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Research Article

Uncertainty Principles for Heisenberg Motion Group

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In this article, we will recall the main properties of the Fourier transform on the Heisenberg motion group $G = \mathbb{H}^n \rtimes K$, where $K = U(n)$ and $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ denote the Heisenberg group. Then, we will present some uncertainty principles associated to this transform as Beurling, Hardy, and Gelfand-Shilov.

1. Introduction

In Harmonic analysis, the uncertainty principle states that a nonzero function and its Fourier transform cannot simultaneously decay very rapidly. This fact is expressed by several versions which were proved by Hardy, Cowling-Price, Morgan, and Gelfand-Shilov [1, 2].

In more recent times, Beurling gave a different approach to expressing this uncertainty principle. The proof of the theorem was given by Hörmander [3], and it states that if $f \in L^2(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)| \left|\hat{f}(y)\right| e^{|x||y|} dx dy < \infty, \qquad (1)$$

then, $f = 0$ almost everywhere.

The above theorem of Hörmander was further generalized by Bonami, Demange, and Jaming [4], as follows:

Theorem 1. *Let $N \geq 0$ and let $f \in L^2(\mathbb{R}^n)$ satisfying*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)| \left|\hat{f}(y)\right|}{(1+|x|+|y|)^N} e^{|x||y|} dx dy < \infty. \qquad (2)$$

Then, $f = 0$ almost everywhere whenever $N \leq n$, and if $N > n$, then $f(x) = P(x) e^{-a|x|^2}$, where a is a positive real number and P is a polynomial on $\mathbb{R}^n$ of degree $< (N-n)/2$.

This last theorem admits another modified version proved by Parui and Sarkar [5]. It is of the following form.

Theorem 2. *Let $\delta \geq 0$ and $f \in L^2(\mathbb{R}^n)$ be such that*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)| \left|\hat{f}(y)\right| |Q(y)|^\delta}{(1+|x|+|y|)^N} e^{|x||y|} dx dy < \infty, \qquad (3)$$

where Q is a polynomial of degree m. Then, $f(x) = P(x) e^{-a|x|^2}$, where a is a positive real number and P is a polynomial with $\deg(P) < (N-n-m\delta)/2$.

Beurling's theorem has been extended to different settings. Huang and Liu established an analogue of Beurling's theorem on the Heisenberg group [6]. An analogue of Beurling's theorem for Euclidean motion groups was also formulated by Sarkar and Thangavelu [7].

In [8], Baklouti and Thangavelu gave an analogue of Hardy's theorem for the Heisenberg motion group by means of the heat kernel and also proved an analogue of Miyachi's theorem and Cowling-Price uncertainty principle. In my paper, we would like to establish other uncertainty principles such as Beurling's theorem and Gelfand-Shilov and prove Hardy's theorem as a consequence of Beurling's theorem.

This paper is organized as follows. In Section 2, we present the group G and the Fourier transform on G, and we will cite some of its fundamental properties. Section 3 is devoted to formulate and prove an analogue of Beurling's theorem associated to the group Fourier transform on the Heisenberg motion group and prove a modified version of this principle. Finally, we derive some other versions of uncertainty principles such as Hardy uncertainty principle and Gelfand-Shilov.

2. Heisenberg Motion Group

Let $\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with the group law

$$(z,t).(w,s) = \left(z + w, t + s + \frac{1}{2}\,\mathrm{Im}\,(z.\bar{w})\right), \tag{4}$$

where $z, w \in \mathbb{C}^n$, $t, s \in \mathbb{R}$.

Let K be the unitary group $U(n)$, we define the Heisenberg motion group G to be the semidirect product of $\mathbb{H}^n$ and K, with the group law

$$(z,t,k)(w,s,h) = ((z,t).(kw,s), kh), \tag{5}$$

where $(z,t), (w,s) \in \mathbb{H}^n$, $k, h \in K$.

The Haar measure on G is given by $dg = dzdtdk$, where $dzdt$ and dk are the normalized Haar measures on $\mathbb{H}^n$ and K , respectively.

For $\lambda \in \mathbb{R} \setminus \{0\}$, we define the Schrödinger representation of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$ by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda\left(x.\xi + \left(\frac{1}{2}\right)x.y\right)} \varphi(\xi + y), \tag{6}$$

where $z = x + iy$ and $\varphi \in L^2(\mathbb{R}^n)$.

