Research Article

Simultaneous Developability of Partner Ruled Surfaces according to Darboux Frame in E^3

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In this paper, we introduce original definitions of Partner ruled surfaces according to the Darboux frame of a curve lying on an arbitrary regular surface in E^3 . It concerns Tg Partner ruled surfaces, Tn Partner ruled surfaces, and gn Partner ruled surfaces. We aim to study the simultaneous developability conditions of each couple of two Partner ruled surfaces. Finally, we give an illustrative example for our study.

1. Introduction

The theory of ruled surfaces forms an important and useful class of theories in differential geometry [1, 2]. This kind of surface is defined by the moving of a straight line along a curve. The various positions of the generating lines are called the rulings of the ruled surface. Such a surface, thus, has a parametric representation of the form

$$\phi : (s; v) \in I \times \mathbb{R} \mapsto c(s) + vX(s), \quad (1)$$

where I is an open interval of $\mathbb{R}$, $c(s)$ is called the base curve, and $X(s)$ are the ruling directors.

One of the most interesting properties related to ruled surface is the property of developability. It defines ruled surfaces that can be transformed into the plane without any deformation and distortion; such surfaces form relatively small subsets that contain cylinders, cones, and tangent surfaces. They are characterized with vanishing Gaussian curvature [3–5].

Many geometers have studied some of the differential geometric concepts of the ruled surfaces by means of different moving frames, such as the Frenet-Serret frame, alternative frame, and Bishop frame [6–8].

Another one of the most important moving frame of the differential geometry is the Darboux frame, which is a natu-

ral moving frame constructed on a surface that contains a curve. It is named after the French mathematician Jean Gaston Darboux, in a four-volume collection of the studies he published between 1887 and 1896. Since that time, there have been many important repercussions of the Darboux frame, having been examined for example in (Darboux, 1896; O'Neill, 1996). One can find studies of ruled surfaces with the Darboux frame realized in Euclidean and non-Euclidean 3-space. For example, in [9], the authors constructed the ruled surface whose rulings are constant linear combinations of the Darboux frame vectors of its base curve along a regular surface of reference; they studied the most important properties of that ruled surface, characterized it, and presented examples with illustrations. Furthermore, in [10], the authors studied the characteristic properties of a ruled surface with the Darboux frame and gave the relationship between the Darboux frame and the Frenet frame. Moreover, in [11], the authors defined the evolute offsets of ruled surface with the Darboux frame and studied its characteristic properties in E^3 .

The main contribution of this work is to introduce new special couples of ruled surfaces defined by means of Darboux frame vectors of a regular curve lying on an arbitrary regular surface in E^3 . Our objective is to study the simultaneous developability of such couples of surfaces. Through our study, we are opening up some avenues for scientists

to apply our approach in some areas such as architectural design, medical science, surface modeling, engineering, and computer-aided geometric design [12–15].

The principle of this study is to consider a unit speed curve $c(s)$ lying on an arbitrary regular surface ϕ , associate the Darboux frame $\{T, g, n\}$ of $c(s)$ on ϕ , and, then, define three couples of ruled surfaces that are generated, reciprocally, by T, g , and n . We call them Tg Partner ruled surfaces, Tn Partner ruled surfaces, and gn Partner ruled surfaces, respectively. We aim to study the simultaneous developability of each couple of Partner ruled surfaces. Indeed, we investigate theorems that reply to our needs. Finally, we present an example with illustrations.

2. Preliminaries

Due to a unit speed curve $\alpha = \alpha(s)$ that lies on a regular surface $\phi = \phi(u, r)$, i.e., $\alpha(s) = \phi(u(s), r(s))$, there exists the Darboux frame and it is denoted by $\{\vec{T}(s), \vec{g}(s), \vec{n}(s)\}$, where $\vec{T}(s) = \alpha'(s)$ is the unit tangent vector of the curve $\alpha = \alpha(s)$, $\vec{n}(s) = ((\phi_u \times \phi_r) / \|\phi_u \times \phi_r\|)(u(s), r(s))$ is the unit normal vector of the surface $\phi = \phi(u, r)$ along the curve $\alpha = \alpha(s)$, and $\vec{g}(s)$ is the unit vector which is defined by $\vec{g}(s) = \vec{n}(s) \times \vec{T}(s)$.

