Research Article

New Solutions for the Generalized BBM Equation in terms of Jacobi and Weierstrass Elliptic Functions

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The Jacobi elliptic function method is applied to solve the generalized Benjamin-Bona-Mahony equation (BBM). Periodic and soliton solutions are formally derived in a general form. Some particular cases are considered. A power series method is also applied in some particular cases. Some solutions are expressed in terms of the Weierstrass elliptic function.

1. Introduction

The regularized long-wave (RLW) equation is a famous nonlinear wave equation which gives the phenomena of dispersion and weak nonlinearity, including magneto hydrodynamic wave in plasma, phonon packets in nonlinear crystals, and nonlinear transverse waves in shallow water or in ion acoustic. This equation is also called the BBM (Benjamin-Bona-Mahony) equation and reads

$$u_t + uu_x + u_x - \mu u_{xxt} = 0. \quad (1)$$

It describes approximately the unidirectional propagation of long waves in certain nonlinear dispersive systems, and it was proposed by Benjamin et al. in 1972 [1] as a more satisfactory model than the KdV equation [2]:

$$u_t + uu_x + u_{xxx} = 0. \quad (2)$$

It is easy to see that Equation (1) can be derived from the equal width equation [3]:

$$u_t + uu_x - \mu u_{xxt} = 0, \quad (3)$$

by means of the change of variable $u \leftarrow u + 1$, that is, by replacing u with $u + 1$. This last equation is considered an

equally valid and accurate model for the same wave phenomena simulated by (1) and (2). On the other hand, some researches analyzed the generalized KdV equation with variable coefficients

$$u_t + \sigma u^n u_x + \mu u_{xxt} = 0, \quad (4)$$

because this model has important applications in several fields of science. Motivated by these facts, we will consider here the generalized equal width (RW) equation with constant coefficients

$$u_t + \sigma u^n u_x - \mu u_{xxt} = 0. \quad (5)$$

Our aim is to present solutions different from those in [2, 4–6].

1.1. Trigonometric and Soliton Solutions. In order to obtain new solutions to Equation (5), we make the traveling wave transformation

$$u(x, t) = u(\xi), \text{ where } \xi = x - \lambda t. \quad (6)$$

Inserting (6) into (5) gives

$$-\lambda u'(\xi) + u(\xi)^n \sigma u'(\xi) + \lambda \mu u'''(\xi) = 0. \quad (7)$$

Integrating once with respect to ξ to get

$$-\lambda u(\xi) + \frac{\sigma}{n+1} u^{n+1}(\xi) + \lambda \mu u''(\xi) - c_1 = 0, \quad (8)$$

where c_1 is the constant of integration. Now, we multiply Equation (8) by $u'(\xi)$, and we integrate the resulting equation with respect to ξ :

$$-\frac{c_1^2}{2\lambda} - c_1 u(\xi) + c_2 + \frac{\sigma u(\xi)^{n+2}}{(n+1)(n+2)} + \frac{1}{2} \mu u'(\xi)^2 - \frac{1}{2} \lambda u(\xi)^2 = 0, \quad (9)$$

where c_2 is the constant of integration. If $c_1 = c_2 = 0$, Equation (9) takes the form

$$\frac{\mu}{2} u'^2(\xi) + \frac{\sigma}{(n+1)(n+2)} u^{n+2}(\xi) - \frac{\lambda}{2} u'(\xi)^2 = 0. \quad (10)$$

Let

$$u(\xi) = v^r(\xi), \quad rv(\xi) \neq 0. \quad (11)$$

Inserting Equation (11) into (10), we obtain

$$v'(\xi)^2 = \frac{\lambda v(\xi)^2}{\mu r^2} - \frac{2\sigma v(\xi)^{nr+2}}{\mu(n+1)(n+2)r^2}. \quad (12)$$

From Equation (12), if $r = 1/n$, then

$$v'(\xi)^2 = \frac{\lambda n^2 v(\xi)^2}{\mu} - \frac{2n^2 \sigma v(\xi)^3}{\mu(n+1)(n+2)}. \quad (13)$$

A sech solution to Equation (13) is

$$v(\xi) = \frac{\lambda(n+1)(n+2)}{2\sigma} \operatorname{sech}^2 \left(\frac{n}{2} \sqrt{\frac{\lambda}{\mu}} (\xi + \xi_0) \right), \quad \xi_0 = \text{const.} \quad (14)$$

It is clear that function

$$u(x, t) = \sqrt[n]{\frac{\lambda(n+1)(n+2)}{2\sigma} \operatorname{sech}^2 \left(\frac{n}{2} \sqrt{\frac{\lambda}{\mu}} (x - \lambda t + \xi_0) \right)}, \quad \lambda > 0, \quad (15)$$

is a solution to equation $u_t + \sigma u^n u_x - \mu u_{xxt} = 0$ for any choice of the parameters λ and ξ_0 .

Finally, if $\mu < 0$, then taking into account that

$$\operatorname{sech} \left(\frac{n}{2} \sqrt{\frac{\lambda}{\mu}} (x - \lambda t + \xi_0) \right) = \sec \left(\frac{n}{2} \sqrt{\frac{\lambda}{-\mu}} (x - \lambda t + \xi_0) \right), \quad (16)$$

we obtain from (15) the trigonometric solution

$$u(x, t) = \sqrt[n]{\frac{\lambda(n+1)(n+2)}{2\sigma} \sec^2 \left(\frac{n}{2} \sqrt{\frac{\lambda}{-\mu}} (x - \lambda t + \xi_0) \right)}. \quad (17)$$

2. The Novel Solutions

In this section, we will give analytical solutions to Equation (5) for two special cases: $n = 1$ and $n = 2$. First of all, observe that Equation (5) is equivalent to Equation (9).

