Research Article

Numerical Treatment on Parabolic Singularly Perturbed Differential Difference Equation via Fitted Operator Scheme

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This paper proposes a new fitted operator strategy for solving singularly perturbed parabolic partial differential equation with delay on the spatial variable. We decomposed the problem into three piecewise equations. The delay term in the equation is expanded by Taylor series, the time variable is discretized by implicit Euler method, and the space variable is discretized by central difference methods. After developing the fitting operator method, we accelerate the order of convergence of the time direction using Richardson extrapolation scheme and obtained $O(h^2 + k^2)$ uniform order of convergence. Finally, three examples are given to illustrate the effectiveness of the method. The result shows the proposed method is more accurate than some of the methods that exist in the literature.

1. Introduction

The spatial delay parabolic singularly perturbed differential equation is a differential equation in which the perturbation parameter multiply the highest order derivative, and it has at least one retarded term on the spatial variable. Some mathematical problems can be treated as singularly perturbed problems such as Navier–Stokes equations, atmospheric pollution, turbulent transport, groundwater flow and solute transport, and Black–Scholes model, as presented in the survey paper by Kadalbajoo and Gupta [1]. There are two principle approaches for solving singular perturbation problems: numerical approach and asymptotic approach by Sharma et al. [2]. The numerical solution of the model problem is challenging due to the existence of a boundary layer. Furthermore, when using the classical finite difference method, we are unable to achieve an accurate solution, and the system becomes unstable. To tackle this issue, we need to create a fitted method for uniform or nonuniform meshes. As a result, ε-uniformly convergent numerical methods were constructed, with the order of convergence and error constant being independent of the perturbation parameter ε. For solving singular perturbation problems, some ε-uniform numerical schemes have been developed in the literature.

Most of the scholars studied on time delayed singularly perturbed problems. Mbroh et al. [3] proposed parameter uniform method for solving a time delay nonautonomous singularly perturbed parabolic differential equation. Clavero and Gracia [4] studied singularly perturbed time-dependent problem of reaction–diffusion type using Richardson extrapolation technique. Woldaregay and Duressa [5] considered a numerical method for both small time delay and large time delay singularly perturbed boundary value problem. Kumar and Kumari [6] show that the influence of a small delay on the solution is extremely sensitive and that a small change in the delay can have a significant impact on the solution. Chakravarthy et al. [7] using the fitted technique, singularly perturbed differential equations with large delays were investigated.

Nowadays, a few scholars have examined numerical solution for spatial delay singularly perturbed parabolic partial differential equations. Bansal and Sharma [8] numerical solutions for a large delay reaction-diffusion problem has been developed. Gupta et al. [9] examined spatial delay parabolic singularly perturbed partial differential equations and

its solution using higher order fitted mesh method. Exponentially fitted operator method was developed to solve differential-difference singularly perturbed problem by Woldaregay and Duressa [10]. Das and Natesan [11] presented the solution of delay singularly perturbed partial differential equation using second-order convergent method. Singh and Srinivasan [12] developed Richardson extrapolation method for solving convection-diffusion equations with retarded term. Chahravarthy and Kumar [13] presented adaptive grid method for solving singularly perturbed convection-diffusion problems with spatial delay. Bansal et al. [14] designed numerical scheme for solving general shift singularly perturbed parabolic convectional diffusion problems.

In this study, a singularly perturbed delayed partial differential equation with small spatial shift right boundary layer problem is decomposed into three piecewise equations which are treated using fitted operator difference methods. To accelerate the order of accuracy in the time variable, the Richardson extrapolation method is applied. The order of convergence of the present method is shown to be second order in both time and spatial variable, whereas the rate of convergence is two. Furthermore, the numerical results of the examples considered shows that the present method has better accuracy compared to some results that appear in the literature.

2. Statement of the Problem

Consider the parabolic singularly perturbed second-order differential equation with small spatial delay, on the domain

$\Omega = D_1 \times D_2 = (0, 1) \times (0, T]$ and $\Omega = \Omega_l \cup \Omega_b \cup \Omega_r$, where $\Omega_l = \{(x, t): -\gamma \leq x \leq 0 \text{ and } 0 \leq t \leq T\}$, $\Omega_b = \{(x, t): 0 \leq x \leq 1 \text{ and } 0 \leq t \leq T\}$, and $\Omega_r = \{(x, t): 1 \leq x \leq 1 + \mu \text{ and } 0 \leq t \leq T\}$ of the form:

$$\{u_t - \varepsilon u_{xx} + A(x)u_x + B(x)u(x, t) + C(x)u(x - \gamma, t) + D(x)u(x + \mu, t) = F(x, t), \ (x, t) \in \Omega, \tag{1}$$

with

$$\begin{cases} u(x, t) = \varphi_l(x, t), & \forall (x, t) \in \Omega_l, \\ u(x, 0) = \varphi_b(x), & x \in [0, 1], \\ u(x, t) = \varphi_r(x, t), & \forall (x, t) \in \Omega_r, \end{cases} \tag{2}$$

where $0 < \varepsilon \ll 1$ and γ, μ denote the small delay parameters. The functions $A(x), B(x), C(x), D(x), \varphi_l(x, t), \varphi_b(x)$, and $\varphi_r(x, t)$ are supposed to be bounded and smooth functions on $\bar{\Omega}$, that satisfy the conditions $B(x) + C(x) + D(x) \geq \beta > 0$ on $\Omega_b = [0, 1]$. When $\gamma = \mu = 0$, the above problem (1) reduced to singularly perturbed parabolic partial differential problem. If $A(x) \geq \alpha > 0$, $C(x) < 0$ and $D(x) < 0$, $\forall x \in \Omega_b = [0, 1]$, then the solution has boundary layer on the right side, i.e., at $x = 1$, where α, β are some constants.

Equation (1) together with initial boundary condition Equation (2) can be rewrite as

$$L^\varepsilon u(x, t) = \begin{cases} u_t - \varepsilon u_{xx} + A(x)u_x + B(x)u + D(x)u(x + \mu, t) = F(x, t) - C(x)\varphi_l(x - \gamma, t), & \text{if } 0 < x \leq \gamma, 0 < t \leq T, \\ u_t - \varepsilon u_{xx} + A(x)u_x + B(x)u + C(x)u(x - \gamma, t) + D(x)u(x + \mu, t) = F(x, t), & \text{if } \gamma < x < 1 - \mu, 0 < t \leq T, \\ u_t - \varepsilon u_{xx} + A(x)u_x + B(x)u + C(x)u(x - \gamma, t) = F(x, t) - D(x)\varphi_r(x + \mu, t), & \text{if } 1 - \mu \leq x < 1, 0 < t \leq T. \end{cases} \tag{3}$$