Let $(\sigma, \mathscr{H}_\sigma)$ be any irreducible, unitary representation of K. For each $\lambda \neq 0$, we consider the representations ρ_σ^λ of G on the tensor product space $L^2(\mathbb{R}^n) \otimes \mathscr{H}_\sigma$ defined by

$$\rho_\sigma^\lambda(z,t,k) = (\pi_\lambda(z,t)\mu_\lambda(k)) \otimes \sigma(k), \tag{7}$$

where μ_λ are the metaplectic representations [9], satisfying

$$\pi_\lambda(kz,t) = \mu_\lambda(k)\pi_\lambda(z,t)\mu_\lambda(k)^*, \text{for all}(z,t,k) \in G. \tag{8}$$

Proposition 1 [9]. *Each ρ_σ^λ is unitary and irreducible.*

For $f \in L^1 \cap L^2(G)$, consider the group Fourier transform

$$\begin{aligned} \widehat{f}(\lambda,\sigma) &= \int_K \int_{\mathbb{R}} \int_{\mathbb{C}^n} f(z,t,k)\rho_\sigma^\lambda(z,t,k)\,dzdtdk \\ &= \int_K \int_{\mathbb{C}^n} f^\lambda(z,k)\rho_\sigma^\lambda(z,k)\,dzdk, \end{aligned} \tag{9}$$

where $\rho_\sigma^\lambda(z,k) = \rho_\sigma^\lambda(z,0,k)$ and the partial Fourier transform $f^\lambda(z,k)$ is defined by

$$f^\lambda(z,k) = \int_{\mathbb{R}} f(z,t,k)e^{i\lambda t}\,dt. \tag{10}$$

For $f \in L^1 \cap L^2(G)$, we have

$$\int_K \int_{\mathbb{C}^n} \left|f^\lambda(z,k)\right|^2 dzdk = (2\pi)^{-n}|\lambda|^n \sum_{\sigma\in\widehat{K}} d_\sigma \left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2, \tag{11}$$

and the Plancherel formula for the Fourier transform on G reads as

$$\int_K \int_{\mathbb{H}^n} |f(z,t,k)|^2 dzdtdk = \sum_{\sigma\in\widehat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2 d\tau(\lambda), \tag{12}$$

where $d\tau(\lambda) = (2\pi)^{-n-1}|\lambda|^n d\lambda$ is the measure defined on $\mathbb{R} \setminus \{0\}$, d_σ is the dimension of the space $\mathscr{H}_\sigma$, and $\|\widehat{f}(\lambda,\sigma)\|_{HS}^2$ denote the Hilbert-Schmidt norm of $\widehat{f}(\lambda,\sigma)$ [9].

At the end of this paragraph, we introduce an orthonormal basis for $L^2(\mathbb{C}^n \times K)$ [10]. Let $H_k(t)$ be the Hermite polynomials defined by

$$H_k(t) = (-1)^k e^{t^2} \frac{d^k}{dt^k}\left(e^{-t^2}\right). \tag{13}$$

The normalized Hemite functions are defined by

$$h_k(t) = \left(2^k\sqrt{\pi}k!\right)^{-1/2} H_k(t)e^{-(1/2)t^2}. \tag{14}$$

The n-dimensional Hermite functions Φ_α are defined on $\mathbb{R}^n$ by taking the tensor products; that is,

$$\Phi_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j), \tag{15}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$.

It is well known that $\{\Phi_\alpha, \alpha \in \mathbb{N}^n\}$ form an orthonormal basis for $L^2(\mathbb{R}^n)$ [2]. Then, an orthonormal basis for $L^2(\mathbb{R}^n) \otimes \mathscr{H}_\sigma$ is given by $B_\sigma = \{\Phi_\alpha \otimes e_i^\sigma : \alpha \in \mathbb{N}^n, 1 \le i \le d_\sigma\}$, where $\{e_i^\sigma : 1 \le i \le d_\sigma\}$ is an orthonormal basis for $\mathscr{H}_\sigma$ and $d_\sigma = \dim \mathscr{H}_\sigma$.

