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Review Article

On a $p(x)$-Biharmonic Kirchhoff Problem with Navier Boundary Conditions

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In this article, we study the existence of solutions for nonlocal $p(x)$-biharmonic Kirchhoff-type problem with Navier boundary conditions. By different variational methods, we determine intervals of parameters for which this problem admits at least one nontrivial solution.

1. Introduction

We consider the problem with Navier boundary conditions.

$$(P\lambda)\begin{cases} M\left(\int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}dx\right)\Delta_p^2 u = \lambda m(x)|u(x)|^{q(x)-2}u(x) \text{ in } \Omega, \\ u = \Delta u = 0 \text{ on } \partial\Omega \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial\Omega$, $\lambda > 0$, and $\Delta^2_{p(x)}$ is the $p(x)$-biharmonic operator defined by $\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$. $p, q \in C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}), h^- > 1\}$, where $h^- = \min_{x\in\bar{\Omega}} h(x)$. We denoted by $p^+ = \max_{x\in\bar{\Omega}} p(x)$.

We assume that the weight m and the Kirchhoff function M satisfy the following conditions:

(m). $m \in L^{\beta(x)}(\Omega)$,$m(x) > 0$ a.e in Ω, with $\beta \in C_+(\bar{\Omega})$ such that

$$q(x) < \frac{\beta(x)-1}{\beta(x)}p_2^*(x), \tag{2}$$

where

$$p_2^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)} & \text{if } p(x) < \frac{N}{2}, \\ +\infty & \text{if } p(x) \geq \frac{N}{2}. \end{cases} \tag{3}$$

(M_1). $M : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function verifying

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1} \forall t > 0, \tag{4}$$

where m_1, m_2, α are real numbers such that $0 < m_1 \leq m_2$ and $\alpha > 1$.

Example 1. A typical example of (P_λ) satisfying the conditions (m)-(M_1) is given by

$$\left(\frac{m_1+m_2}{2}\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}\right)^{\alpha-1}\Delta^2_{p(x)}u = \lambda|x|^{-s(x)}|u|^{q(x)-2}u \text{ in } \Omega,$$
$$u = \Delta u = 0 \text{ on } \partial\Omega, \tag{5}$$

where $s \in C(\bar{\Omega})$ such that $0 < s^- \le s^+ < N/2$, $q^- > 1$, and

$$q(x) < \frac{N - 2s(x)}{N} p_2^*(x), \quad \forall x \in \bar{\Omega}. \tag{6}$$

Put $\beta(x) = N/2s(x)$. Then, we have $q\beta'(x) < p_2^*(x)$, where β' is the conjugate of β.

Furthermore, $|x|^{-s(x)} \in L^{\beta(x)}(\Omega)$. Indeed,

$$\int_\Omega |x|^{-s(x)\beta(x)} dx = \int_\Omega |x|^{-N/2} dx < \infty \text{because } \frac{N}{2} < N. \tag{7}$$

Problem (P_λ) is related to the stationary problem of a model introduced by Kirchhoff [1]. To be more precise, Kirchhoff established a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{8}$$

where ρ, ρ_0, h, E, L are constants, which extends the classical D'Alambert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. In two dimensions, Kirchhoff equations model the oscillations of thin plates and the most usual plate operator is the biharmonic operator $\Delta^2 = \Delta \cdot \Delta$ [2].

Fourth-order equations have various applications in many domains like microelectromechanical systems, surface diffusion on solids, thin film theory, and interface dynamics; for recent contributions concerning this type of equations, we refer to [3–10]. In recent years, the study of variational problems with variable exponent has received considerable attention; these problems arises from nonlinear electrorheological fluids, elastic mechanics, image restoration, and mathematical biology (see [11–15]). The interplay between the fourth-order equation and the variable exponent equation goes to the $p(x)$-biharmonic problems. The $p(x)$-biharmonic operator possesses more complicated structure than the p-biharmonic operator Δ_p^2, where $p > 1$ is a real constant; for example, it is not homogeneous. A study on $p(x)$-biharmonic problems with Navier boundary condition was treated by many authors (see, for example, [16–20]). The authors in [21, 22] proved the existence and multiplicity of weak solutions for the $p(x)$-biharmonic problems under Navier boundary conditions. Their approach is of variational nature and does not require any symmetry of the nonlinearities. In [23], a similar problem to ours has been investigated in the case of $p(x)$-Laplacian and with weight 1. In [17], the authors examined a $p(x)$-biharmonic Kirchhoff-type problem but in the case where the weight is bounded and without parameter λ. Motivated by the above papers and the results in [24, 25], we determine by different variational methods intervals of parameters for which this problem admits at least one nontrivial solution.