The derivative formulae of the Darboux frame are given as follows:

$$\begin{bmatrix} \vec{T}' \\ \vec{g}' \\ \vec{n}' \end{bmatrix} = \begin{bmatrix} 0 & \rho_g & \rho_n \\ -\rho_g & 0 & \theta_g \\ -\rho_n & -\theta_g & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{g} \\ \vec{n} \end{bmatrix}, \quad (2)$$

where ρ_g is the geodesic curvature, ρ_n is the normal curvature, and θ_g is the geodesic torsion of the curve $\alpha = \alpha(s)$ on the surface $\phi = \phi(u, r)$.

Definition 1 (see [16]). The curve $\alpha(s)$ lying on a regular surface is as follows:

- (i) A geodesic curve if its geodesic curvature ρ_g vanishes
- (ii) An asymptotic line if its normal curvature ρ_n vanishes
- (iii) A principal line if its geodesic torsion θ_g vanishes

Let $\varphi : (s, v) \mapsto c(s) + v\vec{X}(s)$ be a ruled surface in E^3 .

Let denote by $\vec{n} = \vec{n}(s, v)$, the unit normal on the ruled surface φ at a regular point $\varphi(s, v)$, we have

$$\vec{n} = \frac{\varphi_s \wedge \varphi_v}{\|\varphi_s \wedge \varphi_v\|} = \frac{(c' + v\vec{X}') \times \vec{X}}{\|(c' + v\vec{X}') \times \vec{X}\|}, \quad (3)$$

where $\varphi_s = \partial\varphi(s, v)/\partial s$ and $\varphi_v = \partial\varphi(s, v)/\partial v$.

The first I and the second II fundamental forms of the ruled surface φ at a regular point $\varphi(s, v)$ are defined, respectively, by

$$\begin{aligned} I(\varphi_s ds + \varphi_v dv) &= E ds^2 + 2F ds dv + G dv^2, \\ II(\varphi_s ds + \varphi_v dv) &= e ds^2 + 2f ds dv + g dv^2, \end{aligned} \quad (4)$$

where

$$\begin{aligned} E &= \|\varphi_s\|^2, & F &= \langle \varphi_s, \varphi_v \rangle, & G &= \|\varphi_v\|^2, \\ e &= \langle \varphi_{ss}, \vec{n} \rangle, & f &= \langle \varphi_{sv}, \vec{n} \rangle, & g &= \langle \varphi_{vv}, \vec{n} \rangle = 0. \end{aligned} \quad (5)$$

Definition 2. The Gaussian curvature K of the ruled surface φ at a regular point $\varphi(s, v)$ is given by

$$K = -\frac{f^2}{EG - F^2}. \quad (6)$$

Proposition 3 (see [16]). A ruled surface is developable if and only if its Gaussian curvature vanishes.

3. Simultaneous Developability of Partner Ruled Surfaces according to the Darboux Frame in E^3

Definition 4. Let $c : s \in I \mapsto c(s)$ be a C^2 -class differentiable unit speed curve lying on a regular surface $\phi = \phi(u, r)$. Let denote by $\{T(s), g(s), n(s)\}$ the Darboux frame of $c = c(s)$ on $\phi = \phi(u, r)$. The two ruled surfaces defined by

$$\begin{cases} {}^{Tg}\varphi : (s, v) \in I \times \mathbb{R} \mapsto T(s) + v g(s), \\ {}^{gT}\varphi : (s, v) \in I \times \mathbb{R} \mapsto g(s) + v T(s), \end{cases} \quad (7)$$

are called Tg Partner ruled surfaces according to the Darboux frame of the curve $c(s)$ on the surface $\phi = \phi(u, r)$.