Define the Fourier-Wigner transform V_f^g of $f, g \in L^2(\mathbb{R}^n) \otimes \mathscr{H}_\sigma$ on $\mathbb{C}^n \times K$ by

$$V_f^g(z,k) = (2\pi)^{-n/2}\left\langle \rho_\sigma^1(z,k)f, g\right\rangle. \tag{16}$$

Lemma 1 [10]. *For $f_l, g_l \in L^2(\mathbb{R}^n) \otimes \mathscr{H}_\sigma$, $l = 1, 2$, the following identity holds.*

$$\int_K \int_{\mathbb{C}^n} V_{f_1}^{g_1}(z,k)\overline{V_{f_2}^{g_2}(z,k)}\,dzdk = \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}. \tag{17}$$

In particular, $V_f^g \in L^2(\mathbb{C}^n \times K)$, for $f, g \in L^2(\mathbb{R}^n) \otimes \mathscr{H}_\sigma$.

Set $\Psi^\sigma_{\alpha,i} = \Phi_\alpha \otimes e^\sigma_i$, then, by Lemma 1, we infer that the set

$$V_{B_\sigma} = \left\{ V^{\Psi^\sigma_{\beta,j}}_{\Psi^\sigma_{\alpha,i}} : \Psi^\sigma_{\alpha,i}, \Psi^\sigma_{\beta,j} \in B_\sigma \right\} \tag{18}$$

is an orthonormal basis for $V_\sigma = \overline{\text{span}}\{V^g_f : f, g \in L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma\}$.

Proposition 2 [10]. *The family $\mathcal{B} = \{V_{B_\sigma} : \sigma \in \widehat{K}\}$ is an orthonormal basis for $L^2(\mathbb{C}^n \times K)$.*

Lemma 2. *The function $(z,k) \mapsto V^{\Psi^\sigma_{\beta,j}}_{\Psi^\sigma_{\alpha,i}}(z,k)$ is a bounded function.*

Proof. Let $(z,k) \in \mathbb{C}^n \times K$ and $\Psi^\sigma_{\alpha,i}, \Psi^\sigma_{\beta,j} \in B_\sigma$, we have

$$\begin{aligned} V^{\Psi^\sigma_{\beta,j}}_{\Psi^\sigma_{\alpha,i}}(z,k) &= (2\pi)^{-n/2} \left\langle \rho^1_\sigma(z,k)\Phi_\alpha \otimes e^\sigma_i, \Phi_\beta \otimes e^\sigma_j \right\rangle \\ &= (2\pi)^{-n/2} \left\langle \pi_1(z)\mu_1(k) \otimes \sigma(k)\Phi_\alpha \otimes e^\sigma_i, \Phi_\beta \otimes e^\sigma_j \right\rangle \\ &= (2\pi)^{-n/2} \langle \pi_1(z)\mu_1(k)\Phi_\alpha, \Phi_\beta \rangle \left\langle \sigma(k)e^\sigma_i, e^\sigma_j \right\rangle. \end{aligned} \tag{19}$$

We know that $\mu_1(k)\Phi_\alpha = \sum_{|\gamma|=|\alpha|} \langle \mu_1(k)\Phi_\alpha, \Phi_\gamma \rangle \Phi_\gamma$ (see [9] p.21); then

$$\left| V^{\Psi^\sigma_{\beta,j}}_{\Psi^\sigma_{\alpha,i}}(z,k) \right| \le (2\pi)^{-n/2} \sum_{|\gamma|=|\alpha|} |\langle \mu_1(k)\Phi_\alpha, \Phi_\gamma \rangle| |\langle \pi_1(z)\Phi_\gamma, \Phi_\beta \rangle| \left| \left\langle \sigma(k)e^\sigma_i, e^\sigma_j \right\rangle \right|. \tag{20}$$

Since π_1, μ_1, and σ are unitary representations, then,

$$|V^{\Psi^\sigma_{\beta,j}}_{\Psi^\sigma_{\alpha,i}}(z,k)| \le (2\pi)^{-n/2} \frac{(m+n-1)!}{m!(n-1)!}, \tag{21}$$

where $m = |\alpha|$. □

3. An Analogue of Beurling's Theorem

In this section, we prove an analogue of Beurling's theorem on the Heisenberg motion group $G = \mathbb{H}^n \rtimes K$ whose statement is as follows:

Theorem 3. *Let $f \in L^2(G)$ and $d \ge 0$. Suppose that*

$$\int_G \int_{\mathbb{R}} \frac{|f(z,t,k)| \left(\sum_{\sigma\in\widehat{K}} d_\sigma \left\| \widehat{f}(\lambda,\sigma) \right\|^2_{HS} \right)^{1/2}}{(1+|t|+|\lambda|)^d} e^{|t||\lambda|} d\tau(\lambda) dz dt dk < \infty. \tag{22}$$

Then,

$$f(z,t,k) = e^{-at^2} \left(\sum_{j=0}^m \varphi_j(z,k) t^j \right), \tag{23}$$

where $a > 0$, $\varphi_j \in L^1 \cap L^2(\mathbb{C}^n \times K)$ and $m < ((d - n/2 - 1)/2)$.

Lemma 3. *If f satisfies the hypotheses of Theorem 3, then, $f \in L^1(G)$.*

Proof. As f is not identically zero, then, there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$\sum_{\sigma\in\widehat{K}} d_\sigma \left\| \widehat{f}(\lambda,\sigma) \right\|^2_{HS} \ne 0. \tag{24}$$

By (22), we have

$$\int_G \frac{|f(z,t,k)|}{(1+|t|+|\lambda|)^d} e^{|t||\lambda|} dz dt dk < \infty. \tag{25}$$

On the other hand, the function $(t,\lambda) \mapsto ((1+|t|+|\lambda|)^d)/e^{|t||\lambda|}$ is bounded, so there exists a constant C such that

$$\frac{(1+|t|+|\lambda|)^d}{e^{|t||\lambda|}} \le C. \tag{26}$$

From where $\int_G |f(z,t,k)| dz dt dk \le C \int_G (|f(z,t,k)|/(1+|t|+|\lambda|)^d) e^{|t||\lambda|} dz dt dk$ and using (25), so we have $f \in L^1(G)$. □

Proof of Theorem 3. For any $\varphi \in \mathcal{S}(\mathbb{C}^n \times K)$, the Schwartz space of $\mathbb{C}^n \times K$, consider the function

$$F_\varphi(t) = \int_{\mathbb{C}^n\times K} f(z,t,k)\varphi(z,k) dz dk. \tag{27}$$

Since $f \in L^1(G)$, then, F_φ is integrable on $\mathbb{R}$, and for any $\lambda \in \mathbb{R} \setminus \{0\}$, the Fourier transform of F_φ is given by

$$\begin{aligned} \widehat{F_\varphi}(\lambda) &= (2\pi)^{-1/2} \int_{\mathbb{R}} F_\varphi(t) e^{-i\lambda t} dt \\ &= (2\pi)^{-1/2} \int_{\mathbb{R}} \int_{\mathbb{C}^n\times K} f(z,t,k)\varphi(z,k) e^{-i\lambda t} dz dk dt \\ &= (2\pi)^{-1/2} \int_{\mathbb{C}^n\times K} f^{(-\lambda)}(z,k)\varphi(z,k) dz dk, \end{aligned} \tag{28}$$

then by (11)

$$|\widehat{F_\varphi}(\lambda)| \leq (2\pi)^{-1/2}\|\varphi\|_2\left(\int_{\mathbb{C}^n\times K}\left|f^{(-\lambda)}(z,k)\right|^2 dzdk\right)^{1/2} \leq (2\pi)^{-n/2-1}\|\varphi\|_2|\lambda|^{n/2}\left(\sum_{\sigma\in\widehat{K}} d_\sigma\left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2\right)^{1/2}. \tag{29}$$

As a result,

$$I=\int_{\mathbb{R}^2}\frac{\left|F_\psi(t)\right|\left|\widehat{F_\varphi}(\lambda)\right|}{(1+|t|+|\lambda|)^d}e^{|t||\lambda|}|\lambda|^{n/2}dtd\lambda \leq (2\pi)^{n/2}\|\varphi\|_2\|\psi\|_\infty \cdot \int_G\int_{\mathbb{R}}\frac{|f(z,t,k)|\left(\sum_{\sigma\in\widehat{K}} d_\sigma\left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2\right)^{1/2}}{(1+|t|+|\lambda|)^d}e^{|t||\lambda|}dzdtdkd\tau(\lambda) < +\infty. \tag{30}$$