Under the assumption that the data are uniformly continuous and also meet relevant compatibility conditions at the corner points $(0, 0), (1, 0), (-\gamma, 0)$, and $(1 + \mu, 0)$, it is possible to establish the uniqueness of a solution to (1) [15]. The compatibility conditions are given as follows:

$$\begin{cases} \varphi_l(0, 0) = \varphi_b(0), \\ \varphi_r(1, 0) = \varphi_b(1), \end{cases}$$

$$\begin{cases} \dfrac{\partial \varphi_l(0, 0)}{\partial t} - \varepsilon \dfrac{\partial^2 \varphi_b(0)}{\partial x^2} + A(0) \dfrac{\partial \varphi_b(0)}{\partial x} + B(0)\varphi_b(0) + C(0)\varphi_l(-\gamma, 0) + D(0)\varphi_b(\mu) = F(0, 0), \\ \dfrac{\partial \varphi_r(1, 0)}{\partial t} - \varepsilon \dfrac{\partial^2 \varphi_b(1)}{\partial x^2} + A(1) \dfrac{\partial \varphi_b(1)}{\partial x} + B(1)\varphi_b(1) + C(1)\varphi_b(1 - \gamma) + D(0)\varphi_r(1 + \mu, 0) = F(1, 0). \end{cases} \tag{4}$$

Using the above compatibility conditions, there exists a constant M independent of the perturbation parameter ε $\forall (x, t) \in \bar{\Omega}$; we have

$$\begin{cases} |u(x, t) - u(x, 0)| = |u(x, t) - \varphi_b(0)| \leq Mt, \\ |u(x, t) - u(0, t)| = |u(x, t) - \varphi_l(0, t)| \leq M(1 - x), \end{cases} \quad (5)$$

for the proof of Equation (5) see [16].

Lemma 1 (Continuous Maximum Principle). *Let the function $\psi(x, t) \in C^{2,1}(\bar{\Omega})$. If $\psi(x, t) \geq 0$, for all $(x, t) \in \partial\Omega$ and $L^\varepsilon \psi(x, t) \geq 0$, for all $(x, t) \in \Omega$, then $\psi(x, t) \geq 0$, for all $(x, t) \in \bar{\Omega}$.*

Proof. Assume $(x_y, t_z) \in \bar{\Omega} - \partial\Omega$ such that $\psi(x_y, t_z) = \min_{(x,t) \in \bar{\Omega}} \psi(x, t)$ and suppose $\psi(x_y, t_z) < 0$. This gives that $\psi_t(x_y, t_z) = 0$, $\psi_x(x_y, t_z) = 0$ and $\psi_{xx}(x_y, t_z) \geq 0$.□

Now, in order to show $L^\varepsilon \psi(x, t) \leq 0$, consider the following cases.

Case 1. $0 < x_y \leq \gamma$ and $0 \leq t_z \leq T$

$$\begin{aligned} L^\varepsilon u(x_y, t_z) &= u_t(x_y, t_z) - \varepsilon u_{xx}(x_y, t_z) + A(x_y)u_x(x_y, t_z) \\ &\quad + B(x_y)u(x_y, t_z) + D(x_y)u(x_y + \mu, t_z) \\ &= -\varepsilon u_{xx}(x_y, t_z) + B(x_y)u(x_y, t_z) + D(x_y)u(x_y + \mu, t_z) \\ &= -\varepsilon u_{xx}(x_y, t_z) + (B(x_y) + D(x_y))u(x_y, t_z) \\ &\quad + D(x_y)(u(x_y + \mu, t_z) - u(x_y, t_z)) \leq 0. \end{aligned}$$
$$(6)$$

Case 2. $1 - \mu < x_y < 1$ and $0 \leq t_z \leq T$

$$\begin{aligned} L^\varepsilon u(x_y, t_z) &= u_t(x_y, t_z) - \varepsilon u_{xx}(x_y, t_z) + A(x_y)u_x(x_y, t_z) \\ &\quad + B(x_y)u(x_y, t_z) + C(x_y)u(x_y - \gamma, t_z) \\ &= -\varepsilon u_{xx}(x_y, t_z) + B(x_y)u(x_y, t_z) + C(x_y)u(x_y - \gamma, t_z) \\ &= -\varepsilon u_{xx}(x_y, t_z) + (B(x_y) + C(x_y))u(x_y, t_z) \\ &\quad + C(x_y)(u(x_y - \gamma, t_z) - u(x_y, t_z)) \leq 0. \end{aligned}$$
$$(7)$$

Case 3. $\gamma < x_y \leq 1 - \mu$ and $0 \leq t_z \leq T$

$$\begin{aligned} L^\varepsilon u(x_y, t_z) &= u_t(x_y, t_z) - \varepsilon u_{xx}(x_y, t_z) + A(x_y)u_x(x_y, t_z) \\ &\quad + B(x_y)u(x_y, t_z) + C(x_y)u(x_y - \gamma, t_z) \\ &\quad + D(x_y)u(x_y + \mu, t_z) = -\varepsilon u_{xx}(x_y, t_z) \\ &\quad + B(x_y)u(x_y, t_z) + C(x_y)u(x_y - \gamma, t_z) \\ &\quad + D(x_y)u(x_y + \mu, t_z) \leq 0. \end{aligned}$$
$$(8)$$

Since $B(x_y) + C(x_y) + D(x_y) \geq \alpha > 0$ and using the above assumption, we have $L^\varepsilon \psi(x, t) \leq 0$ which contradicts our assumption.

Thus, $\psi(x_y, t_z) \geq 0$ which leads to $\psi(x, t) \geq 0 \forall (x, y) \in \bar{\Omega}$.

Lemma 2. *Let $u(x, t)$ be the solution of the continuous problem for Equations (1) and (2). Then, we get the bound*

$$|u(x, t)| \leq \frac{\|f\|}{\theta} + \max \{|\varphi_l(x, t)|, |\varphi_b(x)|, |\varphi_r(x, t)|\}, \quad (9)$$

where $B(x) \geq \theta > 0, \forall x \in [0, 1]$.

Proof. Let the barrier function $\psi^\pm$ and defined as

$$\psi^\pm = \frac{\|f\|}{\theta} + \max \{|\varphi_l(x, t)|, |\varphi_b(x)|, |\varphi_r(x, t)| \pm u(x, t), \quad (10)$$

applying Lemma 1, and we get the above bound.□

Lemma 3. *The derivatives of the exact solution $u(x, t)$ of the Equation (1) fulfill the following bound for $v = 0, 1$, and 2.*

$$\left| \frac{\partial^v u(x, t)}{\partial t^v} \right| \leq M, \quad \forall (x, t) \in \bar{\Omega}, \quad (11)$$

where $M \in \mathbb{R}$ is independent of ε.

Proof. The proof of this lemma can be found in [7].□

Theorem 4. *The analytical solution of Equation (1) satisfies*

$$\left| \frac{d^i u(x, t)}{dx^i} \right| \leq \left(1 + \varepsilon^{-i} \exp \left(-\frac{\alpha(1 - x)}{\varepsilon} \right) \right), \quad 0 \leq i \leq 4. \quad (12)$$

Proof. The proof for the bounds of its derivatives is given in [4].□

3. Numerical Method

In this section, we develop an exponentially fitted operator difference method to solve Equation (1).