2.1. The Case $n = 1$. If $n = 1$, Equation (9) reads

$$\frac{\lambda}{2} u'^2(\xi) + \frac{\sigma}{6} u^3(\xi) - \frac{\lambda \mu}{2} u'(\xi)^2 + c_1 u(\xi) + c_2 = 0. \quad (18)$$

To solve this equation, we consider the reduced Duffing equation

$$y''(\xi) = q y^3(\xi), \quad q \neq 0, \quad (19)$$

where the coefficient q does not depend on ξ . Observe that

$$y'(\xi) y''(\xi) = y'(\xi) q y^3(\xi), \quad (20)$$

and integrating this equation w.r.t. ξ gives

$$\frac{1}{2} y'(\xi)^2 = \frac{q}{4} y^4(\xi) + c_3. \quad (21)$$

Suppose that $y = y(\xi)$ is a solution to Equation (19) and let

$$u(\xi) = A y^2(\xi) + \frac{\lambda}{\sigma}, \quad \text{where } A \text{ is some constant.} \quad (22)$$

We have

$$u'(\xi) = 2A y(\xi) y'(\xi), \quad (23)$$

$$u''(\xi) = 2A y'(\xi)^2 + 2A y(\xi) y''(\xi), \quad (24)$$

$$y'(\xi)^2 = \frac{q}{2} y(\xi)^4 + 2c_3. \quad (25)$$

Taking into account Equations (18) and (23), we must have

$$\begin{aligned} & \left(-A^2 q \lambda \mu + \frac{1}{6} A^3 \sigma\right) y(\xi)^6 + \left(A c_1 - 2 A^2 c_3 \lambda \mu - \frac{A \lambda^2}{2 \sigma}\right) y(\xi)^2 \\ & + c_2 - \frac{\lambda^3}{3 \sigma^2} + \frac{\lambda c_1}{\sigma} = 0. \end{aligned} \quad (26)$$

Equating to zero the coefficients of y^6 , y^2 , and y^0 , we conclude that

$$\begin{aligned} q &= \frac{A \sigma}{6 \lambda \mu}, \\ c_3 &= \frac{2 c_1 \sigma - \lambda^2}{4 A \lambda \mu \sigma}. \end{aligned} \quad (27)$$

We have proved that the general solution to the nonlinear ODE

$$\frac{\lambda}{2} u^2(\xi) + \frac{\sigma}{6} u^3(\xi) - \frac{\lambda \mu}{2} u'(\xi)^2 + c_1 u(\xi) + c_2 = 0 \quad (28)$$

may be written in the form $u(\xi) = A y^2(\xi) + \lambda/\sigma$, where $y = y(\xi)$ is a solution to the Duffing equation

$$y''(\xi) = \frac{A \sigma}{6 \lambda \mu} y^3(\xi), \text{ with } y'(\xi)^2 = \frac{A \sigma}{12 \lambda \mu} y(\xi)^4 + \frac{2 c_1 \sigma - \lambda^2}{2 A \lambda \mu \sigma}. \quad (29)$$

According to [2], the general solution to Equation (29) may be expressed in terms of the Jacobian elliptic functions. More exactly, the general solution to Equation (19) is

$$y(\xi) = y_0 \operatorname{cn} \left(\sqrt{\frac{A \sigma y_0^2}{6 \lambda \mu}} (\xi + \xi_0), \frac{\sqrt{2}}{2} \right) + \frac{\lambda}{\sigma}, \text{ where } q y_0^2 \neq 0. \quad (30)$$

The value of ξ_0 is determined by solving equation $y'(0) = y_1$. In principle, we may consider that ξ_0 is any number as well as y_0 . Taking into account these facts, we have proved that the general solution to Equation (28) is

$$u(\xi) = y_0^2 \operatorname{cn}^2 \left(\sqrt{\frac{A \sigma y_0^2}{6 \lambda \mu}} (\xi + \xi_0), \frac{\sqrt{2}}{2} \right) + \frac{\lambda}{\sigma}, \quad y_0 \neq 0. \quad (31)$$

Finally, an exact solution to the EW-equation $u_t + \sigma u u_x - \mu u_{xxt} = 0$ is

$$u(x, t) = A y_0^2 \operatorname{cn}^2 \left(\sqrt{\frac{A \sigma y_0^2}{6 \lambda \mu}} (x - \lambda t + \xi_0), \frac{\sqrt{2}}{2} \right) + \frac{\lambda}{\sigma}. \quad (32)$$

Solution (32) involves four arbitrary constants: A , y_0 , λ , and ξ_0 (these do not depend on ξ). We choose A so that $A \sigma y_0^2 / 6 \lambda \mu > 0$.