Theorem 5. Tg Partner ruled surfaces (7) are simultaneously developable if and only if at least one of the following statements is verified:

- (i) $c = c(s)$ is a geodesic curve on the surface ϕ
- (ii) $c = c(s)$ is an asymptotic and a principal line on the surface ϕ

Proof. By differentiating the first line of (7) with respect to s and v , respectively, and using Darboux derivative formulae (2), we get

$$\begin{cases} {}^{Tg}\varphi_s = -v\rho_g T + \rho_g g + (\rho_n + v\theta_g)n, \\ {}^{Tg}\varphi_v = g. \end{cases} \quad (8)$$

Then, by considering the cross product of both vectors in (8), we get the normal vector of the ruled surface ${}^Tg\varphi$:

$${}^Tg\varphi_s \times {}^Tg\varphi_v = -(\rho_n + v\theta_g)T - v\rho_g n, \quad (9)$$

which implies that under regularity condition, the unit normal vector of the ruled surface ${}^Tg\varphi$ is given by

$$\frac{{}^Tg\varphi_s \times {}^Tg\varphi_v}{\|{}^Tg\varphi_s \times {}^Tg\varphi_v\|} = -\frac{1}{\sqrt{(\rho_n + v\theta_g)^2 + v^2\rho_g^2}} [(\rho_n + v\theta_g)T + v\rho_g n]. \quad (10)$$

By applying the norms and the scalar product for both vectors in (8), we get the components of the first fundamental form of the ruled surface ${}^Tg\varphi$:

$${}^TgE = (1 + v)\rho_g^2 + (\rho_n + v\theta_g)^2, \quad (11)$$

$${}^TgF = \rho_g, \quad (12)$$

$${}^TgG = 1. \quad (13)$$

By differentiating the second line of (8) with respect to s , using Darboux derivative formulae (2) and making the scalar product with the unit normal (10), we get the second component of the second fundamental form of the ruled surface ${}^Tg\varphi$:

$${}^Tgf = -\frac{\rho_n\rho_g}{\sqrt{(\rho_n + v\theta_g)^2 + v^2\rho_g^2}}. \quad (14)$$

Thus, from (13) and (14), we get the Gaussian curvature of the ruled surface ${}^Tg\varphi$:

$${}^TgK = -\left(\frac{\rho_n\rho_g}{(\rho_n + v\theta_g)^2 + v^2\rho_g^2}\right)^2. \quad (15)$$

On another hand, by differentiating the second line of (7) with respect to s and v and using Darboux derivative formulae (2), we get

$$\begin{cases} {}^gT\varphi_s = -\rho_g T + v\rho_g g + (\theta_g + v\rho_n)n, \\ {}^gT\varphi_v = T. \end{cases} \quad (16)$$

Then, the cross product of both vectors of (16) gives the normal vector of the ruled surface ${}^gT\varphi$:

$${}^gT\varphi_s \times {}^gT\varphi_v = (\theta_g + v\rho_n)g - v\rho_n n, \quad (17)$$

which implies that under regularity condition, the unit normal vector of the ruled surface ${}^gT\varphi$ is given by

$$\frac{{}^gT\varphi_s \times {}^gT\varphi_v}{\|{}^gT\varphi_s \times {}^gT\varphi_v\|} = \frac{1}{(\theta_g + v\rho_n)^2 + v^2\rho_g^2} [(\theta_g + v\rho_n)g - v\rho_g n]. \quad (18)$$

By applying the norms and the scalar product for both vectors (16), we obtain the components of the first fundamental form of the ruled surface ${}^gT\varphi$:

$${}^gTE = \rho_g^2 + v^2\rho_g^2 + (\theta_g + v\rho_n)^2, \quad (19)$$

$${}^gTF = -\rho_g, \quad (20)$$

$${}^gTG = 1. \quad (21)$$

By differentiating the second line of (16) with respect to s , using Darboux derivative formulae (2) and using the unit normal (18), we get the second component of the second fundamental form of the ruled surface ${}^gT\varphi$:

$${}^gTf = \frac{\theta_g\rho_g}{(\theta_g + v\rho_n)^2 - v^2\rho_g^2}. \quad (22)$$

Hence, from (21) and (22), we get the Gaussian curvature of the ruled surface ${}^gT\varphi$:

$${}^gTK = -\left(\frac{\theta_g\rho_g}{(\theta_g + v\rho_n)^2 - v^2\rho_g^2}\right)^2. \quad (23)$$

Consequently, from (15) and (23), we deduce Tg Partner ruled surfaces

${}^gT\varphi$ and ${}^Tg\varphi$ are simultaneously developable if and only if ${}^TgK = {}^gTK = 0$, i.e., $\rho_n\rho_g = \theta_g\rho_g = 0$, which is equivalent to $\rho_g = 0$ or $\rho_n = \theta_g = 0$. Thus, Theorem 5 is proved. $\square$