3.1. Time Discretization. A uniform mesh with a time step of k is used to discretize the time domain $[0, T]$ as follows:

$$t_j = jk, \quad \text{for } j = 0, 1, 2, \cdots, n, \quad (13)$$

where $k = T/n$ and in the interval $[0, T]$, n is the number of subintervals in the time direction.

We utilize the implicit Euler's approach to approximate the time derivative term of Equation (1), which results in a system of boundary value problems.

$$\frac{u^{j+1} - u^j}{k} - \varepsilon u_{xx}^{j+1} + A(x)u_x^{j+1} + B(x)u^{j+1} + C(x)u^{j+1}(x - \gamma, t)$$
$$+ D(x)u^{j+1}(x + \mu, t) = F(x, t). \tag{14}$$

Using Equations (3) and (14), we have

$$L^\varepsilon u^{j+1}(x) = \begin{cases} -\varepsilon u_{xx}^{j+1} + A(x)u_x^{j+1} + E(x)u^{j+1} + D(x)u^{j+1}(x + \mu) = F(x, t) - C(x)\varphi_l\left(x - \gamma, t_{j+1}\right) + \dfrac{u^j}{k}, & \text{if } 0 < x \le \gamma, 0 < t \le T, \\[2mm] -\varepsilon u_{xx}^{j+1} + A(x)u_x^{j+1} + E(x)u^{j+1} + C(x)u^{j+1}(x - \gamma) + D(x)u^{j+1}(x + \mu) = F(x, t) + \dfrac{u^j}{k}, & \text{if } \gamma < x < 1 - \mu, 0 < t \le T, \\[2mm] -\varepsilon u_{xx}^{j+1} + A(x)u_x^{j+1} + E(x)u^{j+1} + C(x)u^{j+1}(x - \gamma) = F(x, t) - D(x)\varphi_r\left(x + \mu, t_{j+1}\right) + \dfrac{u^j}{k}, & \text{if } 1 - \mu \le x \le 1, 0 < t \le T, \end{cases} \tag{15}$$

where $E(x) = B(x) + (1/k)$.

3.2. Spatial Discretization.
The spatial domain $[0, 1]$ is subdivided as follows using a uniform mesh with a step length of h:

$$x_i = ih, \quad \text{for } i = 0, 1, 2, \cdots, m, \tag{16}$$

where $h = 1/m$ and m is the number of subintervals in spatial direction in the interval $[0, 1]$. Using expansion of Taylor series expansion, we have

$$\begin{cases} u^{j+1}(x + \mu) = u^{j+1}(x) + \mu u_x^{j+1} + \dfrac{\mu^2}{2}u^{j+1}(x) + O\left(\mu^3\right), \\[2mm] u^{j+1}(x - \gamma) = u^{j+1}(x) - \gamma u_x^{j+1} + \dfrac{\gamma^2}{2}u^{j+1}(x) + O\left(\gamma^3\right). \end{cases} \tag{17}$$

Substituting Equation (17) into Equation (15) and rearranging gives

$$L^\varepsilon u^{j+1}(x) = \begin{cases} -\varepsilon u_{xx}^{j+1} + p_1(x)u_x^{j+1} + q_1(x)u^{j+1} = g_1(x, t), & \text{if } 0 < x \le \gamma, 0 < t \le T, \\[2mm] -\varepsilon u_{xx}^{j+1} + p_2(x)u_x^{j+1} + q_2(x)u^{j+1} = g_2(x, t), & \text{if } \gamma < x < 1 - \mu, 0 < t \le T, \\[2mm] -\varepsilon u_{xx}^{j+1} + p_3(x)u_x^{j+1} + q_3(x)u^{j+1} = g_3(x, t), & \text{if } 1 - \mu \le x \le 1, 0 < t \le T, \end{cases} \tag{18}$$

with the boundary conditions:

$$\begin{cases} u(0, t) = \varphi_l(0, t), & \forall (x, t) \in \Omega_l, \\ u(1, t) = \varphi_r(1, t), & \forall (x, t) \in \Omega_r, \end{cases} \tag{19}$$

and initial condition

$$u(x, 0) = \varphi_b(x), \quad x \in [0, 1], \tag{20}$$

where

$$p_1(x) = \frac{\varepsilon(A(x) + \mu D(x))}{\varepsilon - (\mu^2/2)D(x)},$$

$$p_2(x) = \frac{\varepsilon(A(x) - \gamma C(x) + \mu D(x))}{\varepsilon - (\gamma^2/2)C(x) - (\mu^2/2)D(x)},$$

$$p_3(x) = \frac{\varepsilon(A(x) - \gamma C(x))}{\varepsilon - (\gamma^2/2)C(x)},$$

$$q_1(x) = \frac{\varepsilon(D(x) + E(x))}{\varepsilon - (\mu^2/2)D(x)},$$

$$q_2(x) = \frac{\varepsilon(C(x) + D(x) + E(x))}{\varepsilon - (\gamma^2/2)C(x) - (\mu^2/2)D(x)},$$

$$q_3(x) = \frac{\varepsilon(C(x) + E(x))}{\varepsilon - (\gamma^2/2)C(x)},$$

$$g_1(x,t) = \frac{\varepsilon\left(F(x,t_{j+1}) - C(x)\varphi_l(x - \gamma, t_{j+1}) + (u^j/k)\right)}{\varepsilon - (\mu^2/2)D(x)},$$

$$g_2(x,t) = \frac{\varepsilon\left(F(x,t_{j+1}) + (u^j/k)\right)}{\varepsilon - (\gamma^2/2)C(x) - (\mu^2/2)D(x)},$$

$$g_3(x,t) = \frac{\varepsilon\left(F(x,t_{j+1}) \quad - D(x)\varphi_r(x + \mu, t_{j+1}) + (u^j/k)\right)}{\varepsilon - (\gamma^2/2)C(x)}. \tag{21}$$

Applying central difference approximation for spatial variable of Equation (18), we obtain

$$\begin{cases} -\varepsilon \dfrac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{h^2} + p_1(x_i)\dfrac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2h} + q_1(x_i)u_i^{j+1} = g_1(x_i,t), & \text{if } 0 < x \leq \gamma, 0 < t \leq T, \\[3mm] -\varepsilon \dfrac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{h^2} + p_2(x_i)\dfrac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2h} + q_2(x_i)u_i^{j+1} = g_2(x_i,t), & \text{if } \gamma < x < 1 - \mu, 0 < t \leq T, \\[3mm] -\varepsilon \dfrac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{h^2} + p_3(x_i)\dfrac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2h} + q_3(x_i)u_i^{j+1} = g_3(x_i,t), & \text{if } 1 - \mu \leq x \leq 1, 0 < t \leq T. \end{cases} \tag{22}$$

To obtain more accurate and ε-uniform solution for Equation (22), introducing fitting factor (σ) as follows:

$$\begin{cases} -\varepsilon\sigma_1 \dfrac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{h^2} + p_1(x_i)\dfrac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2h} + q_1(x_i)u_i^{j+1} = g_1(x_i,t), & \text{for } 0 < x \leq \gamma, 0 < t \leq T, \\[3mm] -\varepsilon\sigma_2 \dfrac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{h^2} + p_2(x_i)\dfrac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2h} + q_2(x_i)u_i^{j+1} = g_2(x_i,t), & \text{for } \gamma < x < 1 - \mu, 0 < t \leq T, \\[3mm] -\varepsilon\sigma_3 \dfrac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{h^2} + p_3(x_i)\dfrac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2h} + q_3(x_i)u_i^{j+1} = g_3(x_i,t), & \text{for } 1 - \mu \leq x \leq 1, 0 < t \leq T. \end{cases} \tag{23}$$

To obtain the value of the fitting factor, multiply both sides of the equations in (23) by h and evaluating limit as $h \longrightarrow 0$.

$$\lim_{h \to 0} \frac{-\sigma_1}{\rho} u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1} + \lim_{h \to 0} p_1(x_i)\frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2} = 0, \tag{24}$$

where $\rho = h/\varepsilon$.