In particular,

$$\int_{\mathbb{R}^2}\frac{\left|F_\varphi(t)\right|\left|\widehat{F_\varphi}(\lambda)\right|}{(1+|t|+|\lambda|)^d}e^{|t||\lambda|}|\lambda|^{n/2}dtd\lambda < \infty. \tag{31}$$

Note that the previous calculations are generalized to a bounded function $\varphi \in L^2(\mathbb{C}^n \times K)$, in particular for the bounded functions φ in the basis $\mathscr{B}$ of $L^2(\mathbb{C}^n \times K)$ defined in (18).

According to Beurling's theorem in the Euclidean case, modified version (Theorem 2), for every function $\varphi \in \mathscr{B}$, there exists a polynomial function P_φ with deg $(P_\varphi) < (d - (n/2) - 1)/2$ and a real $a_\varphi > 0$ such that

$$\int_{\mathbb{C}^n\times K} f(z,t,k)\overline{\varphi(z,k)}dzdk = P_\varphi(t)e^{-a_\varphi|t|^2}, \tag{32}$$

from where

$$f(z,t,k)=\sum_{\varphi\in\mathscr{B}} P_\varphi(t)e^{-a_\varphi|t|^2}\varphi(z,k). \tag{33}$$

Let $\varphi, \psi \in \mathscr{B}$, since

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\left|F_\psi(t)\right|\left|\widehat{F_\varphi}(\lambda)\right|}{(1+|t|+|\lambda|)^d}e^{|t||\lambda|}|\lambda|^{n/2}dtd\lambda < \infty, \tag{34}$$

then by Lemma 2.2 in [5], we obtain that $a_\varphi = a_\psi = a$ are independent of φ.

Let $P_\varphi(t) = \sum_{j=0}^m c_{j,\varphi}t^j$, then

$$f(z,t,k) = e^{-at^2}\sum_{j=0}^m\left(\sum_{\varphi\in\mathscr{B}} c_{j,\varphi}\varphi(z,k)\right)\cdot t^j = e^{-at^2}\sum_{j=0}^m \xi_j(z,k)\cdot t^j, \tag{35}$$

where $\xi_j(z,k) = \sum_{j=0}^m c_{j,\varphi}\varphi_\alpha(z,k) \in L^2(\mathbb{C}^n \times K)$.

The proof of Theorem 3 is completed. □

We will finish this section with a modified version of previous Theorem 3 as follows:

Proposition 3.3. *Let $f \in L^2(G)$ and $p, d, \delta \geq 0$. Suppose that*

$$\int_G\int_{\mathbb{R}}\frac{|f(z,t,k)|\left(\sum_{\sigma\in\widehat{K}} d_\sigma\left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2\right)^{1/2}}{(1+\|z\|)^p(1+|t|+|\lambda|)^d}|\lambda|^\delta e^{|t||\lambda|}d\tau(\lambda)dzdtdk < \infty. \tag{36}$$

Then,

$$f(z,t,k)=e^{-at^2}(1+\|z\|)^p\left(\sum_{j=0}^m \varphi_j(z,k)t^j\right), \tag{37}$$

where $a > 0$, $\varphi_j \in L^1 \cap L^2(\mathbb{C}^n \times K)$ and $m < (d - (n/2) - 1 - \delta)/2$.

Proof. By replacing $f(z,t,k)$ by $f(z,t,k)/(1+\|z\|)^p$ and proceeding as in the proof of Theorem 3, one can apply Theorem 3 to get the result. □

4. Applications to Other Uncertainty Principles

Let us first state and prove the following analogue of Hardy's theorem for G.

Theorem 5 (Hardy type). *Suppose f is a measurable function on G satisfying*

(i) $|f(z,t,k)| \leq g(z,k)e^{-\alpha t^2}$, where $g \in L^1 \cap L^2(\mathbb{C}^n \times K)$ and $\alpha > 0$.

(ii) $|\lambda|^{n/2}\|\widehat{f}(\lambda,\sigma)\|_{HS} \leq c_\sigma e^{-\beta\lambda^2}$, for some $\beta > 0$ and $c_\sigma > 0$ such that $\sum_{\sigma\in\widehat{K}} d_\sigma c_\sigma^2 < +\infty$.