2.2. The Case $n = 2$. This case corresponds to the so-called modified BBM or modified equal width equation. When $n = 2$, Equation (9) reads

$$\frac{\lambda}{2} u^2(\xi) + \frac{\sigma}{12} u^4(\xi) - \frac{\lambda \mu}{2} u'(\xi)^2 + c_1 u(\xi) + c_2 = 0. \quad (33)$$

This equation is harder to solve. We seek solution to Equation (33) in the form

$$u(\xi) = p + \frac{q}{1 + \sqrt{r} \operatorname{cn}(\sqrt{w}(\xi - \xi_0), \sqrt{m})}. \quad (34)$$

Observe that function $y = y(\xi) = r \operatorname{cn}(\sqrt{w}(\xi + \xi_0), \sqrt{m})$ satisfies the nonlinear differential equation

$$y'' + w(1 - 2m)y + \frac{2mw}{r} y^3 = 0, \quad (35)$$

for any constants γ and ξ_0 . This equation is Duffing equation $y'' + \alpha y + \beta y^3 = 0$ with coefficients α and β . The relationship between the frequency ω , the modulus m , and the coefficients is given by the system

$$\begin{aligned} \alpha &= w(1 - 2m), \\ \beta &= \frac{2mw}{r}. \end{aligned} \quad (36)$$

Solving this system for ω and m gives

$$\begin{aligned} \omega &= \alpha + r\beta, \\ m &= \frac{r\beta}{2(\alpha + r\beta)} = \frac{r\beta}{2w}. \end{aligned} \quad (37)$$

To solve the modified BBM equation, we will make use of the power series method (PSM). This is a promising method that also may be applicable to obtain an approximate solution for those ODE's that do not admit solution in closed form. In our case, we consider the function

$$F(\xi) = \frac{\lambda}{2} u^2(\xi) + \frac{\sigma}{12} u^4(\xi) - \frac{\lambda \mu}{2} u'(\xi)^2 + c_1 u(\xi) + c_2, \quad (38)$$

where

$$u(\xi) = p + \frac{q}{1 + \sqrt{r} \operatorname{cn}(\sqrt{w}(\xi - \xi_0), \sqrt{m})}. \quad (39)$$

The values of q and r are obtained from the initial conditions $u(\xi_0) = 0$ and $u(\xi_0 + K(\sqrt{m})/\sqrt{w}) = 0$, that is,

$$\begin{aligned} p + \frac{q}{1 + \sqrt{r}} &= 1, \\ p + \frac{q}{1 - \sqrt{r}} &= 0. \end{aligned} \quad (40)$$

Thus,

$$\sqrt{r} = \frac{1}{2p-1}, \quad (41)$$

$$q = \frac{2p(p-1)}{2p-1}. \quad (42)$$

Inserting the ansatz (39) into (38), we obtain an expression for $F(\xi)$ in terms of the unknowns w , m , p , and λ . Observe that function $u(\xi)$ given by (39) is a solution to Equation (33), if and only if $F(\xi) \equiv 0$. The series method consists in coefficients of Taylor series of $F(\xi)$ around some point. Usually, this is the origin. Since the calculations are enormous, we cannot realize this work by hand. Instead, we may use Mathematica 12 or Maple 16. The Mathematica command has the form Series $[F[\xi], \{\xi, \xi_0, n\}]$. This gives us the n th degree Taylor polynomial of $F(\xi)$ around the point $\xi = \xi_0$. In our case, we will use the command Series $[F[\xi], \{\xi, 0, n\}]$ for $n \geq 3$, since we need at least four equations to determine the five unknowns w , m , p , and λ in terms of the coefficients of Equation (33). Of course, the series method is applicable to solve the BBM when $n = 1$. Our aim is to show two different methods. Let us proceed. We have that $F^{(2n+1)}(\xi_0) = 0$ for all $n = 0, 1, 2, 3, \dots$, and

$$\begin{aligned} F(\xi) = & F(\xi_0) + \frac{1}{2}F''(\xi_0)(\xi - \xi_0)^2 + \frac{1}{24}F''''(\xi_0)(\xi - \xi_0)^4 \\ & + \frac{1}{720}F''''''(\xi_0)(\xi - \xi_0)^6 + \dots, \end{aligned} \quad (43)$$

where

$$F(\xi_0) = \frac{1}{12}(12c_1 + 12c_2 + 6\lambda + \sigma) = 0, \quad (44)$$

$$\lambda = -\frac{1}{6}(\sigma + 12c_1 + 12c_2), \quad (45)$$

$$F''(\xi_0) = -\frac{(p-1)w(6c_1p + 3p\omega\lambda\mu + 6p\lambda + 2p\sigma - 3w\lambda\mu)}{12p^2} = 0, \quad (46)$$

$$p = \frac{3w\lambda\mu}{6c_1 + 6\lambda + 3w\lambda\mu + 2\sigma}, \quad (47)$$

$$\begin{aligned} 0 = & 12p^3F''''(\xi_0) = (p-1)w^3 \\ & \cdot (-18c_1p + 6c_1p^2 + 24c_1mp^2 - 27p\lambda + 15p^2\lambda + 24mp^2\lambda \\ & + 36w\lambda\mu - 48p\omega\lambda\mu - 48mp\omega\lambda\mu + 12p^2w\lambda\mu \\ & + 48mp^2w\lambda\mu - 15p\sigma + 11p^2\sigma + 8mp^2\sigma), \end{aligned} \quad (48)$$

$$w = -\frac{p(6c_1(4mp + p - 3) + 3(8mp + 5p - 9)\lambda + (8mp + 11p - 15)\sigma)}{12(p-1)(4mp + p - 3)\lambda\mu}, \quad (49)$$

$$\begin{aligned} 0 = & -12p^4F''''''(\xi_0) = (p-1)w^3 \\ & \cdot [32p^2(3p\lambda - 24w\lambda\mu + 24p\omega\lambda\mu + p\sigma + 3pc_1)m^2 \\ & + 4p(-135p\lambda + 111p^2\lambda + 450w\lambda\mu - 708p\omega\lambda\mu \\ & + 258p^2w\lambda\mu - 75p\sigma + 67p^2\sigma - 90pc_1 + 66p^2c_1)m \\ & + 270p\lambda - 270p^2\lambda + 51p^3\lambda - 675w\lambda\mu + 1125p\omega\lambda\mu \\ & - 498p^2w\lambda\mu + 48p^3w\lambda\mu + 225p\sigma - 300p^2\sigma + 92p^3\sigma \\ & + 135pc_1 - 90p^2c_1 + 6p^3c_1]. \end{aligned} \quad (50)$$