The solution of Equation (18) comes from the theory of singular perturbation [16] given as

$$u(x,t) = u_0(x,t) + ae^{-p_v(1)((1-x)/\varepsilon)}, \quad v = 1, 2, 3, \tag{25}$$

where $u_0(x,t)$ is outer solution.

This implies the approximate solution at (x_i, t_j) is given as:

$$\lim_{h \to 0} u_i^j = u_o^j(0) + ae^{(-p_v(1))/\varepsilon}e^{p_v(1)i\rho}. \tag{26}$$

Using Equation (26), we have

$$\lim_{h \to 0} u_{i+1}^{j+1} - u_{i-1}^{j+1} = ae^{(-p_v(1))/\varepsilon}e^{p_v(1)i\rho}\left(e^{p_v(1)\rho} - e^{-p_v(1)\rho}\right), \tag{27a}$$

and

$$\lim_{h \to 0} u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1} = ae^{(-p_v(1))/\varepsilon}e^{p_v(1)i\rho}\left(e^{p_v(1)\rho} - 2 + e^{-p_v(1)\rho}\right). \tag{27b}$$

Thus, using Equation (24), (27a), and (27b), we get

$$\sigma_v = \frac{p_v(i)\rho}{2} \coth\left[\frac{p_v(1)\rho}{2}\right], \quad v = 1, 2, 3. \qquad (28)$$

Substituting Equation (28) into Equation (23), we get the following tri-diagonal system of equations that can be solved using Thomas algorithm.

$$E_{i,v}^{j+1} u_{i-1}^{j+1} + F_{i,v}^{j+1} u_i^{j+1} + G_{i,v}^{j+1} u_{i+1}^{j+1} = H_{i,v}^{j+1}, \qquad (29)$$

for $i = 1, 2, \cdots, m-1, j = 0, 1, \cdots, n-1$ $v = 1, 2, 3$, where

$$E_{i,v}^{j+1} = \frac{-\varepsilon\sigma_v}{h^2} - \frac{p_v(x_i)}{2h} \quad F_{i,v}^{j+1} = \frac{2\varepsilon\sigma_v}{h^2} + q_v(x_i),$$
$$G_{i,v}^{j+1} = \frac{-\varepsilon\sigma_v}{h^2} + \frac{p_v(x_i)}{2h} \quad H_{i,v}^{j+1} = g_v(x_i, t). \qquad (30)$$

Since $q_v(x_i)$ is nonnegative, then the system of Equation (29) becomes diagonally dominant.

$$i.e. \left| F_{i,v}^{j+1} \right| > \left| E_{i,v}^{j+1} \right| + \left| G_{i,v}^{j+1} \right|. \qquad (31)$$

Thus, the present method have convergent solution.

3.3. Convergence Analysis

Lemma 5 (Discrete Maximum Principle). *Suppose that the mesh function $w^{j+1}(x_i)$ satisfies $w^{j+1}(x_0) \geq 0$ and $w^{j+1}(x_m) \geq 0$. If $L^\varepsilon w^{j+1}(x_i) \geq 0$ for $1 \leq i \leq m-1$, then $w^{j+1}(x_i) > 0$, for all $i, 0 \leq i \leq m$.*

Proof. Let $w^{j+1}(x_y) = \min_{1 \leq i \leq m-1} w^{j+1}(x_i)$ and suppose that $w^{j+1}(x_y) < 0$, then

$$L^\varepsilon w^{j+1}(x_y) = -\varepsilon\sigma_v \frac{w^{j+1}(x_{y-1}) - 2w^{j+1}(x_y) + w^{j+1}(x_{y+1})}{h^2}$$
$$+ p_v(x_y) \frac{w^{j+1}(x_{y+1}) - w^{j+1}(x_{y-1})}{2h}$$
$$+ q_v w^{j+1}(x_y) < 0, \qquad (32)$$

which contradicts our assumption.□   □

Thus, $w^{j+1}(x_i) > 0$, for all $i, 0 \leq i \leq m$.

Lemma 6 (Stability Estimate). *The solution U_i^{j+1} of the discrete method satisfies the bound*

$$\left| U_i^{j+1} \right| \leq \theta^{-1} \max \left| L^\varepsilon U_i^{j+1} \right| + \max \left| \varphi_l(0, t_{j+1}), \varphi_r(1, t_{j+1}) \right|, \qquad (33)$$

where $q(x_i) \geq \theta > 0$.

Proof. Define the barrier function $\psi_{i,j+1}^\pm$ as

$$\psi_{i,j+1}^\pm = \theta^{-1} \max |L^\varepsilon U_i^{j+1}| + \max |\varphi_l(0, t_{j+1}), \varphi_r(1, t_{j+1})| \pm U_i^{j+1}. \qquad (34)$$

In the barrier function at the boundary condition, we obtain

$$\psi_{j+1}^\pm(0) = \theta^{-1} \max |L^\varepsilon U_i^{j+1}| + \max |\varphi_l(0, t_{j+1}), \varphi_r(1, t_{j+1})| \pm \varphi_l(0, t_{j+1}) \geq 0,$$
$$\psi_{j+1}^\pm(1) = \theta^{-1} \max |L^\varepsilon U_i^{j+1}| + \max |\varphi_l(0, t_{j+1}), \varphi_r(1, t_{j+1})| \pm \varphi_r(1, t_{j+1}) \geq 0. \qquad (35)$$

The barrier function at spatial domain x_i, $1 \leq i \leq m-1$; we obtain

$$L^\varepsilon \psi_{i,j+1}^\pm = L^\varepsilon \left[\theta^{-1} \max | L^\varepsilon U_i^{j+1} | + \max | \varphi_l(0, t_{j+1}), \varphi_r(1, t_{j+1}) | \right]$$
$$\pm L^\varepsilon U_i^{j+1} = q_v(x_i) \left[\theta^{-1} \max | L^\varepsilon U_i^{j+1} | \right.$$
$$\left. + \max | \varphi_l(0, t_{j+1}), \varphi_r(1, t_{j+1}) | \right] \pm L^\varepsilon U_i^{j+1} \geq 0. \qquad (36)$$