2. Preliminaries

We state some definitions and basic properties of variable exponent Lebesgue-Sobolev spaces. We refer the reader to [26–29] for details.

For any $p(x) \in C_+(\bar{\Omega})$, we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u/u : \Omega \longrightarrow \mathbb{R} \text{ is measurable}, \int_\Omega |u|^{p(x)} dx < \infty \right\}, \tag{9}$$

with the norm

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}. \tag{10}$$

Proposition 1 (see [29]). *The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is separable, uniformly convex, and reflexive and its conjugate space is $L^{p_0(x)}(\Omega)$, where $(1/p(x)) + (1/p_0(x)) = 1$ for all $x \in \bar{\Omega}$. For $u \in L^{p(x)}$ and $v \in L^{p_0(x)}$, we have*

$$\left| \int_\Omega uv dx \right| \le \left(\frac{1}{p^-} + \frac{1}{p_0^-} \right) |u|_{p(x)} |v|_{p_0(x)}. \tag{11}$$

Proposition 2 (see [26]). *Let $\rho(u) = \int_\Omega |u|^{p(x)} dx$ be the modular of the $L^{p(x)}(\Omega)$ space. For $u, u_n \in L^{p(x)}(\Omega), n = 1, 2, \cdots$, we have*

$$\begin{gathered}
|u|_{p(x)} < (=;>) 1 \Rightarrow \rho(u) < (=;>) 1, \\
|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}, \\
|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}, \\
\lim_{n \longrightarrow +\infty} |u_n|_{p(x)} = 0 \Leftrightarrow \lim_{n \longrightarrow +\infty} \rho(u_n) = 0, \\
\lim_{n \longrightarrow +\infty} |u_n|_{p(x)} = +\infty \Leftrightarrow \lim_{n \longrightarrow +\infty} \rho(u_n) = +\infty.
\end{gathered} \tag{12}$$

For $k \ge 1$, we define the variable exponent Sobolev space

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \colon D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k \right\}, \tag{13}$$

where $D^\alpha u = \partial^{|\alpha|} u / \partial^{\alpha_1} x_1 \cdots \partial^{\alpha_N} x_N$, where $\alpha = (\alpha_1, \cdots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$ equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \le k} |D^\alpha u|_{p(x)} \tag{14}$$

becomes a separable, reflexive, and uniformly convex Banach space.

Proposition 3 ([26]). *For $p, r \in C_+(\bar{\Omega})$ such that $r(x) \le p_k^*(x)$ for all $x \in \Omega$, there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega). \tag{15}$$

If we replace $\le$ with $<$, the embedding is compact.

We denote

$$X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), \tag{16}$$

where $W_0^{k,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

For $u \in X$, we define

$$\|u\| = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\Delta u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}. \tag{17}$$

X endowed with the above norm is a separable and reflexive Banach space.

Remark 4. From [30], the norms $|u|_{2,p(x)}$ and $\|u\|$ are equivalent in X.

Let $d : \Omega \longrightarrow \mathbb{R}$ be a measurable real function $d(x) > 0$ a.e. $x \in \Omega$. We define the weighted variable exponent Lebesgue space

$$L_{d(x)}^{p(x)}(\Omega) = \left\{u/u : \Omega \longrightarrow \mathbb{R} \text{ is measurable}, \int_\Omega d(x)|u|^{p(x)} dx < \infty\right\}. \tag{18}$$

$L_{d(x)}^{p(x)}(\Omega)$ equipped with the norm $|u|_{d(x),p(x)} = \inf\{\lambda > 0 : \int_\Omega d(x)(|u(x)|/\lambda) dx \le 1\}$ is a Banach space which has similar properties with the usual variable exponent Lebesgue spaces. The modular of this space is $\rho_{d(x),p(x)} : L_{d(x)}^{p(x)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\rho_{d(x),p(x)} = \int_\Omega d(x)|u(x)|^{p(x)} dx. \tag{19}$$

Proposition 5 ([31]). *For $u, u_n \in L_{d(x)}^{p(x)}(\Omega), n = 1, 2, \cdots,$ we have*

(1) $|u|_{d(x),p(x)} < (=;>) 1 \Rightarrow \rho_{d(x),p(x)}(u) < (=;>) 1$

(2) $|u|_{d(x),p(x)} > 1 \Rightarrow |u|_{d(x),p(x)}^{p^-} \le \rho_{d(x),p(x)}(u) \le |u|_{d(x),p(x)}^{p^+}$