Solving this last equation for m and taking into account (45), (47), and (49), we obtain after long algebraic calculations following solution:

$$\lambda = \frac{1}{6}(-12c_1 - 12c_2 - \sigma), \quad (51)$$

$$w = \frac{2\sqrt{6\Delta}}{\mu(12c_1 + 12c_2 + \sigma)}, \quad (52)$$

$$p = -\frac{-6c_1 - 24c_2 + \sqrt{6\Delta}}{12c_2 + \sigma}, \quad (53)$$

$$m = \frac{216c_1c_2 + 432c_2^2 + 6c_1\sigma - \sigma^2 + (12\sqrt{6}c_2 - 24c_2\sigma + 3\sqrt{6}\sigma)\sqrt{\Delta}}{8(6c_1 + 24c_2 - \sqrt{6\Delta})(6c_1 + 12c_2 - \sigma)}, \quad (54)$$

$$\Delta = (c_1 + 4c_2)(6c_1 + 12c_2 - \sigma), \sigma = \sigma, \mu = \mu. \quad (55)$$

Direct calculations show that function $u = u(\xi)$ defined by (39) with parameters given by (51)-(55) and (41) is a solution to Equation (41). Then, function $u = u(x - \lambda t - \xi_0)$ is a solution to the modified BBM equation

$$u_t + \sigma u^2 u_x - \mu u_{xxx} = 0. \quad (56)$$

Observe that $\text{cn}(x, 1) = \text{sech}(x)$. It may be verified that the following function is a solution to Equation (56):

$$u(x, t) = \alpha \left(1 + \frac{\beta}{1 + \gamma \cosh(k(x + \lambda t) + C)} \right), \quad (57)$$

for the choice

$$\begin{aligned}
c_1 &= -\frac{2\alpha^3\gamma^4\sigma}{3(\gamma^2-2)^3}, \\
c_2 &= \frac{\alpha^4(\gamma-2)\gamma^6(\gamma+2)\sigma}{12(\gamma^2-2)^4}, \\
k &= \frac{\sqrt{2}\sqrt{1-\gamma^2}}{\sqrt{(\gamma^2+2)\mu}}, \\
\lambda &= -\frac{\alpha^2\gamma^2(\gamma^2+2)\sigma}{3(\gamma^2-2)^2}, \\
\beta &= -\frac{2(\gamma-1)(\gamma+1)}{\gamma^2-2}.
\end{aligned} \quad (58)$$

We obtain soliton solutions for $(1-\gamma^2)\mu > 0$ and trigonometric solutions for $(1-\gamma^2)\mu < 0$.

2.3. Solution for a Generalized BBM Equation. Let us consider the following generalized BBM equation:

$$u_t + au_x + bu^p u_x - \delta u_{xxt} = 0. \quad (59)$$

In order to obtain new solutions to Equation (59), we make the travelling wave transformation

$$u(x, t) = v^{-2/p}(\xi), \text{ where } \xi = x - \lambda t + \xi_0. \quad (60)$$

Inserting (60) into (59) gives

$$\begin{aligned}
pv'(\xi)(\xi) &\left(p(a-\lambda)v(\xi)^2 + bp - 3\delta\lambda(p+2)v(\xi)v''(\xi) \right) \\
&+ \delta\lambda p^2 v(\xi)^2 v^{(3)}(\xi) + 2\delta\lambda(p+1)(p+2)v'(\xi)(\xi)^3 = 0.
\end{aligned} \quad (61)$$

Equation (61) is hard to integrate. We seek for an exact solution of the form

$$v(\xi) = A + \frac{B}{1 + C\wp(\xi; g_2, g_3)}. \quad (62)$$

Inserting ansatz (62) into (61) gives $R(\xi) = 0$, where

$$\begin{aligned}
R(\xi) &= c_4\wp(\xi; g_2, g_3)^4 + c_3\wp(\xi; g_2, g_3)^3 + c_2\wp(\xi; g_2, g_3)^2 \\
&+ c_1\wp(\xi; g_2, g_3) + c_0,
\end{aligned} \quad (63)$$

being

$$\begin{aligned}
c_0 &= 2aA^2p^2 + 4aABp^2 + 2aB^2p^2 - 12A^2C^2\delta g_3\lambda p^2 \\
&+ 6A^2C\delta g_2\lambda p^2 - 2A^2\lambda p^2 - 12ABC^2\delta g_3\lambda p^2 \\
&+ 24ABC^2\delta g_3\lambda p + 9ABC\delta g_2\lambda p^2 - 6ABC\delta g_2\lambda p \\
&- 4AB\lambda p^2 + 2bp^2 - 8B^2C^2\delta g_3\lambda - 4B^2C^2\delta g_3\lambda p^2 \\
&+ 12B^2C^2\delta g_3\lambda p + 3B^2C\delta g_2\lambda p^2 - 6B^2C\delta g_2\lambda p - 2B^2\lambda p^2,
\end{aligned}$$