Then,

(1) If $\alpha\beta > 1/4$, $f = 0$ almost everywhere on G.

(2) If $\alpha\beta = 1/4$, $f(z,t,k) = e^{-\alpha t^2}f(z,0,k)$, for all $(z,t,k) \in \mathbb{C}^n \times \mathbb{R} \times K$.

Proof. From (i) and (ii), we have

$$\int_G\int_{\mathbb{R}}\frac{|f(z,t,k)|\left(\sum_{\sigma\in\widehat{K}} d_\sigma\left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2\right)^{1/2}}{(1+|t|+|\lambda|)^d}e^{|t||\lambda|}d\tau(\lambda)dzdtdk \leq C^{te} \times M, \tag{38}$$

where

$$M=\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|\lambda|^{n/2}}{(1+|t|+|\lambda|)^d}e^{-\alpha(|t|-(|\lambda|/2\alpha))^2}e^{-(\beta-(1/4\alpha))|\lambda|^2}dtd\lambda. \tag{39}$$

Then, if $\alpha\beta \geq 1/4$, the expression above for $d > (n/2) + 2$ is finite.

According to Theorem 3, there exist $a > 0$ and $\varphi_j \in L^1 \cap L^2(\mathbb{C}^n \times K)$ such that

$$f(z,t,k) = e^{-at^2}\left(\sum_{j=0}^{m} \varphi_j(z,k)t^j\right), \tag{40}$$

where $m < (d - (n/2 + 1))/2$. En particular for $(n/2) + 2 < d \leq (n/2) + 3$, we have $m = 0$ and

$$f(z,t,k) = e^{-at^2} f(z,0,k). \tag{41}$$

From condition (i), we have

$$\int_{\mathbb{C}^n\times K} |f(z,0,k)|\,dzdk \leq K \times e^{(a-\alpha)t^2}, \tag{42}$$

where K is a positive constant.

We have

$$f^\lambda(z,k) = \int_{\mathbb{R}} f(z,t,k)e^{i\lambda t}dt = \sqrt{\frac{\pi}{a}}e^{-\lambda^2/4a} f(z,0,k). \tag{43}$$

Then,

$$\int_{\mathbb{C}^n\times K} \left|f^\lambda(z,k)\right|^2 dzdk = \frac{\pi}{a}e^{-\lambda^2/2a}\int_{\mathbb{C}^n\times K} |f(z,0,k)|^2 dzdk. \tag{44}$$

From (11), we have

$$(2\pi)^{-n}|\lambda|^n \sum_{\sigma\in\widehat{K}} d_\sigma \left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2 = \frac{\pi}{a}e^{-\lambda^2/2a}\int_{\mathbb{C}^n\times K} |f(z,0,k)|^2 dzdk. \tag{45}$$

From condition (ii), we obtain

$$\int_{\mathbb{C}^n\times K} |f(z,0,k)|^2 dzdk \leq Ce^{((1/2a)-2\beta)\lambda^2}, \tag{46}$$

where C is a positive constant.

(1) Case $\alpha\beta > (1/4)$:

(i) Suppose that $a < \alpha$; then, $\lim_{t\longrightarrow+\infty} e^{(a-\alpha)t^2} = 0$ and $\int_{\mathbb{C}^n\times K} |f(z,0,k)|\,dzdk = 0$, by (42). We conclude that $f = 0$.

(ii) Suppose that $a \geq \alpha$; we have $(1/2a) < 2\beta$ and $\lim_{t\longrightarrow+\infty} e^{((1/2a)-2\beta)\lambda^2} = 0$. From (46), we have $\int_{\mathbb{C}^n\times K} |f(z,0,k)|^2 dzdk = 0$, and finally $f = 0$.

(2) Case $\alpha\beta = 1/4$:

(i) Suppose that $f \neq 0$; then (42) and (46) hold if and only if $a = \alpha$. We conclude that $f(z,t,k) = e^{-at^2} f(z,0,k)$.