(3) $|u|_{d(x),p(x)} < 1 \Rightarrow |u|_{d(x),p(x)}^{p^+} \le \rho_{d(x),p(x)}(u) \le |u|_{d(x),p(x)}^{p^-}$

(4) $\lim_{n \longrightarrow +\infty} |u_n|_{d(x),p(x)} = 0 \Leftrightarrow \lim_{n \longrightarrow +\infty} \rho_{d(x),p(x)}(u_n) = 0$

(5) $\lim_{n \longrightarrow +\infty} |u_n|_{d(x),p(x)} = \infty \Leftrightarrow \lim_{n \longrightarrow \infty} \rho_{d(x),p(x)}(u_n) = \infty$

In the same way as in [32], we show the following proposition.

Proposition 6. *Assume that the boundary of Ω possesses the cone property and $p \in C(\bar{\Omega})$. Suppose that $c \in L^{\beta(x)}(\Omega)$, $d(x) > 0$ for a.e $x \in \Omega$,*

$\beta \in C_+(\bar{\Omega})$. If $q \in C_+(\bar{\Omega})$ and

$$q(x) < \frac{\beta(x) - 1}{\beta(x)} p_2^*(x), \quad \forall x \in \bar{\Omega}, \tag{20}$$

then there is a compact embedding $X \hookrightarrow L_{d(x)}^{q(x)}(\Omega)$.

Denote $I : X \longrightarrow X^*$ the operator defined by $\langle I(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx$ for all $u, v \in X$.

Proposition 7 ([20]). *The operator I satisfies the following assertions:*

(i) I is continuous, bounded, and strictly monotone

(ii) I is a mapping of $(S+)$ type, namely, $u_n u$ and $\limsup_{n \longrightarrow +\infty} \langle I(u_n), u_n - u\rangle \le 0$, which imply $u_n \longrightarrow u$

(iii) I is a homeomorphism

3. The Main Result

We say that $u \in X$ is a weak solution of (P_λ) if

$$M\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} dx\right) \int_\Omega |\Delta u|^{p-2} \Delta u \Delta v dx = \lambda \int_\Omega m(x)|u|^{q(x)-2} u v dx \tag{21}$$

for every $v \in X$.

For any $\lambda > 0$, the energy functional corresponding to problem (P_λ) is defined as $J_\lambda : X \longrightarrow \mathbb{R}$,

$$J_\lambda(u) = \widehat{M}\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} dx\right) - \lambda \int_\Omega m(x) \frac{|u|^{q(x)}}{q(x)} dx, \tag{22}$$

where

$$\widehat{M}(t) = \int_0^t M(s) ds. \tag{23}$$

Standard arguments imply that $J_\lambda \in C^1(X, \mathbb{R})$ and

$$\begin{aligned} \left\langle J_\lambda'(u), v\right\rangle = {} & M\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} dx\right) \int_\Omega |\Delta u|^{p-2} \Delta u \Delta v dx \\ & - \lambda \int_\Omega m(x)|u|^{q(x)-2} u v dx, \end{aligned} \tag{24}$$

for any $u, v \in X$. Hence, we can infer that critical points of functional J_λ are the weak solutions for problem (P_λ).

In the sequel, we use fountain theorem to study the existence of multiple solutions of (P_λ). We obtain the following result.

Theorem 8. *Assume that (M_1), (m), and $\alpha(p^+)^\alpha < q^-$; then, for every $\lambda > 0$, problem (P_λ) has a sequence of weak solutions $(\pm u_n)$ such that $J_\lambda(\pm u_n) \longrightarrow +\infty$ as $n \longrightarrow +\infty$.*

Before proving Theorem 8, we give some preliminary results.