Using Lemma 5, we have that $\psi_{i,j+1}^\pm \geq 0$ for all $(x_i, t_{j+1}) \in \bar{\Omega}$. Thus, the required bound in Equation (33) is satisfied. □

Consistency of the method: Consider the Taylor series expansion of

$$u_{i+1}^{j+1} = u_i^{j+1} + hu_{x,i}^{j+1} + \frac{h^2}{2} u_{xx,i}^{j+1} + \frac{h^3}{3!} u_{xxx,i}^{j+1} + \cdots u_{i-1}^{j+1}$$
$$= u_i^{j+1} - hu_{x,i}^{j+1} + \frac{h^2}{2} u_{xx,i}^{j+1} - \frac{h^3}{3!} u_{xxx,i}^{j+1} + \cdots u_i^j \qquad (37)$$
$$= u_i^{j+1} - ku_{t,i}^{j+1} + \frac{k^2}{2} u_{tt,i}^{j+1} - \frac{k^3}{3!} u_{ttt,i}^{j+1} + \cdots.$$

Using the above expansion of Equation (22) becomes

$$
\begin{aligned}
&\frac{1}{k}\left[u_i^{j+1} - \left(u_i^{j+1} - ku_{t,i}^{j+1} + \frac{k^2}{2}u_{tt,i}^{j+1} - \cdots\right)\right] \\
&- \frac{\varepsilon}{h^2}\left[\left(u_i^{j+1} + hu_{x,i}^{j+1} + \frac{h^2}{2}u_{xx,i}^{j+1} + \frac{h^3}{3!}u_{xxx,i}^{j+1} + \cdots\right)\right. \\
&\quad - 2u_i^{j+1} + \left.\left(u_i^{j+1} - hu_{x,i}^{j+1} + \frac{h^2}{2}u_{xx,i}^{j+1} - \frac{h^3}{3!}u_{xxx,i}^{j+1} + \cdots\right)\right] \\
&+ \frac{A(x_i)}{2h}\left[\left(u_i^{j+1} + hu_{x,i}^{j+1} + \frac{h^2}{2}u_{xx,i}^{j+1} + \frac{h^3}{3!}u_{xxx,i}^{j+1} + \cdots\right)\right. \\
&\quad - \left.\left(u_i^{j+1} - hu_{x,i}^{j+1} + \frac{h^2}{2}u_{xx,i}^{j+1} - \frac{h^3}{3!}u_{xxx,i}^{j+1} + \cdots\right)\right] \\
&+ B(x_i)u_i^{j+1} + C(x_i)u\left(x_i - \gamma, t_{j+1}\right) \\
&+ D(x_i)u\left(x_i + \mu, t_{j+1}\right) - F\left(x_i, t^{j+1}\right).
\end{aligned}
\tag{38}
$$

After simplifying the terms, the truncation error becomes

$$
\begin{aligned}
T.E. &= u_{i,t}^{j+1} - \varepsilon u_{xx,i}^{j+1} + A(x_i)u_{x,i}^{j+1} + B(x_i)u_i^{j+1} \\
&\quad + C(x_i)u\left(x_i - \gamma, t_{j+1}\right) + D(x_i)u\left(x_i + \mu, t_{j+1}\right)u_i^{j+1} \\
&\quad - F\left(x_i, t^{j+1}\right) - \frac{k}{2}u_{tt,i}^{j+1} - \frac{\varepsilon h^2}{12}u_{xxxx,i}^{j+1} + A(x_i)\frac{h^2}{6}u_{xxx,i}^{j+1},
\end{aligned}
\tag{39}
$$

which gives

$$
T.E. = -\frac{k}{2}u_{tt,i}^{j+1} - \frac{\varepsilon h^2}{12}u_{xxxx,i}^{j+1} + A(x_i)\frac{h^2}{6}u_{xxx,i}^{j+1}.
\tag{40}
$$

Thus, $\lim_{(h,k)\rightarrow(0,0)} T.E. = 0$, which shows that the present method is consistent. Since the method is consistent and stable, then using Lax equivalence theorem, the present method is convergent.

Theorem 7. *Let the exact solution and numerical solution of Equation (1), respectively, are u and U. Then,*

$$
\sup_{0<\varepsilon\leq 1}\max_{i,j}|u(x_i, t_j) - U_i^j| \leq M(h^2 + k).
\tag{41}
$$

Proof. For the proof, one can refer [17].□ □

4. Richardson Extrapolation Approach

The Richardson extrapolation approach has been described, and it is designed to improve the accuracy of the computed solutions in the basic scheme.

Let $D_2^n \subseteq D_2^{2n}$, where D_2^{2n} is the mesh obtained from bisecting the step size k. Denote the numerical solution obtained from D_2^{2n} by $\bar{U}^j(x)$, we have

$$
u^j(x) - U^j(x) \leq Ck + R_j^n(x), \left(x, t_j\right) \in D_1 \times D_2^n,
\tag{42a}
$$

$$
u^j(x) - \bar{U}^j(x) \leq C\left(\frac{k}{2}\right) + R_j^{2n}(x), \left(x, t_j\right) \in D_1 \times D_2^{2n},
\tag{42b}
$$

where $R_j^n(x)$ and $R_j^{2n}(x)$ are the remainder terms of the error. Now, subtracting the inequality (42b) from (42a) to obtain the extrapolation formula.

$$
u^j(x) - U^j(x) - 2\left(u^j(x) - \bar{U}^j(x)\right) = R_j^n(x) - R_j^{2n},
\tag{43}
$$

which gives that

$$
U_j^{\text{ext}}(x) = 2\bar{U}^j(x) - U^j(x),
\tag{44}
$$

is an approximate solution [12].

Theorem 8. *Let $u(x_i, t_{j+1})$ and $U_{i,j+1}^{\text{ext}}$ be the solution of problems in (1) and (44), respectively, then the proposed scheme satisfies the following error estimate*

$$
\sup_{0<\varepsilon<1}\max_{x_i, t_{j+1}}|u\left(x_i, t_{j+1}\right) - U_{i,j+1}^{\text{ext}}| \leq C\left(h^2 + k^2\right).
\tag{45}
$$

Proof. Using the error for the temporal and spatial discretization gives the required bound.□ □

5. Numerical Examples

To determine the efficacy of the current scheme, we looked at model problems that had been addressed in the literature and had approximate solutions that could be compared.

We used the double-mesh principle to estimate the absolute maximum error of the current approach when the exact solution for the given problem was unknown. We use the following formula to approximate the absolute maximum error at the selected mesh points:

Case 1. If the exact solution is known,

$$
E_\varepsilon^{M,N} = \max_{\left(x_i, t_j\right)\in\Omega}\left|u\left(x_i, t_j\right) - u_i^{\text{ext},j}\right|.
\tag{46}
$$

Case 2. If the exact solution is unknown,

$$
E_\varepsilon^{M,N} = \max_{\left(x_i, t_j\right)\in\Omega}\left|\left(u_i^{\text{ext},j}\right)^{M,N} - \left(u_i^{\text{ext},j}\right)^{2M,2N}\right|.
\tag{47}
$$

TABLE 1: Maximum absolute point-wise error and rate of convergence for Example 1 before and after Richardson extrapolation method applied, where $\gamma = 0.5\varepsilon$, $\mu = 0.6\varepsilon$, and $L \geq 8$.

