\tan\theta=|y/x|$. Then, (X,d_c) is a complete complex-valued b -metric space. Consider the mappings S, $T:X\to CB(X)$ be defined by

$$Sx=\left\{t\in X:0\le t\le\frac{x}{7}\right\},\quad Tx=\left\{t\in X:0\le t\le\frac{x}{3}\right\}.\tag{36}$$

Observe that by setting $x=y=0$ in Theorem 11, the contractive condition becomes trivial. Now, consider the case for which x and y are nonzeros, we obtain by using the definition of d_c that

$$\begin{aligned}d_c(x,y)&=|x-y|e^{i\theta},\\ d_c(x,Sx)&=|x-\frac{x}{7}|e^{i\theta},\\ d_c(y,Ty)&=\left|y-\frac{y}{3}\right|e^{i\theta},\\ d_c(y,Sx)&=|y-\frac{x}{7}|e^{i\theta},\\ d_c(x,Ty)&=\left|x-\frac{y}{3}\right|e^{i\theta},\\ (d_c(Sx,Ty))&=\left(\left|\frac{x}{7}-\frac{y}{3}\right|e^{i\theta}\right).\end{aligned}\tag{37}$$

Furthermore, for all $x,y\in X$ and fixed $a=1/5\in X$. Define the functions $\lambda,\mu,\gamma,\delta:X\times X\times X\to[0,1)$ by $\lambda(x,y,a)=(x/7+y/4+a)$,$\mu(x,y,a)=xya/14$, $\gamma(x,y,a)=2xya^2/14$, and $\delta(x,y,a)=3xya^3/14$. Choose $\tau\in[1,2]$, it is clear that $\tau\lambda(x,y,a)+\mu(x,y,a)+\gamma(x,y,a)+\delta(x,y,a)<1$ for each $x,y\in X$ and fixed $a=1/5\in X$. Now, define

$$x_n=\frac{1}{n+1},\quad n=0,1,2,3,\cdots,\tag{38}$$

then,

$$\lambda(x_{2n},y,a)=\frac{1}{7(2n+1)}+\frac{y}{5}+\frac{1}{5}\le1+\frac{y}{5}+\frac{1}{5},\quad n=0,1,2,\cdots=\lambda(x_0,y,a).\tag{39}$$

Also, consider

$$\lambda(x,x_{2n},a)=\frac{x}{7}+\frac{1}{5(2n+1)}+\frac{1}{5}\le\frac{x}{7}+1+\frac{1}{5},\quad n=0,1,2,\cdots=\lambda(x,x_0,a).\tag{40}$$

Similarly, we can show that

$$\begin{cases}\mu(x_{2n},y,a)\le\mu(x_0,y,a), & \mu(x,y_{2n},a)\le\mu(x,y_0,a),\\ \gamma(x_{2n},y,a)\le\gamma(x_0,y,a), & \gamma(x,y_{2n},a)\le\mu(x,y_0,a),\\ \delta(x_{2n},y,a)\le\delta(x_0,y,a), & \delta(x,y_{2n},a)\le\delta(x,y_0,a).\end{cases}\tag{41}$$

To conclude the example, we only need to show that $|d_c(Sx,Ty)|\le\tau\lambda(x,y,a)|d_c(x,y)|$. Indeed, for $x,y\in X$ and $a=1/2\in X$,

$$\begin{aligned}\left|\frac{x}{7}-\frac{y}{3}\right|e^{i\theta}&\le\frac{2}{7}|x-y|e^{i\theta}\le\frac{2}{5}|x-y|e^{i\theta}\\ &\le2\left(\frac{x}{7}+\frac{y}{5}+\frac{1}{5}\right)|x-y|e^{i\theta},\forall x,y\in X\\ &=\tau\lambda(x,y,a)|d_c(x,y)|,\end{aligned}\tag{42}$$

with $\tau\in[1,2]$ and $a=1/5\in X$. Thus, $|d_c(Sx,Ty)|\le\tau\lambda(x,y,a)|d_c(x,y)|$, for all $x,y\in X$ and for $a=1/5\in X$.

Therefore, all conditions of Theorem 11 are satisfied and 0 is a common fixed point of S and T.

4. Application

4.1. Application to Riemann-Liouville Equation. In this section, we establish the existence of a solution of a Riemann-Liouville of the form

$${}^{RL}c_1I_t^u x(t)=\Gamma(u)\int_{c_1}^t(t-s)^{u-1}x(s)ds,\mathscr{R}(u)>0,\tag{43}$$

where $u\in\mathbb{C}$, $x(t)\in X=C([0,1],\mathbb{R})$ and $t,s\in[0,1]$ which is the fractional integral. Let $X=C([0,1],\mathbb{R})$ be the space of continuous function and $d_c:X\times X\to\mathbb{C}$ be defined as

$$d_c(u,v)=\left[\max_{t\in[0,1]}\|u(t)-v(t)\|\sqrt{1+a^2}e^{i\tan^{-1}a}\right]^2,\tag{44}$$

with $\tau=2$. It is well known that (X,d_c) is a complete complex-valued b-metric space. Define $T : X \to X$ by

$$Tx(t) = \Gamma(u)\int_{c_1}^{t} (t-s)^{u-1} x(s)ds. \tag{45}$$

Theorem 19. *Let $X = C([0,1], \mathbb{R})$, for all $x, y \in X$ and $a \in \mathbb{R}$, let $\lambda : X \times X \times X \to [0,1)$ be given by $\lambda(x, y, a) \leq 1/16$ and suppose that*

$$\left[max_{t\in[0,1]} \frac{1}{\Gamma(u+1)} \frac{(t-s)^{u-1}(t-c_1)^u}{|(t-s)^{u-1}|}\right]^2 \precsim \frac{1}{64}, \tag{46}$$

then Equation (43) has a solution.