$$\begin{aligned}
c_1 &= 8aA^2Cp^2 + 12aABCp^2 + 4aB^2Cp^2 - 24A^2C^3\delta g_3\lambda p^2 \\
&+ 6A^2C^2\delta g_2\lambda p^2 - 8A^2C\lambda p^2 + 24A^2\delta\lambda p^2 \\
&- 12ABC^3\delta g_3\lambda p^2 + 24ABC^3\delta g_3\lambda p + 6ABC^2\delta g_2\lambda p^2 \\
&+ 12ABC^2\delta g_2\lambda p - 12ABC\lambda p^2 + 48AB\delta\lambda p^2 + 8bCp^2 \\
&- 8B^2C^2\delta g_2\lambda - B^2C^2\delta g_2\lambda p^2 + 6B^2C^2\delta g_2\lambda p - 4B^2C\lambda p^2 \\
&+ 24B^2\delta\lambda p^2,
\end{aligned}$$

$$\begin{aligned}
c_2 &= Cp(12aA^2Cp + 12aABCp + 2aB^2Cp - 12A^2C^3\delta g_3\lambda p \\
&- 6A^2C^2\delta g_2\lambda p - 12A^2C\lambda p + 24A^2\delta\lambda p + 18ABC^2\delta g_2\lambda \\
&- 3ABC^2\delta g_2\lambda p - 12ABC\lambda p + 72AB\delta\lambda + 36AB\delta\lambda p \\
&+ 12bCp - 2B^2C\lambda p + 72B^2\delta\lambda + 12B^2\delta\lambda p),
\end{aligned}$$

$$\begin{aligned}
c_3 &= 2C^2(4aA^2Cp^2 + 2aABCp^2 - 3A^2C^2\delta g_2\lambda p^2 - 4A^2C\lambda p^2 \\
&- 12A^2\delta\lambda p^2 - 2ABC\lambda p^2 - 12AB\delta\lambda p^2 + 24AB\delta\lambda p \\
&+ 4bCp^2 + 16B^2\delta\lambda + 2B^2\delta\lambda p^2 + 12B^2\delta\lambda p),
\end{aligned}$$

$$c_4 = 2C^3p(aA^2Cp - A^2C\lambda p - 12A^2\delta\lambda p - 12AB\delta\lambda - 6AB\delta\lambda p + bCp). \quad (64)$$

Equating to zero the coefficients of $\wp^j(\xi; g_2, g_3)$ gives a nonlinear algebraic system. Solving it with the aid of Wolfram Mathematica, we obtain

$$\begin{aligned}
B &= -6A, \\
C &= -\frac{24\delta(aA^2p^2 + 3aA^2p + 2aA^2 + 2b)}{bp^2}, \\
\lambda &= \frac{a(p+2)(p+1)A^2 + 2b}{A^2(p+1)(p+2)}, \\
g_2 &= \frac{b^2p^4}{48\delta^2(a(p+2)(p+1)A^2 + 2b)^2}, \\
g_3 &= -\frac{b^3p^6}{1728\delta^3(a(p+2)(p+1)A^2 + 2b)^3}.
\end{aligned} \quad (65)$$

Other solutions may be obtained choosing B from the condition

$$\begin{aligned}
&12A^2p^2(30aA^2 + 30b + 15aA^2p + bp - bp^2) \\
&- 2Ap(28aA^2 + 28b - 166aA^2p - 118bp - 88aA^2p^2 \\
&- 4bp^2 + aA^2p^3 + 4bp^3)B - (p-1)(p+2) \\
&\cdot (-24aA^2 - 24b - 38aA^2p - 2bp + aA^2p^2 + bp^2)B^2 = 0.
\end{aligned} \quad (66)$$

Then, a solution to (61) is

$$v(\xi) = A - \frac{6A}{1 - (24\delta(a(p+2)(p+1)A^2 + 2b)/bp^2)\wp(\xi; g_2, g_3)}, \quad (67)$$

where g_2, g_3 , and λ are obtained from (65). The obtained solution is valid for

$$A\delta b(a(p+2)(p+1)A^2 + 2b)p(p+1)(p+2) \neq 0. \quad (68)$$

We have solved the generalized BBM equation for any p such that $p(p+1)(p+2) \neq 0$. The exact traveling wave solution is given by

$$u(x, t) = \left(A - \frac{6A}{1 - (24\delta(aA^2p^2 + 3aA^2p + 2aA^2 + 2b)/bp^2)\wp(x - (a(p+2)(p+1)A^2 + 2b/A^2(p+1)(p+2))t + \xi_0; g_2, g_3)} \right)^{-2/p}. \quad (69)$$

The numbers A and ξ_0 are arbitrary. Particular cases are

(1) $p = 1$:

$$u(x, t) = \left(A + \frac{6Ab}{48\delta(3aA^2 + b)\wp((b + 3A^2a/3A^2)t - x - \xi_0; b^2/192(3aA^2 + b)^2\delta^2, -(b^3/13824(3aA^2 + b)^3\delta^3))} - b \right)^{-2} \quad (70)$$

(2) $p = 2$:

$$u(x, t) = \left(A + \frac{6Ab}{12\delta(6aA^2 + b)\wp((b + 6A^2a/6A^2)t - x - \xi_0; b^2/12(6aA^2 + b)^2\delta^2, -(b^3/216(6aA^2 + b)^3\delta^3))} - b \right)^{-1} \quad (71)$$

(3) $p = 1/2$:

$$u(x, t) = \left(A + \frac{6Ab}{24\delta(15aA^2 + 8b)\wp((8b + 15A^2a/15A^2)t - x - \xi_0; b^2/48(15aA^2 + 8b)^2\delta^2, -(b^3/1728(15aA^2 + 8b)^3\delta^3))} - b \right)^{-4} \quad (72)$$