Since X is a reflexive and separable Banach space, then X^* is too. There exist (see [33]) $\{e_n : n = 1, 2, \cdots\} \subset X$ and $\{e_n^* : n = 1, 2, \cdots\} \subset X^*$ such that

$$\begin{aligned} X &= \overline{\operatorname{span}\{e_j : j = 1, 2, \cdots\}}, \\ X^* &= \overline{\operatorname{span}\{e_j^* : j = 1, 2, \cdots\}}, \\ \left\langle e_i, e_j^* \right\rangle &= \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \end{aligned} \quad (25)$$

where $\langle .,. \rangle$ denote the duality product between X and X^*. We define

$$\begin{aligned} X_j &= \operatorname{span}\{e_j\}, Y_n = \overset{n}{\underset{j=1}{\oplus}} X_j, \\ Z_n &= \overline{\underset{j=n}{\oplus} \infty X_j}. \end{aligned} \quad (26)$$

Theorem 9 (fountain theorem, see [34]). *X is a Banach space; $J \in C^1(X, \mathbb{R})$ is an even functional. If for every $n = 1, 2, \cdots$, there exist $\rho_n > r_n > 0$ such that*

(a) $\inf\{J(u): u \in Z_n, \|u\| = r_n\} \longrightarrow +\infty$ as $n \longrightarrow +\infty$

(b) $\max\{J(u): u \in Y_n, \|u\| = \rho_n\} < 0$

(c) J satisfies the (PS) condition for every $c > 0$

Then, J has a sequence of critical values tending to $+\infty$.

Lemma 10. *If $\theta_n = \sup\{\int_\Omega m(x)|u|^{q(x)} dx : \|u\| = 1, u \in Z_n\}$, then under condition (m), $\lim_{n \longrightarrow +\infty} \theta_n = 0$*

Proof. For every $n \geq 1$, there exist $u_n \in Z_n$, $\|u_n\| = 1$ such that

$$\left| \theta_n - \int_\Omega m(x)|u_n|^{q(x)} dx \right| < \frac{1}{n}. \quad (27)$$

There exists a subsequence of (u_n) such that $u_n u$. For every $j \geq 1$ and for every $n > j$, we obtain $e_j^*(u_n) = 0$ and we conclude that $e_j^*(u) = 0 \forall j > 1$; hence, $u = 0$. On the other hand, by condition (m) and Proposition 5, there is a compact embedding $X \hookrightarrow L^{q(x)}_{m(x)}(\Omega)$; hence, $u_n \longrightarrow u = 0$ in $L^{q(x)}_{m(x)}(\Omega)$. By relation (27) we conclude that $\lim_{n \longrightarrow +\infty} \theta_n = 0$.□ □

Proof of Theorem 8. $J_\lambda \in C^1(X, \mathbb{R})$, and J_λ is even. According to Lemmas 17 and 18, J_λ satisfies $(PS)_c$ condition for every $c > 0$. We will prove that if n is large enough, then there exist $\rho_n > r_n > 0$ such that (a) and (b) hold.

(a) By condition (M_1), we have for any $u \in Z_n$ such that $\|u\| > 1$

$$\begin{aligned} J_\lambda(u) &= \widehat{M}\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} dx \right) - \lambda \int_\Omega m(x) \frac{|u|^{q(x)}}{q(x)} dx \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha} \|u\|^{p^- \alpha} - \frac{\lambda}{q^-} \int_\Omega m(x) \left(\frac{|u|}{\|u\|} \right)^{q(x)} dx \|u\|^{q^+} \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha} \|u\|^{p^- \alpha} - \frac{\lambda}{q^-} \theta_n \|u\|^{q^+}. \end{aligned} \quad (28)$$

For n large enough, choose u such that

$$\frac{\lambda}{q^-} \theta_n \|u\|^{q^+} = \frac{m_1}{q^-} \|u\|^{\alpha p^-}, \quad (29)$$

and we take

$$r_n = \|u\| = \left(\frac{m_1}{\lambda \theta_n} \right)^{1/(q^+ - \alpha p^-)}. \quad (30)$$

Then, we have

$$J_\lambda(u) \geq m_1 \left(\frac{1}{\alpha(p^+)^\alpha} - \frac{1}{q^-} \right) r_n^{\alpha p^-}. \quad (31)$$

By Lemma 10 and the fact that $\alpha(p^+)^\alpha < q^-$, the assertion (a) is verified.

(b) For any $\omega \in Y_n$ with $\|\omega\| = 1$ and $t > 1$, using (M_1), we have

$$\begin{aligned} J_\lambda(t\omega) &= \widehat{M}\left(\int_\Omega \frac{|t\Delta\omega|^{p(x)}}{p(x)} dx \right) - \lambda \int_\Omega m(x) \frac{|t\omega|^{q(x)}}{q(x)} dx \\ &\leq \frac{m_2}{\alpha(p^-)^\alpha} t^{\alpha p^+} - \frac{\lambda}{q^+} t^{q^-} \int_\Omega m(x)|\omega|^{q(x)} dx. \end{aligned} \quad (32)$$

By $q^- > \alpha p^+$ and $\dim Y_n < \infty$, it is easy to see that $J_\lambda(u) \longrightarrow -\infty$ as $\|u\| \longrightarrow +\infty$ for $u \in Y_n$. □

Now, a nontrivial solution of (P_λ) is given by using the coercivity and the weakly lower semicontinuity of J_λ.