Proof. It is well-known that $x \in X$ is a fixed point of T if and only if x is a solution of problem (43). Note that at some point. Now, observe that for all $u, v \in X$, we have that

$$\begin{aligned} d_c(Tx, Ty) &= \left[\max_{t\in[0,1]} | Tx(t) - Ty(t) | \sqrt{1+a^2} e^{i \tan^{-1} a}\right]^2 \\ &= \left[\max_{t\in[0,1]} | \frac{1}{\Gamma(u)} \int_{c_1}^{t} (t-s)^{u-1} x(s) \right. \\ &\quad \left. - \frac{1}{\Gamma(u)} \int_{c_1}^{t} (t-s)^{u-1} y(s) | \sqrt{1+a^2} e^{i \tan^{-1} a}\right]^2 \\ &\precsim \left[\max_{t\in[0,1]} \frac{1}{\Gamma(u)} | \int_{c_1}^{t} (t-s)^{u-1} ds | \right. \\ &\quad \left. \cdot (|x(s) - y(s)|) \sqrt{1+a^2} e^{i \tan^{-1} a}\right]^2 \\ &\precsim \left[\max_{t\in[0,1]} \frac{1}{\Gamma(u)} \int_{c_1}^{t} | (t-s)^{u-1} ds | \right. \\ &\quad \left. \cdot (|x(s) - y(s)|) \sqrt{1+a^2} e^{i \tan^{-1} a}\right]^2 \\ &\precsim \left[\max_{t\in[0,1]} \frac{1}{\Gamma(u)} \frac{(t-s)^{u-1}}{|(t-s)^{u-1}|} \int_{c_1}^{t} | (t-s)^{u-1} ds | \right. \\ &\quad \left. \cdot (|x(s) - y(s)|) \sqrt{1+a^2} e^{i \tan^{-1} a}\right]^2. \end{aligned} \tag{47}$$

Let $q = t - s$, we get

$$\begin{aligned} d_c(Tx, Ty) &\precsim \left[\max_{t\in[0,1]} \frac{1}{\Gamma(u)} \frac{(t-s)^{u-1}}{|(t-s)^{u-1}|} \right. \\ &\quad \left. \cdot \int_{t-c_1}^{0} q^{u-1} dq (|x(s) y(s)|) \sqrt{1+a^2} e^{i \tan^{-1} a}\right]^2 \\ &\cdot \precsim \left[\max_{t\in[0,1]} \frac{1}{\Gamma(u)} \frac{(t-s)^{u-1}}{|(t-s)^{u-1}|} \left(\frac{q^u}{u}\right)_{t-c_1}^{0} \right. \\ &\quad \left. \cdot (|x(s) - y(s)|) \sqrt{1+a^2} e^{i \tan^{-1} a}\right]^2 \\ &= \left[\max_{t\in[0,1]} \frac{1}{\Gamma(u)} \frac{(t-s)^{u-1}}{|(t-s)^{u-1}|} \frac{(t-c_1)^u}{u} \right. \\ &\quad \left. \cdot (|x(s) - y(s)|) \sqrt{1+a^2} e^{i \tan^{-1} a}\right]^2 \\ &\precsim \left[\max_{t\in[0,1]} \frac{1}{\Gamma(u+1)} \frac{(t-s)^{u-1}(t-c_1)^u}{|(t-s)^{u-1}|} \right. \\ &\quad \left. \cdot (|x(s) - y(s)|) \sqrt{1+a^2} e^{i \tan^{-1} a}\right]^2 \\ &= \left[\max_{t\in[0,1]} (|x(s) - y(s)|) \sqrt{1+a^2} e^{i \tan^{-1} a}\right]^2 \\ &\quad \cdot \left[\max_{t\in[0,1]} \frac{1}{\Gamma(u+1)} \frac{(t-s)^{u-1}(t-c_1)^u}{|(t-s)^{u-1}|}\right]^2 \precsim d(x, y) \cdot \frac{1}{64}, \end{aligned} \tag{48}$$

that is,

$$|d_c(Tx, Ty)| \leq \frac{1}{16} |d_c(x, y)|. \tag{49}$$

Thus, we have that $d(Tx, Ty) \precsim \lambda d(x, y)$. Clearly, all conditions of Corollary 17 are satisfied and guarantee the existence of the fixed point $x \in X$. Thus, x is the solution of the integral Equation (43).

Data Availability

No data was required for the research work.

Disclosure

Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

Conflicts of Interest

The authors declare that there is no competing interest on the paper.

Authors' Contributions

All authors worked equally on the results and approved the final manuscript.

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