2.4. Forbidden Values. Let us examine the cases when $p(p+1)(p+2) = 0$.

Case 1 ($p = 0$). For this value, Equation (59) takes the form

$$u_t + au_x + bu_x - \delta u_{xxt} = 0. \quad (73)$$

After doing a traveling wave transformation, we get the linear ode

$$(a + b - \lambda)v'(\xi) + \delta v'''(\xi) = 0, \quad (74)$$

whose general solution is given by

$$v(\xi) = \frac{\sqrt{\delta\lambda}}{\sqrt{\lambda - a - b}} \left(c_1 e^{(\sqrt{\lambda - a - b}/\sqrt{\delta\lambda})\xi} - c_2 e^{-(\sqrt{\lambda - a - b}/\sqrt{\delta\lambda})\xi} \right) + c_3. \quad (75)$$

Case 2 ($(p+1)(p+2) = 0$). Making the traveling wave transformation $\xi = x - \lambda t + \xi_0$ and making use of the ansatz (62), we obtain that $B = 0$, which is a trivial solution. So, in the cases $p = -1$ and $p = -2$, we cannot get nontrivial solutions to the generalized BBM equation.

3. Analysis and Discussion

We obtained the traveling wave solutions to BBM equation $u_t + \sigma uu_x - \mu u_{xxt} = 0$ and modified BBM equation $u_t + \sigma u^2 u_x - \mu u_{xxt} = 0$. Other solutions were obtained in [7] using the extended Jacobi elliptic function expansion method. However, in our approach, we got nonzero integration constants. Most authors get zero integration constants to simplify matters, but this may cause loss of solutions. We also may consider the following combined BBM equation:

$$u_t + (au + \sigma u^2)u_x - \mu u_{xxt} = 0. \quad (76)$$

Making the traveling wave transformation $u = v(\xi)$, $\xi = x - \lambda t + \xi_0$ gives

$$-\lambda v'(\xi) + (av(\xi) + \sigma v^2(\xi))v'(\xi) + \delta \lambda v'''(\xi) = 0. \quad (77)$$

We now integrate once with respect to ξ , and denoting the constant of integration by c_1 , we obtain the ode

$$v''(\xi) + \frac{c_1}{\delta \lambda} - \frac{1}{\delta} v(\xi) + \frac{a}{2\delta \lambda} v(\xi)^2 + \frac{\sigma}{3\delta \lambda} v(\xi)^3 = 0. \quad (78)$$

This last equation is a Duffing-Helmholtz equation. The general form of the undamped and unforced Duffing-Helmholtz equation reads

$$v''(\xi) + n + pv(\xi) + qv^2(\xi) + rv^3(\xi) = 0. \quad (79)$$

The general solution to Equation (79) may be expressed in the ansatz form

$$v(\xi) = A + \frac{B}{1 + C\wp(\xi + D; g_2, g_3)}. \quad (80)$$

Indeed, given the initial conditions

$$v(0) = v_0, \quad (81)$$

$$v'(0) = \dot{v}_0, \quad (82)$$

the solution to the initial value problem (80)-(81) is given by (80), where

$$B = -\frac{6(A^3r + A^2q + Ap + n)}{3A^2r + 2Aq + p},$$

$$C = \frac{12}{3A^2r + 2Aq + p},$$

$$g_2 = \frac{1}{12}(-3A^4r^2 - 4A^3qr - 6A^2pr - 12Anr - 4nq + p^2),$$

$$g_3 = \frac{1}{216}(9A^4pr^2 - 3A^4q^2r + 12A^3pqr - 4A^3q^3 + 18A^2p^2r - 6A^2pq^2 + 36Anpr - 12Anq^2 + 27n^2r - 6npq + p^3),$$

$$D = \pm \wp^{-1}\left(\frac{v_0 - A - B}{C(A - v_0)}; g_2, g_3\right). \quad (83)$$

The number A is a solution to the quartic

$$3rA^4 + 4qA^3 + 6pA^2 + 12nA - (12nv_0 + 6pv_0^2 + 4qv_0^3 + 3rv_0^4 + 6\dot{v}_0^2) = 0. \quad (84)$$

In soliton theory, we are interested in sech or tanh solutions to Equation (79). The soliton solutions may occur only when

$$27nr^2 - 9pqr + 2q^3 = 0. \quad (85)$$

In this case, we will have the following soliton solution to Equation (79):

$$v(\xi) = -\frac{q}{3r} + \frac{3Ar + q}{3r} \operatorname{sech}\left(\frac{3Ar + q}{3\sqrt{2r}}\xi\right), \text{ for } 3Ar + q \neq 0. \quad (86)$$

Let us examine condition (85) for (78). We must have

$$a^3 + 6a\lambda\sigma + 12c_1\sigma^2 = 0. \quad (87)$$

Solving this equation for c_1 , we will have the following soliton solution to equation (18):

$$u(x, t) = -\frac{a}{2\sigma} + \frac{a + 2A\sigma}{2\sigma} \operatorname{sech}\left(\frac{a + 2A\sigma}{\sqrt{\delta(4aA\sigma + 4A^2\sigma^2 - 5a^2)}}\right) \cdot \left(x + \frac{1}{24}\left(\frac{5a^2}{\sigma} - 4aA - 4A^2\sigma\right)t + \xi_0\right). \quad (88)$$

In (88), the constants A and ξ_0 are arbitrary. In the case when $\sigma = 0$, Equation (18) converts into usual BBM equation $u_t + au_x - \mu u_{xxt} = 0$.

