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Research Article

Existence of Solutions for Superlinear Second-Order System with Noninstantaneous Impulses

Yucheng Bu

Danyang Normal School, Zhenjiang College, Zhenjiang 212300, China

Correspondence should be addressed to Yucheng Bu; ychbu@126.com

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Variational methods are used in order to establish the existence of nontrivial weak solution for superlinear second-order system with noninstantaneous impulses. The main result is obtained when a kind of definition of the weak solution for this system is introduced. Meanwhile, an example is presented to illustrate the main result.

1. Introduction

In this paper, we consider the following superlinear system with noninstantaneous impulsive effects:

$$\begin{cases} -\ddot{u} = \lambda b_i(t)\nabla F_i(u - u(t_{i+1})), & t \in]s_i, t_{i+1}], i \in M, \\ \dot{u}(t) = \alpha_i, & t \in]t_i, s_i], i \in M \setminus \{0\}, \\ \dot{u}(s_i^+) = \dot{u}(s_i^-), & i \in M \setminus \{0\}, \\ u(0) = 0 = u(T), & \dot{u}(0) = \alpha_0, \end{cases} \tag{1}$$

where $u(t) = (u^1(t), u^2(t), \cdots, u^N(t))$, $\lambda > 0$, $b_i \in \mathscr{L}]s_i, t_{i+1}]$, $F_i \in \mathscr{F}$, $\alpha_i \in \mathbf{R}^N (i \in M)$, $0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < t_m < s_m < t_{m+1} = T$, the impulses start abruptly at points t_i and keep the derivative constant on a finite time interval $]t_i, s_i]$, $\dot{u}(s_i^{\pm}) = \lim_{s \longrightarrow s_i^{\pm}} \dot{u}(s) (i \in M \setminus \{0\})$. Here, m is a given positive integer and $T > 0$. Denote

$$M \coloneqq \{0, 1, 2, \cdots, m\},$$
$$\mathscr{F} \coloneqq \left\{F \in C^1\left(\mathbf{R}^N, \mathbf{R}\right) \colon F(0) = 0\right\}, \tag{2}$$
$$\mathscr{L}]s, t] \coloneqq \left\{b \in L^1]s, t] \setminus \{0\} \colon b \geq 0 \, a.e. \text{in} \,]s, t]\right\},$$

where $0 \leq s < t \leq T$.

This paper is mainly motivated by two facts. For the past thirty years, impulsive differential equations have been studied extensively in the literature. There are many approaches to investigate solutions for impulsive differential equations, such as the method of lower and upper solutions, fixed point theory, coincidence degree theory, and geometric approach (see for example [1–8]). In addition, variational methods have been also dealt with impulsive differential equations since papers [9, 10] appeared. Meanwhile, many existence and multiplicity results of solutions for various impulsive differential equations have been obtained. We refer the reader to a large quantity of the references for papers [9, 10]. It is worth noting that in all these references, most of impulsive effects are instantaneous. However, in [11, 12], the authors considered a kind of noninstantaneous impulsive effects, which is motivated by paper [13]. On the other hand, in recent years, some new critical point theorems are used to study nonlinear differential equations with no impulsive effects (see for example, [14–21]). Based on these two facts, we investigate the existence of nontrivial solution for (1).

This paper is organized as follows. In Section 2, we recall a critical point theorem in [16] and introduce a definition of the weak solution for (1), which generalizes Definition 2.3 in [12] to an N-dimensional case. In Section 3, we present our result and complete its proof via variational methods. The result is new in the field of studying noninstantaneous impulsive problem. In Section 4, we present an example in order to illustrate our result.

2. Preliminaries

The following critical point theorem is crucial in our arguments.

Theorem 1 (see [16], Theorem 3.2). *Let X be a real Banach space and let $\Phi, \Psi : X \longrightarrow \mathbf{R}$ be two continuously Gâteaux differentiable functions such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u) < +\infty$ and assume that for each $\lambda \in]0, r/(\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u))[$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below. Then, for each $\lambda \in]0, r/(\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u))[$, the functional I_λ admits at least two distinct critical points.*

Consider the Hilbert space:

$$H_0^1 := \{u \in H^1(]0, T[; \mathbf{R}^N) \mid u(0) = 0 = u(T)\}, \quad (3)$$

with the inner product

$$\langle u, v\rangle_{H_0^1} = \int_0^T (\dot{u}(t), \dot{v}(t))dt + \int_0^T (u(t), v(t))dt, \forall u, v \in \mathrm{H}_0^1, \quad (4)$$

where $(\cdot, \cdot)$ is the inner product in $\mathbf{R}^N$. The corresponding norm is defined by

$$\|u\|_{H_0^1} = \left(\int_0^T |\dot{u}(t)|^2 dt + \int_0^T |u(t)|^2 dt\right)^{1/2}, \forall u \in H_0^1. \quad (5)$$