$\varepsilon \downarrow$ M,N $\longrightarrow$	32,32	64,64	128,128	256,256	512,512
Before					
10^{-2}	5.5909e-03	2.7792e-03	1.3847e-03	6.9081e-04	3.4501e-04
10^{-4}	6.2590e-03	3.3063e-03	1.5449e-03	7.7016e-04	3.8448e-04
10^{-6}	6.2656e-03	3.1096e-03	1.5465e-03	7.7096e-04	3.8488e-04
10^{-8}	6.2657e-03	3.1096e-03	1.5465e-03	7.7097e-04	3.8488e-04
10^{-10}	6.2657e-03	3.1096e-03	1.5465e-03	7.7097e-04	3.8488e-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-L}	6.2657e-03	3.1096e-03	1.5465e-03	7.7097e-04	3.8488e-04
Rate of convergence					
10^{-L}	1.0107	1.0077	1.0043	1.0022	
After					
10^{-2}	1.8542e-04	5.5542e-05	1.5119e-05	3.9402e-06	1.0055e-06
10^{-4}	2.2699e-04	6.6325e-05	1.7858e-05	4.6297e-06	1.1786e-06
10^{-6}	2.2742e-04	6.6437e-05	1.7886e-05	4.6369e-06	1.1804e-06
10^{-8}	2.2743e-04	6.6438e-05	1.7887e-05	4.6370e-06	1.1804e-06
10^{-10}	2.2743e-04	6.6438e-05	1.7887e-05	4.6370e-06	1.1804e-06
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-L}	2.2743e-04	6.6438e-05	1.7887e-05	4.6370e-06	1.1804e-06
Rate of convergence					
10^{-L}	1.7753	1.8931	1.9476	1.9739	

We also evaluate the corresponding rate of convergence.

$$R^{M,N} = \frac{\log E_\varepsilon^{M,N} - \log E_\varepsilon^{2M,2N}}{\log 2}. \quad (48)$$

Example 1. Let $A(x) = (2 - x^2)$, $B(x) = (x - 3)$, $C(x) = -2$, $D(x) = -1$, $F(x,t) = 10t^2 e^{-t} x(1 - x)$, where $(x,t) \in (0,1) \times (0,1]$, and with initial boundary condition,

$$u(x,t) = 0, \quad \forall (x,t) \in \Omega_l = \{(x,t): -\gamma \leq x \leq 0, \text{and } 0 \leq t \leq 1\},$$
$$u(x,t) = 0, \quad \forall (x,t) \in \Omega_r = \{(x,t): 1 \leq x \leq 1 + \mu, \text{and } 0 \leq t \leq 1\},$$
$$u(x,t) = 0, \quad \forall (x,t) \in \Omega_b = \{(x,t): 0 \leq x \leq 1, \text{and } 0 \leq t \leq 1\}. \quad (49)$$

Example 2. Let $A(x) = (1 + x + x^2)$, $B(x) = (1 + x^2)$, $C(x) = -(0.25 + 0.5x^2)$, $D(x) = -0.25$, $F(x,t) = \sin(\pi x)(1 - x)$, where $(x,t) \in (0,1) \times (0,1]$, with initial and boundary condition,

$$u(x,t) = 0, \quad \forall (x,t) \in \Omega_l = \{(x,t): -\gamma \leq x \leq 0, \text{and } 0 \leq t \leq 1\},$$
$$u(x,t) = 0, \quad \forall (x,t) \in \Omega_r = \{(x,t): 1 \leq x \leq 1 + \mu, \text{and } 0 \leq t \leq 1\},$$
$$u(x,t) = 0, \quad \forall (x,t) \in \Omega_b = \{(x,t): 0 \leq x \leq 1, \text{and } 0 \leq t \leq 1\}. \quad (50)$$

Example 3. Let $A(x) = (1 - x^2/2)$, $B(x) = (x + 6)$, $C(x) = -4$, $D(x) = -1$, $F(x,t) = x(1 - x)$, where $(x,t) \in (0,1) \times (0,3]$, with initial and boundary condition,

$$u(x,t) = 0, \quad \forall (x,t) \in \Omega_l = \{(x,t): -\gamma \leq x \leq 0, \text{and } 0 \leq t \leq 3\},$$
$$u(x,t) = 0, \quad \forall (x,t) \in \Omega_r = \{(x,t): 1 \leq x \leq 1 + \mu, \text{and } 0 \leq t \leq 3\},$$
$$u(x,t) = 0, \quad \forall (x,t) \in \Omega_b = \{(x,t): 0 \leq x \leq 1, \text{and } 0 \leq t \leq 3\}. \quad (51)$$

6. Discussions and Results

We have presented the method for solving spatial delayed singularly perturbed parabolic partial differential equation. The basic mathematical procedures are defining the model problem, decomposing into three equations, approximating time variable using implicit Euler's method, approximating the delay term using Taylor series expansion of order two, approximating the spatial variable using the central difference method, and finding fitting factor. Finally, apply Richardson extrapolation method to accelerate the accuracy of the method.

Three model examples are used to exemplify the performance of the proposed method. The maximum error and rate of convergence are shown in Tables 1–3 with different values of ε, delay parameters, and mesh length. The physical

TABLE 2: Maximum absolute point-wise error rate of convergence for Example 2 before and after Richardson extrapolation method applied, where $\gamma = 0.5\varepsilon$, $\mu = 0.6\varepsilon$, and $L \geq 8$.

$\varepsilon\downarrow$ M,N $\longrightarrow$	16,16	32,32	64,64	128,128	256,256
Before					
10^{-2}	8.9343e-03	4.4707e-03	2.2286e-03	1.1098e-03	5.5384e-04
10^{-4}	9.3800e-03	4.7241e-03	2.3512e-03	1.1687e-03	5.8202e-04
10^{-6}	9.3844e-03	4.7269e-03	2.3526e-03	1.1693e-03	5.8233e-04
10^{-8}	9.3845e-03	4.7269e-03	2.3526e-03	1.1694e-03	5.8233e-04
10^{-10}	9.3845e-03	4.7269e-03	2.3526e-03	1.1694e-03	5.8233e-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-L}	9.3845e-03	4.7269e-03	2.3526e-03	1.1694e-03	5.8233e-04
Rate of convergence					
10^{-L}	0.9894	1.0066	1.0085	1.0058	
After					
10^{-2}	1.8103e-03	5.1433e-04	1.5297e-04	3.9939e-05	1.0095e-05
10^{-4}	2.0571e-03	6.2361e-04	1.9584e-04	5.1831e-05	1.3148e-05
10^{-6}	2.0599e-03	6.2482e-04	1.9636e-04	5.1977e-05	1.3185e-05
10^{-8}	2.0599e-03	6.2483e-04	1.9637e-04	5.1979e-05	1.3186e-05
10^{-10}	2.0599e-03	6.2483e-04	1.9637e-04	5.1979e-05	1.3186e-05
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-L}	2.0599e-03	6.2483e-04	1.9637e-04	5.1979e-05	1.3186e-05
Rate of convergence					
10^{-L}	1.7210	1.6699	1.9176	1.9789	