In a more general fashion, we solved Equation (59). This equation was solved in [2, 5, 8] using the substitution $u = v^{1/p}$. In the present paper, we solved it by using the substitution $u = v^{-2/p}$. Let us examine the odes obtained by means of these different substitutions. Using $u = v^{1/p}$ gives

$$\begin{aligned} n^2 v(\xi)^2 \left((a - \lambda)v'(\xi) + \delta \lambda v^{(3)}(\xi) \right) + b n^2 v(\xi)^3 v'(\xi) \\ + \delta \lambda (n - 1)(2n - 1)v'(\xi)^3 - 3\delta \lambda (n - 1)nv(\xi)v'(\xi)v''(\xi) = 0. \end{aligned} \quad (89)$$

On the other hand, letting $u = v^{-2/p}$ gives a different ode given by (61). So, our method gives other than that obtained in [2, 5, 8] solutions.

The authors in [5] claim that there are no periodic solutions for the parameter values of different form $p = 1$, $p = 2$,

or $p = 1/2$. However, we obtained periodic solutions for $(p-1)(p-2)(p-1/2) \neq 0$.

On the other hand, we may solve the ode (89) by means of the ansatz (62). That was the approach the author employed in [8] by means of the substitution $u = v^{1/p}$. Indeed, let us consider the equation [8]:

$$u_t + u_x + au^p u_x + u_{xxx} = 0. \quad (90)$$

Let $u = v^{1/p}(x - \lambda t + \xi_0)$. Then, the ode (90) converts into the ode

$$\begin{aligned} -ap^2 v(\xi)^3 v'(\xi) + p^2 v(\xi)^2 \left((\lambda - 1)v'(\xi) - v^{(3)}(\xi) \right) \\ + ((3-2p)p-1)v'(\xi)^3 + 3(p-1)p v(\xi)v'(\xi)v''(\xi) = 0. \end{aligned} \quad (91)$$

In order to solve equation (91), we will assume the ansatz form (62). Plugging this ansatz into (91), we get

$$\begin{aligned} d_4 \wp(\xi; g_2, g_3)^4 + d_3 \wp(\xi; g_2, g_3)^3 + d_2 \wp(\xi; g_2, g_3)^2 \\ + d_1 \wp(\xi; g_2, g_3) + d_0 = 0, \end{aligned} \quad (92)$$

where

$$d_4 = 2AC^3 p(aA^2 Cp - AC\lambda p + ACp - 12Ap - 6Bp + 6B),$$

$$\begin{aligned} d_3 = 2C^2 (4aA^3 Cp^2 + 3aA^2 BCp^2 - 3A^2 C^2 g_2 p^2 - 4A^2 C\lambda p^2 \\ + 4A^2 Cp^2 - 12A^2 p^2 - 2ABC\lambda p^2 + 2ABCp^2 - 12ABp^2 \\ - 12ABp + 2B^2 p^2 - 6B^2 p + 4B^2), \end{aligned}$$

$$\begin{aligned} d_2 = Cp(12aA^3 Cp + 18aA^2 BCp + 6aAB^2 Cp - 12A^2 C^3 g_3 p \\ - 6A^2 C^2 g_2 p - 12A^2 C\lambda p + 12A^2 Cp + 24A^2 p \\ - 3ABC^2 g_2 p - 9ABC^2 g_2 - 12ABC\lambda p + 12ABCp \\ + 36ABp - 36AB - 2B^2 C\lambda p + 2B^2 Cp + 12B^2 p - 36B^2), \end{aligned}$$

$$\begin{aligned} d_1 = 8aA^3 Cp^2 + 18aA^2 BCp^2 + 12aAB^2 Cp^2 + 2aB^3 Cp^2 \\ - 24A^2 C^3 g_3 p^2 + 6A^2 C^2 g_2 p^2 - 8A^2 C\lambda p^2 + 8A^2 Cp^2 \\ + 24A^2 p^2 - 12ABC^3 g_3 p^2 - 12ABC^3 g_3 p + 6ABC^2 g_2 p^2 \\ - 6ABC^2 g_2 p - 12ABC\lambda p^2 + 12ABCp^2 + 48ABp^2 \\ - B^2 C^2 g_2 p^2 - 3B^2 C^2 g_2 p - 2B^2 C^2 g_2 - 4B^2 C\lambda p^2 \\ + 4B^2 Cp^2 + 24B^2 p^2, \end{aligned}$$

$$\begin{aligned} d_0 = 2aA^3 p^2 + 6aA^2 Bp^2 + 6aAB^2 p^2 + 2aB^3 p^2 - 12A^2 C^2 g_3 p^2 \\ + 6A^2 Cg_2 p^2 - 2A^2 \lambda p^2 + 2A^2 p^2 - 12ABC^2 g_3 p^2 \\ - 12ABC^2 g_3 p + 9ABCg_2 p^2 + 3ABCg_2 p - 4AB\lambda p^2 \\ + 4ABp^2 - 4B^2 C^2 g_3 p^2 - 6B^2 C^2 g_3 p - 2B^2 C^2 g_3 \\ + 3B^2 Cg_2 p^2 + 3B^2 Cg_2 p - 2B^2 \lambda p^2 + 2B^2 p^2. \end{aligned} \quad (93)$$