Theorem 11. *If we assume that (M_1), (m), and $q^+ < \alpha p^-$ hold, then there exists $\lambda_* > 0$ such that for any $\lambda > \lambda_*$, problem (P_λ) possesses a nontrivial weak solution.*

We start with the following auxiliary result

Lemma 12. *Assume (M_1), (m), and $q^+ < \alpha p^-$. Then, the functional J_λ is coercive on X.*

Proof. By Proposition 5 and the compact imbedding $X \hookrightarrow L^{p(x)}_{m(x)}(\Omega)$, there exists C_0 such that

$$\int_\Omega \mathrm{m}(x)|u|^{q(x)}dx \le |u|^{q^-}_{m(x),q(x)} + |u|^{q^+}_{m(x),q(x)} \le {C_0}^{q^-}\|u\|^{q^-} + {C_0}^{q^+}\|u\|^{q^+}. \quad (33)$$

Therefore, for $\|u\| > 1$ and under condition (M_1), we obtain

$$J_\lambda(u) \ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-} - \frac{\lambda}{q^-}\left({C_0}^{q^-}\|u\|^{q^-} + {C_0}^{q^+}\|u\|^{q^+}\right). \quad (34)$$

Since $q^+ < \alpha p^-$, we infer that $J_\lambda(u) \longrightarrow +\infty$ as $\|u\| \longrightarrow +\infty$.□ □

Proof of Theorem 11. J_λ is a coercive functional and weakly lower semicontinuous on X. Then, there exists $v_\lambda \in X$ a global minimizer of J_λ (cf. Theorem 1.2 [35]). Thus, v_λ is a weak solution of (P_λ). Now, we prove that v_λ is a nontrivial solution for λ large enough. Letting $t_0 > 1$ be a constant and Ω_1 be an open subset of Ω with $|\Omega_1| > 0$, we assume that $\varphi \in C_0^\infty(\bar{\Omega})$ is such that $\varphi(x) = t_0$ for any $x \in \bar{\Omega}_1$ and $0 \le \varphi(x) \le t_0$ in $\Omega \setminus \Omega_1$. There exists $\lambda_* > 0$ such that $\forall \lambda \in (\lambda_*, +\infty)$,

$$J_\lambda(\varphi) \le \widehat{M}\left(\int_\Omega \frac{|\Delta\varphi|^{p(x)}}{p(x)}dx\right) - \frac{\lambda}{q^+}|t_0|^{q^-}\int_{\Omega_1} m(x)dx < 0. \quad (35)$$

Hence, for $\lambda > \lambda_*$, v_λ is a nontrivial weak solution of (P_λ). □

In the following theorem, we apply Ekeland variational principle [36] to get a nontrivial solution to problem (P_λ).

Theorem 13. *Assume that the conditions (M_1), (m), and $q^- < \alpha p^- < q^+$ are satisfied. Then, there exists $\lambda_{**} > 0$ such that for any $\lambda \in (0, \lambda_{**})$, problem (P_λ) has at last one nontrivial weak solution.*

We start with two auxiliary results.

Lemma 14. *Assume (M_1), (m), and $q^- < \alpha p^-$. There exists $\lambda_{**} > 0$ and two positive real numbers r, a such that for any $\lambda \in (0, \lambda_{**})$, $J_\lambda(u) \ge a > 0$ for any $u \in X$ with $\|u\| = r$.*

Proof. By using the condition (m) and Proposition 6, the embedding from X to $L^{q(x)}_{m(x)}(\Omega)$ is compact. Then, there exists $C_1 > 0$ such that for all $u \in X$,

$$|u|_{q(x),m(x)} \le C_1\|u\|. \quad (36)$$

□ □

Consider $r \in (0,1)$ such that $r < 1/C_1$. Then, we get

$$|u|_{q(x),m(x)} < 1, \quad \forall u \in X \text{ with } \|u\| = r. \quad (37)$$