TABLE 3: Maximum absolute point-wise error and rate of convergence for Example 3 before and after Richardson extrapolation method applied, where $\gamma = 0.1\varepsilon$, $\mu = 0.2\varepsilon$, and $L \geq 6$.

| $\varepsilon\downarrow$ M,N $\longrightarrow$	8,8	16,16	32,32	64,64	128,128
Before					
10^{-2}	6.1105e-03	4.0192e-03	2.4412e-03	1.4312e-03	7.9252e-04
10^{-4}	6.5144e-03	4.2051e-03	2.5534e-03	1.4901e-03	8.2602e-04
10^{-6}	6.5190e-03	4.2076e-03	2.5547e-03	1.4908e-03	8.2641e-04
10^{-8}	6.5190e-03	4.2076e-03	2.5547e-03	1.4908e-03	8.2641e-04
10^{-10}	6.5190e-03	4.2076e-03	2.5547e-03	1.4908e-03	8.2641e-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-L}	6.5190e-03	4.2076e-03	2.5547e-03	1.4908e-03	8.2641e-04
After					
10^{-2}	1.9128e-03	9.9156e-04	4.3385e-04	2.1411e-04	4.8046e-05
10^{-4}	2.0383e-03	1.0236e-03	4.5811e-04	2.1921e-04	5.4179e-05
10^{-6}	2.0407e-03	1.0240e-03	4.5837e-04	2.1924e-04	5.4255e-05
10^{-8}	2.0407e-03	1.0240e-03	4.5837e-04	2.1924e-04	5.4256e-05
10^{-10}	2.0407e-03	1.0240e-03	4.5837e-04	2.1924e-04	5.4256e-05
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-L}	2.0407e-03	1.0240e-03	4.5837e-04	2.1924e-04	5.4256e-05
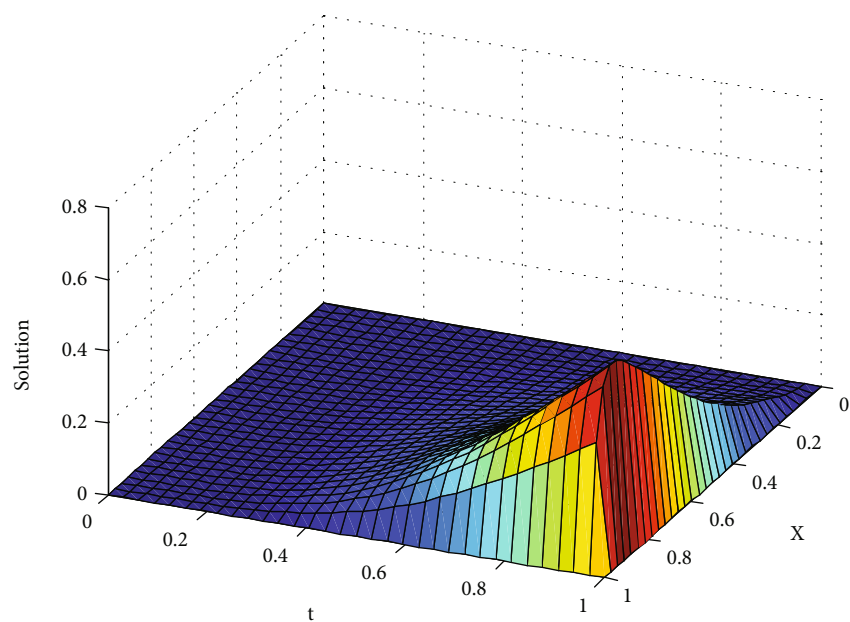
FIGURE 1: The physical behavior of the solutions for Example 1 at $m = n = 32$, $\varepsilon = 10^{-2}$, $\gamma = 0.5\varepsilon$, and $\mu = 0.6\varepsilon$.
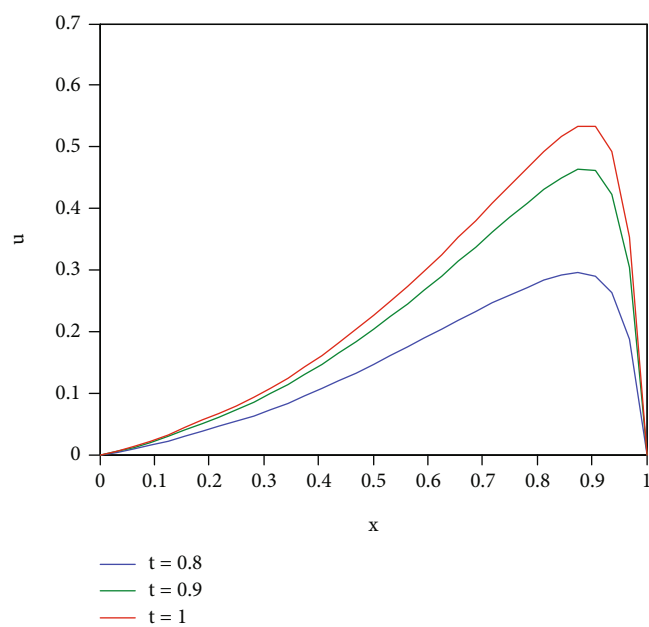

FIGURE 2: The physical behavior of the solutions for Example 1 at $m = n = 32$, $\varepsilon = 10^{-2}$, $\gamma = 0.5\varepsilon$, and $\mu = 0.6\varepsilon$ at different time level.