Definition 6. Let $c : s \in I \mapsto c(s)$ be a C^2 -class differentiable unit speed curve lying on a regular surface $\phi = \phi(u, r)$. Let denote by $\{T(s), g(s), n(s)\}$ the Darboux frame of $c = c(s)$ on $\phi = \phi(u, r)$. The two ruled surfaces defined by

$$\begin{cases} {}^{Tn}\varphi : (s, v) \in I \times \mathbb{R} \mapsto T(s) + vn(s), \\ {}^{nT}\varphi : (s, v) \in I \times \mathbb{R} \mapsto n(s) + vT(s), \end{cases} \quad (24)$$

are called Tn Partner ruled surfaces according to the Darboux frame of the curve $c = c(s)$ on the surface $\phi = \phi(u, r)$.

Theorem 7. Tn Partner ruled surfaces (24) are simultaneously developable if and only if at least one of the following statements is verified:

- (i) $c = c(s)$ is an asymptotic line on the surface ϕ
- (ii) $c = c(s)$ is a geodesic curve and a principal line on the surface ϕ

Proof. Differentiating the first line of (24) with respect to s and v and using Darboux derivative formulae (2), we get

$$\begin{cases} {}^{Tn}\varphi_s = -v\rho_n T + (\rho_g - v\theta_g)g + \rho_n n, \\ {}^{Tn}\varphi_v = n, \end{cases} \quad (25)$$

then, the cross product of both last vectors gives us the normal vector of the ruled surface ${}^{Tn}\varphi$:

$${}^{Tn}\varphi_s \times {}^{Tn}\varphi_v = (\rho_g - v\theta_g)T + v\rho_n g, \quad (26)$$

which implies that under regularity condition, the unit normal vector of the ruled surface ${}^{Tn}\varphi$ takes the following form:

$$\frac{{}^{Tn}\varphi_s \times {}^{Tn}\varphi_v}{\|{}^{Tn}\varphi_s \times {}^{Tn}\varphi_v\|} = \frac{1}{\sqrt{(\rho_g - v\theta_g)^2 + v^2\rho_n^2}} \left[(\rho_g - v\theta_g)T + v\rho_n g \right]. \quad (27)$$

By applying the norms and the scalar product for both vectors (25), we get the components of the first fundamental form of the ruled surface ${}^{Tn}\varphi$:

$${}^{Tn}E = v^2\rho_n^2 + (\rho_g - v\theta_g)^2 + \rho_n^2, \quad (28)$$

$${}^{Tn}F = \rho_n, \quad (29)$$

$${}^{Tn}G = 1. \quad (30)$$

By differentiating the second line of (25) with respect to s , using (2) and (27), we get the second component of the second fundamental form of the ruled surface ${}^{Tn}\varphi$:

$${}^{Tn}f = -\frac{\rho_n \rho_g}{\sqrt{(\rho_g - v\theta_g)^2 + v^2\rho_n^2}}. \quad (31)$$

Thus, from (30) and (31), we obtain the Gaussian curvature of the ruled surface ${}^{Tn}\varphi$:

$${}^{Tn}K = -\left(\frac{\rho_n \rho_g}{(\rho_g - v\theta_g)^2 + v^2\rho_n^2} \right)^2. \quad (32)$$

Let us now differentiate the second line of (24) with respect to s and v , respectively, and using Darboux derivative formulae (2), we get

$$\begin{cases} {}^{nT}\varphi_s = -\rho_n T + (-\theta_g + v\rho_g)g + v\rho_n n, \\ {}^{nT}\varphi_v = T. \end{cases} \quad (33)$$

The cross product of both vectors (33) gives the normal vector of the ruled surface ${}^{nT}\varphi$:

$${}^{nT}\varphi_s \times {}^{nT}\varphi_v = v\rho_n g - (-\theta_g + v\rho_g)n, \quad (34)$$

which gives, under regularity condition, the unit normal vector of the ruled surface ${}^{nT}\varphi$ as follows:

$$\frac{{}^{nT}\varphi_s \times {}^{nT}\varphi_v}{\|{}^{nT}\varphi_s \times {}^{nT}\varphi_v\|} = \frac{1}{\sqrt{v^2\rho_n^2 + (-\theta_g + v\rho_g)^2}} \left[v\rho_n g - (-\theta_g + v\rho_g)n \right]. \quad (35)$$