Solving the system $d_0 = d_1 = d_2 = d_3 = d_4 = 0$ gives the following solution:

$$\begin{aligned} B &= -\frac{3A}{2}, \\ C &= \frac{3(p+1)(p+2)}{aAp^2}, \\ g_2 &= \frac{a^2 A^2 p^4}{3(p+1)^2(p+2)^2}, \\ g_3 &= -\frac{a^3 A^3 p^6}{27(p+1)^3(p+2)^3}, \\ \lambda &= \frac{2aA + p^2 + 3p + 2}{(p+1)(p+2)}. \end{aligned} \quad (94)$$

Thus, a traveling wave solution to Equation (90) reads

$$u(x, t) = \sqrt[p]{A - \frac{3A/2}{1 + (3(p+1)(p+2)/aAp^2)\wp(x - \lambda t + \xi_0; a^2 A^2 p^4/3(p+1)^2(p+2)^2, -(a^3 A^3 p^6/27(p+1)^3(p+2)^3))}}. \quad (95)$$

In a similar way, we may consider the following more general equation than that considered in [2]:

$$u_t + au_x + bu^p u_x - \delta u_{xxt} + \gamma u_{xxx} = 0. \quad (96)$$

In [9], the author solved the following generalization of the BBM equation:

$$u_t + \alpha u_x + (\beta u^n + \gamma u^{2n})u_x - \delta u_{xxt} = 0. \quad (97)$$

Let us consider the following more general variant of BBM equation:

$$u_t + \alpha u_x + (\beta u^n + \gamma u^{2n})u_x - \delta u_{xxt} - \kappa u_{xxx} = 0. \quad (98)$$

We again may use the ansatz (62) to solve this equation. Making the traveling wave transformation $u = v^{1/p}(x - \lambda t + \xi_0)$ gives the ode

$$\begin{aligned}
& n^2 v(\xi)^2 \left(v'(\xi)(\alpha - \lambda) + v'''(\xi)(\delta\lambda - \kappa) \right) + \beta n^2 v'(\xi) v(\xi)^3 \\
& + \gamma n^2 v'(\xi) v(\xi)^4 + 3(n-1)nv'(\xi)v'(\xi)v(\xi)(\kappa - \delta\lambda) \\
& + (1-n)(2n-1)v'(\xi)^3(\kappa - \delta\lambda) = 0.
\end{aligned} \tag{99}$$

Plugging the ansatz (62) into the ode (99) and equating to zero the coefficients of ϕ^j will give a very large algebraic system. After some hard algebraic calculations, we get the following solution:

$$\begin{aligned}
B &= -\frac{6A(A\gamma(n+2) + \beta + 2\beta n)}{5A\gamma(n+2) + \beta(8n+4)}, \\
C &= \frac{12(A^2\gamma\delta(n+2) + 2A\beta\delta(2n+1) + \alpha\delta(2n^3 + 7n^2 + 7n+2) - \kappa(2n^3 + 7n^2 + 7n+2))}{An^2(5A\gamma(n+2) + \beta(8n+4))}, \\
g_2 &= \frac{A^2n^4(A\gamma(n+2) + \beta(4n+2))^2}{12(A^2\gamma\delta(n+2) + 2A\beta\delta(2n+1) + \alpha\delta(2n^3 + 7n^2 + 7n+2) - \kappa(2n^3 + 7n^2 + 7n+2))^2}, \\
g_3 &= -\frac{A^3n^6(A\gamma(n+2) + \beta(4n+2))^3}{216(A^2\gamma\delta(n+2) + 2A\beta\delta(2n+1) + \alpha\delta(2n^3 + 7n^2 + 7n+2) - \kappa(2n^3 + 7n^2 + 7n+2))^3}, \\
\lambda &= \frac{A(A\gamma(n+2) + \beta(4n+2)) + \alpha(2n^3 + 7n^2 + 7n+2)}{(n+1)(n+2)(2n+1)}.
\end{aligned} \tag{100}$$

The forbidden values are $(n+1)(n+2)(2n+1) = 0$. The soliton solutions may be obtained under the condition

$$(2n-1)(n-2) = 0. \tag{101}$$

For $n = 2$, we have the following soliton solution to Equation (98):

$$u(x, t) = \sqrt{-\frac{5\beta}{8\gamma} \pm \frac{5\beta}{8\gamma} \operatorname{sech} \left(2\sqrt{5} \sqrt{\frac{\beta^2}{192\alpha\gamma\delta - 25\beta^2\delta - 192\gamma\kappa}} \left(x - \frac{192\alpha\gamma - 25\beta^2}{192\gamma} t + \xi_0 \right) \right)}. \tag{102}$$

In the case when $n = 1/2$, we have the following solution to Equation (98):

$$\begin{aligned}
u(x, t) &= \left(-\frac{2\beta}{5\gamma} \pm \frac{2\sqrt{2}\beta}{5\gamma} \operatorname{sech} \left(\sqrt{\frac{2\beta^2}{75\alpha\gamma\delta - 16\beta^2\delta - 75\gamma\kappa}} \right. \right. \\
&\quad \left. \left. \cdot \left(x - \frac{75\alpha\gamma - 16\beta^2}{75\gamma} t + \xi_0 \right) \right) \right)^2.
\end{aligned} \tag{103}$$

Finally, more solutions may be obtained by means of the transformation $u = v^{-2/n}$. We will not derive them here.

4. Conclusions

We successfully obtained exact solutions to regularized long-wave and generalized BBM equation by using different approaches. We showed the way to derive all traveling wave

solutions to all known until now variants of BBM. We compared the known previously published solutions with the solutions obtained in this work. Other methods to find exact solutions to nonlinear differential equations may be found in [1–3, 7, 9–39].

Data Availability

Data statement is not applicable.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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