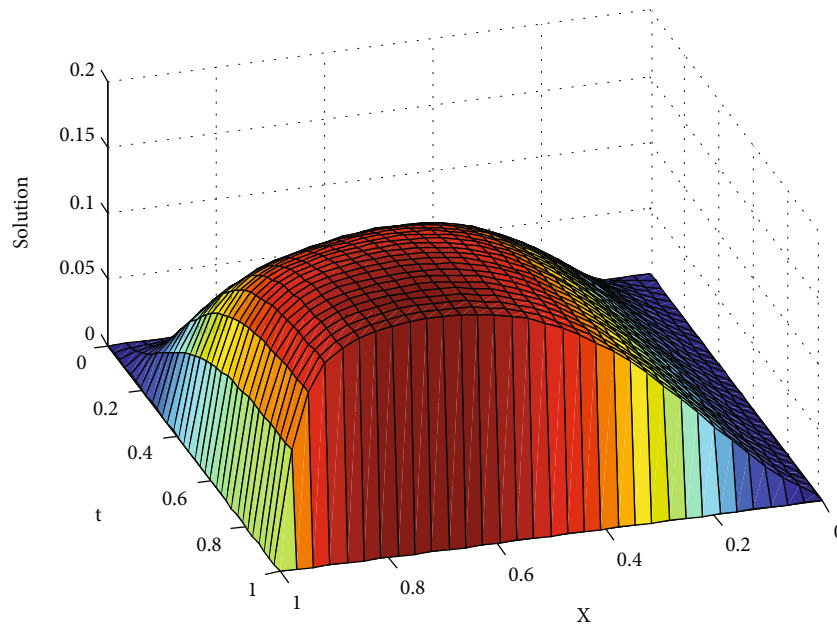Figure 3: The physical behavior of the solutions for Example 2 at $m = n = 64$, $\varepsilon = 10^{-2}$, $\gamma = 0.5\varepsilon$, and $\mu = 0.6\varepsilon$.
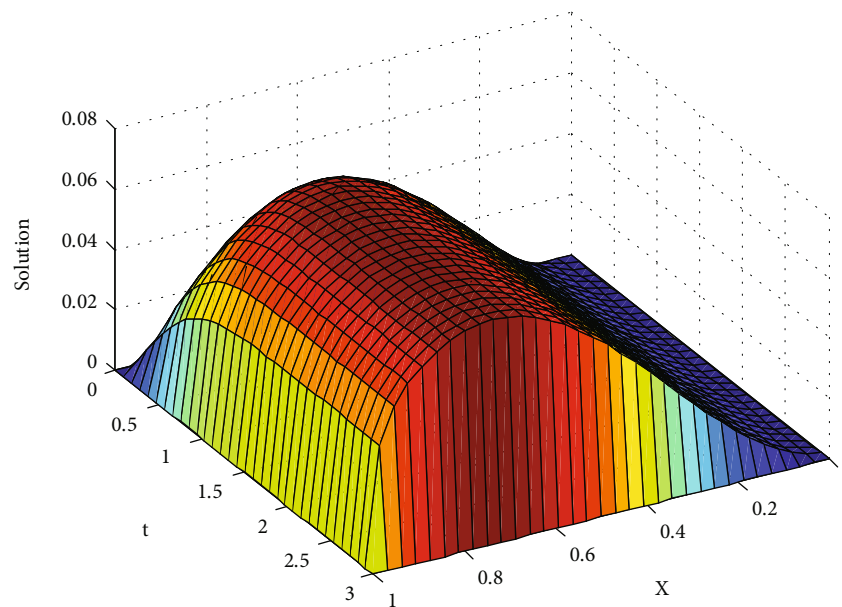

Figure 4: The physical behavior of the solutions for Example 3 at $m = n = 32$, $\varepsilon = 10^{-2}$, $\gamma = 0.1\varepsilon$, and $\mu = 0.2\varepsilon$.

behavior of the solution are shown in Figures 1–4. We examined the suggested numerical scheme for stability, consistency, and ε uniform convergence. As shown in the result, the current method is second-order convergent with respect to time and spatial variables, the rate of convergence is two and more accurate than some of the methods that appear in the literature.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] M. K. Kadalbajoo and V. Gupta, "A brief survey on numerical methods for solving singularly perturbed problems," *Applied Mathematics and Computation*, vol. 217, no. 8, pp. 3641–3716, 2010.

[2] K. K. Sharma, P. Rai, and K. C. Patidar, "A review on singularly perturbed differential equations with turning points and

interior layers," *Applied Mathematics and Computation*, vol. 219, no. 22, pp. 10575–10609, 2013.

[3] N. Mbroh, S. C. Oukouomi Noutchie, and R. Y. M'pika Massoukou, "A robust method of lines solution for singularly perturbed delay parabolic problem," *Alexandria Engineering Journal*, vol. 59, no. 4, pp. 2543–2554, 2020.

[4] C. Clavero and J. L. Gracia, "A higher order uniformly convergent method with Richardson extrapolation in time for singularly perturbed reaction-diffusion parabolic problems," *Journal of Computational and Applied Mathematics*, vol. 252, pp. 75–85, 2013.

[5] M. Woldaregay and G. Duressa, "Uniformly convergent hybrid numerical method for singularly perturbed delay convection-diffusion problems," *Arab Journal of Mathematical Sciences*, vol. 2021, article 6654495, pp. 1–20, 2021.

[6] D. Kumar and P. Kumari, "A parameter-uniform scheme for singularly perturbed partial differential equations with a time lag," *Numerical Methods for Partial Differential Equations*, vol. 36, no. 4, pp. 868–886, 2020.

[7] P. Pramod Chakravarthy, S. Dinesh Kumar, and R. Nageshwar Rao, "An exponentially fitted finite difference scheme for a class of singularly perturbed delay differential equations with large delays," *Ain Shams Engineering Journal*, vol. 8, no. 4, pp. 663–671, 2017.

[8] K. Bansal and K. K. Sharma, "Parameter-robust numerical scheme for time-dependent singularly perturbed reaction–diffusion problem with large delay," *Numerical Functional Analysis and Optimization*, vol. 39, no. 2, pp. 127–154, 2018.

[9] V. Gupta, M. Kumar, and S. Kumar, "Higher order numerical approximation for time dependent singularly perturbed differential-difference convection-diffusion equations," *Numerical Methods for Partial Differential Equations*, vol. 34, no. 1, pp. 357–380, 2018.

[10] M. M. Woldaregay and G. F. Duressa, "Parameter uniform numerical method for singularly perturbed parabolic differential difference equations," *Journal of the Nigerian Mathematical Society*, vol. 38, 2019.

[11] A. Das and S. Natesan, "Second-order uniformly convergent numerical method for singularly perturbed delay parabolic partial differential equations," *International Journal of Computer Mathematics*, vol. 95, no. 3, pp. 490–510, 2018.

[12] M. Singh and S. Natesan, "Richardson extrapolation technique for singularly perturbed system of parabolic partial differential equations with exponential boundary layers," *Applied Mathematics and Computation*, vol. 333, pp. 254–275, 2018.

[13] P. P. Chakravarthy and K. Kumar, "An adaptive mesh method for time dependent singularly perturbed differential-difference equations," *Nonlinear Engineering*, vol. 8, 2019.

[14] K. Bansal, P. Rai, and K. K. Sharma, "Numerical treatment for the class of time dependent singularly perturbed parabolic problems with general shift arguments," *Differential Equations and Dynamical Systems*, vol. 25, no. 2, pp. 327–346, 2017.

[15] O. Ladyženskaja, V. Solonnikov, N. Ural'ceva, and S. Smith, "Linear and quasilinear equations of parabolic type," *Translations of Mathematical Monographs*, vol. 23, 1968.

[16] H.-G. Roos, "Robust numerical methods for singularly perturbed differential equations," *Springer Series in Computational Mathematics*, vol. 24, 2008.

[17] E. B. M. Bashier and K. C. Patidar, "A second-order fitted operator finite difference method for a singularly perturbed delay parabolic partial differential equation," *Journal of Difference Equations and Applications*, vol. 17, no. 5, pp. 779–794, 2011.