The norms and scalar product applied on both vectors in (33) give us the components of the first fundamental form of the ruled surface ${}^{nT}\varphi$:

$${}^{nT}E = (\rho_n^2 + v^2\rho_n^2) + (-\theta_g + v\rho_g)^2, \quad (36)$$

$${}^{nT}F = -\rho_n^{nT}, \quad (37)$$

$$G = 1. \quad (38)$$

Differentiating the second line of (33) with respect to s , using Darboux derivative formulae (2), and using the unit normal (35), we get the second component of the second fundamental form of the ruled surface ${}^{nT}\varphi$:

$${}^{nT}f = \frac{\rho_n \theta_g}{\sqrt{v^2\rho_n^2 + (-\theta_g + v\rho_g)^2}}. \quad (39)$$

Thus, from (38) and (39), we obtain the Gaussian curvature of the ruled surface ${}^{nT}\varphi$ as follows:

$${}^{nT}K = -\left(\frac{\rho_n \theta_g}{v^2\rho_n^2 + (-\theta_g + v\rho_g)^2} \right)^2. \quad (40)$$

Consequently, from (32) and (40), we deduce Tn Partner ruled surfaces

${}^{Tn}\varphi$ and ${}^{nT}\varphi$ are simultaneously developable if and only if ${}^{Tn}K = {}^{nT}K = 0$, i.e., $\rho_n \rho_g = \rho_n \theta_g = 0$, which is equivalent to $\rho_n = 0$ or $\rho_g = \theta_g = 0$. Thus, Theorem 7 is proved. $\square$

Definition 8. Let $c : s \in I \mapsto c(s)$ be a C^2 -class differentiable unit speed curve lying on a regular surface $\phi = \phi(u, r)$. Let denote by $\{T(s), g(s), n(s)\}$ the Darboux frame of $c = c(s)$ on $\phi = \phi(u, r)$. The two ruled surfaces defined by

$$\begin{cases} {}^{gn}\varphi : (s, v) \in I \times \mathbb{R} \mapsto g(s) + vn(s), \\ {}^{ng}\varphi : (s, v) \in I \times \mathbb{R} \mapsto n(s) + vg(s), \end{cases} \quad (41)$$

are called *gn* Partner ruled surfaces according to the Darboux frame of the curve $c = c(s)$ on the surface $\phi = \phi(u, r)$.

Theorem 9. *gn Partner ruled surfaces are simultaneously developable if and only if at least one of the following statements is verified:*

- (i) $c = c(s)$ is a principal line on the surface ϕ
- (ii) $c = c(s)$ is a geodesic curve and an asymptotic line on the surface ϕ

Proof. Differentiating the first line of (41) with respect to s and v , respectively, and using Darboux derivative formulae (2), we get

$$\begin{cases} {}^{gn}\varphi_s = -(\rho_g + v\rho_n)T - v\theta_g g + \theta_g n, \\ {}^{gn}\varphi_v = n. \end{cases} \quad (42)$$

We get the normal vector of the ruled surface ${}^{gn}\varphi$ by realizing the cross product of both vectors (42):

$${}^{gn}\varphi_s \times {}^{gn}\varphi_v = -v\theta_g T + (\rho_g + v\rho_n)g, \quad (43)$$

which implies that under regularity condition, the unit normal vector of the ruled surface ${}^{gn}\varphi$ is given by

$$\frac{{}^{gn}\varphi_s \times {}^{gn}\varphi_v}{\|{}^{gn}\varphi_s \times {}^{gn}\varphi_v\|} = \frac{1}{\sqrt{v^2\theta_g^2 + (\rho_g + v\rho_n)^2}} [-v\theta_g T + (\rho_g + v\rho_n)g]. \quad (44)$$

By applying the norms and the scalar product for both vectors of (42), we obtain

$${}^{gn}E = (\rho_g + v\rho_n)^2 + v^2\theta_g^2 + \theta_g^2, \quad (45)$$

$${}^{gn}F = \theta_g, \quad (46)$$

$${}^{gn}G = 1. \quad (47)$$

By differentiating the second line of (42) with respect to s , using (2) and (44), we obtain

$${}^{gn}f = -\frac{\theta_g \rho_g}{\sqrt{v^2\theta_g^2 + (\rho_g + v\rho_n)^2}}. \quad (48)$$