Furthermore, Proposition 5 yields

$$\int_\Omega m(x)|u|^{q(x)}dx \le |u|^{q^-}_{q(x),m(x)}, \quad (38)$$

and we conclude that

$$\int_\Omega m(x)|u|^{q(x)}dx \le {C_1}^{q^-}\|u\|^{q^-}. \quad (39)$$

For $\lambda > 0$, using (M_1), (39), and Proposition 2, we get

$$J_\lambda(u) \ge \frac{m_1}{\alpha p^+}\|u\|^{\alpha p^+} - \frac{\lambda {C_1}^{q^-}}{q^-}\|u\|^{q^-} = r^{q^-}\left(\frac{m_1}{\alpha p^+}r^{\alpha p^+ - q^-} - \frac{\lambda {C_1}^{q^-}}{q^-}\right). \quad (40)$$

By the above inequality, we remark that if we define

$$\lambda_{**} = \frac{m_1}{2\alpha p^+}r^{\alpha p^+ - q^-}\frac{q^-}{{C_1}^{q^-}}, \quad (41)$$

then for any $\lambda \in (0, \lambda_{**})$ and any $u \in X$ with $\|u\| = r$, there exists $a = (m_1/2\alpha p^+)r^{\alpha p^+} > 0$ such that $J_\lambda(u) \ge a > 0$, which end the proof of Lemma 14.

Lemma 15. *Assume (M_1) and $q^- < \alpha p^-$. There exists $e \in X$ such that $e \ge 0$, $e \ne 0$, and $J_\lambda(te) < 0$ for $t > 0$ small enough.*

Proof. Since $q^- < \alpha p^-$, there exists $\varepsilon_0 > 0$ such that $q^- + \varepsilon_0 < \alpha p^-$. On the other hand, we have $q \in C(\bar{\Omega})$; then, there exists an open ball $B \subset \Omega$ such that $|q(x) - q^-| \le \varepsilon_0$ for all $x \in B$. Thus, we conclude that $q(x) \le q^- + \varepsilon_0 < \alpha p^-$ for all $x \in B$. Let $\phi \in C_0^\infty(\Omega)$ be such $B \subset \operatorname{supp}(\phi)$, $\phi(x) = 1$ for all $x \in \bar{B}$ and $0 \le \phi(x) \le 1$ for all $x \in \Omega$. Then, using the above information and (M_1), for all $0 < t < 1$, we obtain

$$\begin{aligned} J_\lambda(t\phi) &\le \frac{m_2}{\alpha(p^-)^\alpha}t^{\alpha p^-}\left(\int_\Omega |\Delta\phi|^{p(x)}dx\right)^\alpha - \frac{\lambda}{q^+}\int_\Omega m(x)|t\phi|^{q(x)}dx \\ &\le \frac{m_2}{\alpha(p^-)^\alpha}t^{\alpha p^-}\left(\int_\Omega |\Delta\phi|^{p(x)}dx\right)^\alpha - \frac{\lambda}{q^+}\int_B m(x)|t\phi|^{q(x)}dx \\ &\le \frac{m_2}{\alpha(p^-)^\alpha}t^{\alpha p^-}\left(\int_\Omega |\Delta\phi|^{p(x)}dx\right)^\alpha - \frac{\lambda}{q^+}t^{q^- + \varepsilon_0}\int_B m(x)|\phi|^{q(x)}dx < 0, \end{aligned} \quad (42)$$

for all $t < \delta$ with

$$0 < \delta < \min\left\{1, \left(\frac{\lambda\alpha(p^-)^\alpha}{m_2 q^+}\frac{\int_B m(x)|\phi|^{q(x)}dx}{\left(\int_\Omega |\Delta\phi|^{p(x)}dx\right)^\alpha}\right)^{1/(\alpha p^- - q^- - \varepsilon_0)}\right\}. \quad (43)$$

□ □

Proof of Theorem 13. Let $\lambda_{**} > 0$ be defined as in (41) and $\lambda \in (0,\lambda_{**})$. By Lemma 14, it follows that on the boundary of the ball centered at the origin and of radius r in X, denoted by $B_r(0)$, we have

$$\inf_{\partial B_r(0)} J_\lambda > 0. \quad (44)$$

On the other hand, by Lemma 15, there exists $\phi \in X$ such that $J_\lambda(t\phi) < 0$ for all $t > 0$ small enough. Moreover, by relations (53), we have for any $u \in B_r(0)$

$$J_\lambda(u) \geq \frac{m_1}{\alpha p^+}\|u\|^{\alpha p^+} - \frac{\lambda C^{q^-}}{q^-}\|u\|^{q^-}. \quad (45)$$