□

Theorem 6 (Gelfand-Shilov type). *Let $f \in L^2(G)$ and $d \geq 0$ satisfy*

$$\int_G \frac{|f(z,t,k)|\,e^{\alpha t^2}}{(1+|t|)^d} dzdtdk < +\infty,$$
$$\int_{\mathbb{R}} \frac{\left(\sum_{\sigma\in\widehat{K}} d_\sigma \left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2\right)^{1/2} e^{\beta\lambda^2}}{(1+|\lambda|)^d} d\tau(\lambda) < +\infty, \tag{47}$$

for some positive constants α and β.

Then, $\alpha\beta > 1/4$ implies $f = 0$.

Proof. Suppose that $\alpha\beta > 1/4$, and consider the following numerical sequences $(\alpha_p)_{p\in\mathbb{N}^*}$ and $(\beta_p)_{p\in\mathbb{N}^*}$ defined by

$$\alpha_p = \alpha - \frac{1}{p},\ \beta_p = \beta - \frac{1}{p}. \tag{48}$$

(i) $\lim_{p\longrightarrow+\infty} \alpha_p = \alpha > 0$; therefore, there exists $N_\alpha \in \mathbb{N}^*$ such that $\alpha_p > 0$ for $p > N_\alpha$.

(ii) $\lim_{p\longrightarrow+\infty} \beta_p = \beta > 0$; therefore, there exists $N_\beta \in \mathbb{N}^*$ such that $\beta_p > 0$ for $p > N_\beta$.

(iii) $\lim_{p\longrightarrow+\infty} \alpha_p\beta_p = \alpha\beta > 1/4$; therefore, there exists $N_{\alpha\beta} \in \mathbb{N}^*$ such that $\alpha_p\beta_p > 1/4$ for $p > N_{\alpha\beta}$.

Let $N = \max(N_\alpha, N_\beta, N_{\alpha\beta})$, then, for all $p > N$.

$$\alpha_p > 0,\ \beta_p > 0,\ \alpha_p\beta_p > \frac{1}{4},$$
$$|t||\lambda| \leq \alpha_p t^2 + \beta_p\lambda^2. \tag{49}$$

We have

$$\frac{|f(z,t,k)|\left(\sum_{\sigma\in\widehat{K}} d_\sigma \left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2\right)^{1/2}}{(1+|t|+|\lambda|)^d} e^{|t||\lambda|} \leq \frac{|f(z,t,k)|e^{\alpha t^2}}{(1+|t|)^d} \frac{\left(\sum_{\sigma\in\widehat{K}} d_\sigma \left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2\right)^{1/2} e^{\beta\lambda^2}}{(1+|\lambda|)^d} h(t,\lambda), \quad (50)$$

where $h(t,\lambda) = (1+|t|)^d(1+|\lambda|)^d/(1+|t|+|\lambda|)^d e^{-(1/p)t^2} e^{-(1/p)\lambda^2}$. The function $h(t,\lambda)$ is a bounded function, then there exists a positive constant K such that

$$\frac{|f(z,t,k)|\left(\sum_{\sigma\in\widehat{K}} d_\sigma \left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2\right)^{1/2}}{(1+|t|+|\lambda|)^d} e^{|t||\lambda|} \leq K\frac{|f(z,t,k)|e^{\alpha t^2}}{(1+|t|)^d} \frac{\left(\sum_{\sigma\in\widehat{K}} d_\sigma \left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2\right)^{1/2} e^{\beta\lambda^2}}{(1+|\lambda|)^d}. \quad (51)$$

According to the hypotheses of the theorem, the integral

$$\int_G \int_{\mathbb{R}} \frac{|f(z,t,k)|\left(\sum_{\sigma\in\widehat{K}} d_\sigma \left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2\right)^{1/2}}{(1+|t|+|\lambda|)^d} e^{|t||\lambda|} d\tau(\lambda)dzdtdk \quad (52)$$

is finite. According to Theorem 3, there exist $a > 0$ and $\psi_j \in L^2(\mathbb{C}^n \times K)$ such that

$$f(z,t,k) = e^{-at^2}\left(\sum_{j=0}^m \varphi_j(z,k)t^j\right), \quad (53)$$

where $m < (d - (n/2 + 1))/2$.