Hence, from (47) and (48), we obtain the Gaussian curvature of ruled surface ${}^{gn}\varphi$:

$${}^{gn}K = -\left(\frac{\theta_g \rho_g}{v^2\theta_g^2 + (\rho_g + v\rho_n)^2}\right)^2. \quad (49)$$

Let us now differentiate the second line of (41) with respect to s and v , respectively, and use Darboux derivative formulae (2), we get

$$\begin{cases} {}^{ng}\varphi_s = -(\rho_n + v\rho_g)T - \theta_g g + v\theta_g n, \\ {}^{ng}\varphi_v = g. \end{cases} \quad (50)$$

By realizing the cross product of both vectors of (50), we get the normal vector of the ruled surface ${}^{ng}\varphi_v$:

$${}^{ng}\varphi_s \times {}^{ng}\varphi_v = -v\theta_g T - (\rho_n + v\rho_g)n, \quad (51)$$

which implies that under regularity condition, the unit normal vector of the ruled surface ${}^{ng}\varphi_v$ is given by

$$\frac{{}^{ng}\varphi_s \times {}^{ng}\varphi_v}{\|{}^{ng}\varphi_s \times {}^{ng}\varphi_v\|} = -\frac{1}{v^2\theta_g^2 + (\rho_n + v\rho_g)^2} [v\theta_g T + (\rho_n + v\rho_g)n]. \quad (52)$$

By applying the norms and the scalar product for (50), we obtain

$${}^{ng}E = (\rho_n + v\rho_g)^2 + \theta_g^2 + v^2\theta_g^2, \quad (53)$$

$${}^{ng}F = -\theta_g, \quad (54)$$

$${}^{ng}G = 1. \quad (55)$$

By differentiating the second line of (50) with respect to s , using (2) and (52), we get

$${}^{ng}f = -\frac{\theta_g \rho_n}{v^2\theta_g^2 + (\rho_n + v\rho_g)^2}. \quad (56)$$

Hence, from (55) and (56), we obtain the Gaussian curvature of the ruled surface ${}^{ng}\varphi_v$ as follows:

$${}^{ng}K = -\left(\frac{\theta_g \rho_n}{v^2\theta_g^2 + (\rho_n + v\rho_g)^2}\right)^2. \quad (57)$$

Consequently, from (49) and (57), we deduce that *gn* Partner ruled surfaces ${}^{gn}\varphi$ and ${}^{ng}\varphi$ are simultaneously developable if and only if ${}^{gn}K = {}^{ng}K = 0$, i.e., $\theta_g \rho_g = \theta_g \rho_n = 0$, which is equivalent to $\theta_g = 0$ or $\rho_g = \rho_n = 0$. Thus, Theorem 9 is proved. $\square$

Here follows, we give an example of our study and present corresponding illustrations.

FIGURE 1: Ruled surface $Tg\varphi$ of (60). FIGURE 2: Ruled surface $gT\varphi$ of (60).

Example 1. Let us consider the regular surface parameterized by

$${}^e\varphi(u, r) = \left(2 \cos \frac{u}{2} - \frac{r}{\sqrt{2}} \sin \frac{u}{2}, 2 \sin \frac{u}{2} - \frac{r}{\sqrt{2}} \cos \frac{u}{2}, \frac{r}{\sqrt{2}} \right). \quad (58)$$

It is easy to see that the curve ${}^ec(s) = (2 \cos(s/2), 2 \sin(s/2), 0)$ lies on the regular surface ${}^e\varphi$; indeed, we have ${}^ec(s) = {}^e\varphi(s, 0)$.