It follows that

$$-\infty < \underline{c} = \inf_{u \in \bar{B_r}(0)} J_\lambda(u) < 0. \quad (46)$$

Choose $0 < \varepsilon < \inf_{u \in \partial B_r(0)} J_\lambda(u) - \inf_{u \in B_r(0)} J_\lambda(u)$. Applying the Ekeland variational principle [36] to the functional $J_\lambda : \bar{B_r}(0) \longrightarrow \mathbb{R}$, we find $u_\varepsilon \in \bar{B_r}(0)$ such that

$$\begin{aligned} &J_\lambda(u_\varepsilon) < \underline{c} + \varepsilon, \\ &\left\|J'_\lambda(u_\varepsilon)\right\| \leq \varepsilon. \end{aligned} \quad (47)$$

The fact that $J_\lambda(u_\varepsilon) \leq \underline{c} + \varepsilon \leq \inf_{B_r(0)} J_\lambda + \varepsilon < \inf_{\partial B_r(0)} J_\lambda$ implies that $u_\varepsilon \in B_r(0)$. We deduce that there exists a sequence $(u_n) \subset B_r(0)$ such that

$$\lim_{n\longrightarrow\infty} J_\lambda(u_n) = \underline{c}, \quad (48)$$

$$\lim_{n\longrightarrow\infty} J'_\lambda(u_n) = 0. \quad (49)$$

It is clear that (u_n) is bounded in X. Thus, there exists $v_1 \in X$ and a subsequence still denoted by (u_n) such that u_n v_1 in X. Moreover, by the Hölder inequality, we get

$$\begin{aligned} &\left|\int_\Omega m(x)|u_n|^{q(x)-2}u_n(u_n - v_1)dx\right| \\ &= \left|\int_\Omega m(x)^{1/q_0(x)}|u_n|^{q(x)-2}u_n m(x)^{1/q(x)}(u_n - v_1)dx\right| \\ &\leq \left||u_n|^{q(x)-1}\right|_{m(x),q_0(x)}|u_n - v_1|_{m(x),q(x)}, \end{aligned} \quad (50)$$

where $q_0(x)$ is the conjugate exponent of $q(x)$, i.e., $(1/q(x)) + (1/q_0(x)) = 1$ for all $x \in \bar{\Omega}$.

By the compact embedding $X \hookrightarrow L^{q(x)}_{m(x)}(\Omega)$, $u_n \longrightarrow v_1$ in $L^{q(x)}_{m(x)}(\Omega)$. Using Proposition 5, we obtain $\rho_{m(x),q_0(x)}(|u_n|^{q(x)-1}) = \rho_{m(x),q(x)}(u_n) \longrightarrow \rho_{m(x),q(x)}(v_1) = \rho_{m(x),q_0(x)}(|v_1|^{q(x)-1})$ and we deduce that

$$\lim_{n\longrightarrow\infty}\left||u_n|^{q(x)-1}\right|_{m(x),q_0(x)} = \left||v_1|^{q(x)-1}\right|_{m(x),q_0(x)}. \quad (51)$$

Hence, (50) and (51) imply that $|\int_\Omega m(x)|u_n|^{q(x)-2}u_n(u_n - v_1)dx| \longrightarrow 0$ as $n \longrightarrow 0$.

Using (48), we infer that

$$M\left(\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}\right)\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n - \Delta v_1)dx \longrightarrow 0. \quad (52)$$

From (M_1) and assertion (ii) of Proposition 7, it follows that $u_n \longrightarrow v_1$ in X. Thus, $J_\lambda(v_1) = \underline{c} < 0$ and $J'_\lambda(v_1) = 0$. □

If in addition we have the following condition on

(M_2). There exists $0 < \gamma < 1$ such that $\widehat{M}(t) \geq \gamma t M(t) \forall t > 1$.

Then by help of the Mountain Pass theorem, we obtain the following.

Theorem 16. *Assume that (M_1), (M_2), (m), and $\alpha p^+ < \gamma q^-$ hold. For every $\lambda > 0$, problem (P_λ) has a nontrivial weak solution.*

To prove Theorem 16, we need the two following lemmas.