We have

$$\begin{aligned} f^\lambda(z,k) &= \sum_{j=0}^m \varphi_j(z,k)\int_{\mathbb{R}} t^j e^{-at^2} e^{i\lambda t} \\ &= \sum_{j=0}^m \varphi_j(z,k)P_j(\lambda)e^{-(1/4a)\lambda^2} \\ &\quad \cdot \left(\text{where } P_j(\lambda) \text{ is a polynomial function}\right) \\ &= \left(\sum_{j=0}^m \varphi_j(z,k)P_j(\lambda)\right)e^{-(1/4a)t^2}. \end{aligned} \quad (54)$$

Then $\|f^\lambda\|_2 = h(\lambda)e^{-(1/4a)\lambda^2}$, where $h(\lambda)$ is a polynomial function, and from (11), we have

$$\left(\sum_{\sigma\in\widehat{K}} d_\sigma \left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2\right)^{1/2} = \frac{(2\pi)^{n/2}}{|\lambda|^{n/2}} h(\lambda)e^{-(1/4a)\lambda^2}. \quad (55)$$

From the hypotheses of the theorem, we obtain

$$\int_{\mathbb{R}} \frac{|\lambda|^{n/2}}{(1+|\lambda|)^d} h(\lambda)e^{(\beta-(1/4a))\lambda^2} d\lambda < +\infty. \quad (56)$$

(i) Suppose that $a \geq \alpha$; then $\beta - (1/4a) > 0$, so $h(\lambda) = 0$ implies $\widehat{f}(\lambda,\sigma) = 0$

By Plancherel formula (12), we conclude that $f = 0$.

(ii) Suppose that $a < \alpha$;, according to the hypotheses of the theorem, we have

$$\int_{\mathbb{R}} \frac{e^{(\alpha-a)}t^2}{(1+|t|)^d}\left(\int_{\mathbb{C}^n\times K} \left|\sum_{j=0}^m \varphi_j(z,k)t^j\right| dzdk\right)dt < +\infty. \quad (57)$$

Since $\alpha - a > 0$ and $\sum_{j=0}^m \varphi_j(z,k)t^j \in L^1(\mathbb{C}^n \times K)$ for almost every $t \in \mathbb{R}$, then, $\varphi_j = 0$ for all $j = 0, \cdots, m$ and $f = 0$. □

Corollary 1 (Another version of Hardy). *Suppose $f \in L^1 \cap L^2(G)$ satisfy*

(i) $|f(z,t,k)| \leq g(z,k)e^{-\alpha t^2}$, where $g \in L^1 \cap L^2(\mathbb{C}^n \times K)$ and $\alpha > 0$.

(ii) $\|\widehat{f}(\lambda,\sigma)\|_{HS} \leq c_\sigma e^{-\beta\lambda^2}$, for some $\beta > 0$ and $c_\sigma > 0$ such that $\sum_{\sigma\in\widehat{K}} d_\sigma c_\sigma^2 < +\infty$.

Then:

(1) If $\alpha\beta > 1/4$, $f = 0$ almost everywhere on G.

(2) If $\alpha\beta < 1/4$, there exist an infinite number of linearly independent functions meeting hypotheses (i) and (ii).

Proof.

(1) Case $\alpha\beta > 1/4$: if $\alpha\beta > 1/4$ and $d > n + 1$, then, from the hypothesis of the theorem, we have

$$\int_G \frac{|f(z,t,k)|e^{\alpha t^2}}{(1+|t|)^d} dzdtdk \leq C\int_{\mathbb{R}} \frac{1}{(1+|t|)^d} dt < +\infty,$$
$$\int_{\mathbb{R}} \frac{\left(\sum_{\sigma\in\widehat{K}} d_\sigma \left\|\widehat{f}(\lambda,\sigma)\right\|_{HS}^2\right)^{1/2} e^{\beta\lambda^2}}{(1+|\lambda|)^d} d\tau(\lambda) \leq K\int_{\mathbb{R}} \frac{|\lambda|^n}{(1+|\lambda|)^d} d\lambda < +\infty, \quad (58)$$

where $C, K > 0$. Since $\alpha\beta > 1/4$, then, the Gelfand-Shilov theorem implies that $f = 0$.

(2) Case $\alpha\beta < 1/4$: If $\alpha\beta < 1/4$, then, any function of the form $g(z,k)h_k(t)$ where h_k are the one-dimensional

Hermite functions satisfies the hypotheses of the theorem.

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Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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