The Darboux frame vectors and the Darboux invariants of ${}^ec(s)$ on ${}^e\varphi$ are given, respectively, by

$$T(s) = \left(-\sin \left(\frac{s}{2} \right), \cos \left(\frac{s}{2} \right), 0 \right),$$

$$g(s) = \frac{1}{\sqrt{1 + \sin^2(s)}} \left(-\sin(s) \cos \left(\frac{s}{2} \right), -\sin(s) \sin \left(\frac{s}{2} \right), 1 \right),$$

$$n(s) = \frac{1}{\sqrt{1 + \sin^2(s)}} \left(\cos \left(\frac{s}{2} \right), \sin \left(\frac{s}{2} \right), \sin(s) \right),$$

$$\rho_g(s) = \frac{\sin(s)}{\sqrt{1 + \sin^2(s)}},$$

$$\rho_n(s) = -\frac{1}{2\sqrt{1 + \sin^2(s)}},$$

$$\theta_g(s) = -\frac{\cos(s)}{1 + \sin^2(s)}. \quad (59)$$

Hence, Tg Partner ruled surfaces, Tn Partner ruled surfaces, and gn Partner ruled surfaces according to the

FIGURE 3: Ruled surface $T^n\varphi$ of (61).

FIGURE 4: Ruled surface $n^T\varphi$ of (61).

FIGURE 5: Ruled surface $g^m\varphi$ of (62).

FIGURE 6: Ruled surface $^{ng}\varphi$ of (62).

Darboux frame of ${}^e c(s)$ on ${}^e\varphi$ are given, respectively, as follows:

$$\begin{cases} {}^{Tg}\varphi = \left(-\sin\left(\frac{s}{2}\right), \cos\left(\frac{s}{2}\right), 0 \right) + \nu R(s) \left(-\sin(s) \cos\left(\frac{s}{2}\right), -\sin(s) \sin\left(\frac{s}{2}\right), 1 \right), \\ {}^{gT}\varphi = R(s) \left(-\sin(s) \cos\left(\frac{s}{2}\right), -\sin(s) \sin\left(\frac{s}{2}\right), 1 \right) + \nu \left(-\sin\left(\frac{s}{2}\right), \cos\left(\frac{s}{2}\right), 0 \right), \end{cases} \quad (60)$$

$$\begin{cases} {}^{Tn}\varphi = \left(-\sin\left(\frac{s}{2}\right), \cos\left(\frac{s}{2}\right), 0 \right) + \nu R(s) \left(\cos\left(\frac{s}{2}\right), \sin\left(\frac{s}{2}\right), \sin(s) \right), \\ {}^{nT}\varphi = R(s) \left(\cos\left(\frac{s}{2}\right), \sin\left(\frac{s}{2}\right), \sin(s) \right) + \nu \left(-\sin\left(\frac{s}{2}\right), \cos\left(\frac{s}{2}\right), 0 \right), \end{cases} \quad (61)$$

$$\begin{cases} {}^{gn}\varphi = R(s) \left[\left(-\sin(s) \cos\left(\frac{s}{2}\right), -\sin(s) \sin\left(\frac{s}{2}\right), 1 \right) + \nu \left(\cos\left(\frac{s}{2}\right), \sin\left(\frac{s}{2}\right), \sin(s) \right) \right], \\ {}^{ng}\varphi = R(s) \left[\left(\cos\left(\frac{s}{2}\right), \sin\left(\frac{s}{2}\right), \sin(s) \right) + \nu \left(-\sin(s) \cos\left(\frac{s}{2}\right), -\sin(s) \sin\left(\frac{s}{2}\right), 1 \right) \right], \end{cases} \quad (62)$$

where $R(s) = 1/\sqrt{1 + \sin^2(s)}$.

Here follows, we present the illustrations of the Tg Partner ruled surface (60) represented by Figures 1 and 2, the Tn Partner ruled surfaces (61) represented by Figures 3 and 4, and the gn Partner ruled surfaces (62) represented by Figures 5 and 6, respectively.

4. Conclusion

In this paper, we presented a novel method to construct three special couples of ruled surfaces according to the Darboux frame of a regular curve $c(s)$ on a regular surface in E^3 . These three couples of surfaces were called Tg Partner ruled surfaces, Tn Partner ruled surfaces, and gn Partner ruled surfaces, respectively. We investigated theorems that give necessary and sufficient conditions for each couple of two Partner ruled surfaces to be simultaneously developable. The obtained results reveal that the simultaneous developability conditions are related to the properties of the curve $c(s)$ on the surface. Finally, we presented an illustrative example.

On the other hand, our approach can also provide excellent support for architectural design, surface modeling, computer-aided geometric design, and engineering application.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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