Lemma 17. *For $\lambda > 0$ and under conditions (M_1), (m), and $\alpha p^+ < q^-$, there exist $r > 0$ and $a > 0$ such that $J_\lambda(u) > a$ for any $u \in X$ with $\|u\| = r$.*

Proof. Let $0 < r < 1$. Under conditions (M_1), (m), and as in Lemma 14, we show the existence of a constant $C_2 > 0$ such that for all $u \in X$ with $\|u\| = r$, we have

$$\begin{aligned} J_\lambda(u) &\geq \frac{m_1}{\alpha p^+}\|u\|^{\alpha p^+} - \frac{\lambda C_2{}^{q^-}}{q^-}\|u\|^{q^-} \\ &= \left(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{\lambda C_2{}^{q^-}}{q^-}\|u\|^{q^- - \alpha p^+}\right)\|u\|^{\alpha p^+}. \end{aligned} \quad (53)$$

Since $q^- > p^+\alpha$, then we can choose $0 < r < \min(1, 1/C_2)$ such that

$$a = \left(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{\lambda C_2{}^{q^-}}{q^-}r^{q^- - \alpha p^+}\right)r^{\alpha p^+} > 0, \quad (54)$$

and we have $J_\lambda(u) \geq a > 0$ for every $u \in X$, $\|u\| = r$.□ □

Lemma 18. *For $\lambda > 0$ and under conditions (M_1) and $\alpha p^+ < q^-$, there exists $e \in X$ with $\|e\| > r$ where r is given in Lemma 17, such that $J_\lambda(e) < 0$.*

Proof. Let $\phi \in C_0^{+\infty}(\Omega)$ such that $\phi \geq 0$ and $\phi \not\equiv 0$ and $t > 1$. By (M_1), we have

$$J_\lambda(t\phi) \leq \frac{m_2}{\alpha (p^-)^\alpha} t^{p^+\alpha} \left(\int_\Omega |\Delta \phi|^{p(x)} dx \right)^\alpha - \frac{\lambda}{q^+} t^{q^-} \int_\Omega m(x) |\phi|^{q(x)} dx. \quad (55)$$

Since $\alpha p^+ < q^-$, we obtain $\lim_{t \longrightarrow +\infty} J_\lambda(t\phi) = -\infty$. Then, for $t > 1$ large enough, we can take $e = t\phi$ such that $\|e\| > r$ and $J_\lambda(e) < 0$.□ □

Proof of Theorem 16. By Lemmas 17 and 18 and the mountain pass theorem of Ambrosetti and Rabinowitz [37], we deduce the existence of a sequence $(u_n) \subset X$ and positive real number c such that

$$\begin{gathered} J_\lambda(u_n) \longrightarrow c > 0, \\ J'_\lambda(u_n) \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \end{gathered} \quad (56)$$

Firstly, we prove that (u_n) is bounded in X. Arguing by contradiction and passing to a subsequence, we have $\|u_n\| \longrightarrow +\infty$ as $n \longrightarrow \infty$. Considering $\|u_n\| > 1$, for n large enough and using (M_1), (M_2), (48), and Proposition 2, we have

$$\begin{aligned} 1 + c + \|u_n\| &\geq J_\lambda(u_n) - \frac{1}{q^-} \left\langle J'_\lambda(u_n), u_n \right\rangle \\ &\geq \gamma M \left(\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)} dx \right) \int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)} dx \\ &\quad - \lambda \int_\Omega m(x) \frac{|u_n|^{q(x)}}{q(x)} dx \\ &\quad - \frac{1}{q^-} M \left(\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)} dx \right) \int_\Omega |\Delta u_n|^{p(x)} dx \\ &\quad + \frac{\lambda}{q^-} \int_\Omega m(x) |u_n|^{q(x)} dx \\ &\geq \frac{m_1}{(p^+)^{\alpha-1}} \left(\frac{\gamma}{p^+} - \frac{1}{q^-} \right) \|u_n\|^{\alpha p^-} \\ &\quad + \lambda \int_\Omega \left(\frac{1}{q^-} - \frac{1}{q(x)} \right) m(x) |u_n|^{q(x)} dx \\ &\geq \frac{m_1}{(p^+)^{\alpha-1}} \left(\frac{\gamma}{p^+} - \frac{1}{q^-} \right) \|u_n\|^{\alpha p^-}. \end{aligned} \quad (57)$$

But, this cannot hold true since $\alpha p^- > 1$ and $(\gamma/p^+) > (1/q^-)$. Hence, (u_n) is bounded in X. The fact that X is reflexive implies that there exists a subsequence, still denoted by (u_n), and $v_2 \in X$ such that $u_n v_1$ in X. Actually, with similar arguments as those used in the proof of Theorem 13, we can show that $u_n \longrightarrow v_2$ in X. Thus, $J_\lambda(v_2) = c > 0$ and $J'_\lambda(v_2) = 0$. □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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