Meanwhile, for every $u, v \in H_0^1$, we define

$$\langle u, v\rangle_1 = \int_0^T (\dot{u}(t), \dot{v}(t))dt, \quad (6)$$

which is also an inner product in H_0^1, whose corresponding norm is that

$$\|u\| := \langle u, u\rangle_1^{1/2}. \quad (7)$$

Poincare's inequality [22] implies that

$$\int_0^T |u(t)|^2 dt \le \frac{1}{\lambda_1}\int_0^T |\dot{u}(t)|^2 dt, \quad (8)$$

where $\lambda_1 = \pi^2/T^2$ is the first eigenvalue of the Dirichlet problem

$$-\ddot{u}(t) = \lambda u(t), t \in [0, T]; u(0) = 0 = u(T). \quad (9)$$

Hence,

$$\sqrt{\frac{\lambda_1}{\lambda_1 + 1}}\|u\|_{H_0^1} \le \|u\| \le \|u\|_{H_0^1}. \quad (10)$$

That is, $\|\cdot\|$ is equivalent to $\|\cdot\|_{H_0^1}$. It is well known that $(H_0^1, \|\cdot\|_{H_0^1})$ is compactly embedded in $C([0, T], \mathbf{R}^N); \|\cdot\|_\infty)$, where $\|u\|_\infty = \max_{t\in[0,T]}|u(t)|$. Since the norms $\|\cdot\|_{H_0^1}$ and $\|\cdot\|$ are equivalent, there exists a positive number γ such that

$$\|u\|_\infty \le \gamma\|u\|, \quad (11)$$

where $\gamma \le k := \sqrt{2(1 + (1/\lambda_1))} \max\{\sqrt{T}, 1/\sqrt{T}\}$.

Following the idea of the variational approach for impulsive differential equation of [11, 12] and combining with (1), one has that for every $v \in H_0^1$,

$$\begin{aligned}\int_0^T (-\ddot{u}(t), v(t))dt &= \sum_{i=0}^{M}\int_{s_i}^{t_{i+1}} (-\ddot{u}(t), v(t))dt + \sum_{i=1}^{m}\int_{t_i}^{s_i}(-\ddot{u}(t), v(t))dt \\ &= \int_0^T (\dot{u}(t), \dot{v}(t))dt - \sum_{i=1}^{m}(\dot{u}(t_i^-) - \dot{u}(t_i^+), v(t_i)) \\ &\quad - \sum_{i=1}^{m}((\dot{u}(s_i^-) - \dot{u}(s_i^+)), v(s_i)) \\ &= \int_0^T (\dot{u}(t), \dot{v}(t))dt + \sum_{i=1}^{m}(-\dot{u}(t_i^-) + \alpha_i, v(t_i)).\end{aligned} \quad (12)$$

In addition, combing with (1) again, one has

$$\begin{aligned}&\sum_{i=0}^{m-1}\int_{s_i}^{t_{i+1}} \lambda(b_i(t)F_i(u - u(t_{i+1})), v(t_{i+1}))dt \\ &= \sum_{i=0}^{m-1}\int_{s_i}^{t_{i+1}} (-\ddot{u}(t), v(t_{i+1}))dt \\ &= \sum_{i=1}^{m}\int_{s_{i-1}}^{t_i} (-\ddot{u}(t), v(t_i))dt \\ &= \sum_{i=1}^{m}(-\dot{u}(t_i^-) + \alpha_{i-1}, v(t_i)).\end{aligned} \quad (13)$$

(12) and (13) show that

$$\begin{aligned}\int_0^T (-\ddot{u}(t), v(t))dt = &\int_0^T (\dot{u}(t), \dot{v}(t))dt + \sum_{i=1}^{m}(\alpha_i - \alpha_{i-1}, v(t_i)) \\ &+ \sum_{i=0}^{m-1}\int_{s_i}^{t_{i+1}} \lambda(b_i(t)F_i(u - u(t_{i+1})), v(t_{i+1}))dt.\end{aligned} \quad (14)$$

On the other hand,

$$\int_0^T (-\ddot{u}(t), v(t))dt = \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \lambda(b_i(t)F_i(u-u(t_{i+1})), v(t))dt + \sum_{i=1}^{m}\int_{t_i}^{s_i}\left(\frac{d}{dt}(\alpha_i), v(t)\right)dt = \lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}(b_i(t)F_i(u-u(t_{i+1})), v(t))dt. \quad (15)$$

Hence, (14) and (15), together with $v(t_{m+1}) = v(T) = 0$, imply that

$$\int_0^T (\dot{u}(t), \dot{v}(t))dt = \sum_{i=1}^{m}(\alpha_{i-1}-\alpha_i, v(t_i)) + \lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}(b_i(t)\nabla F_i(u(t)-u(t_{i+1})), v(t)-v(t_{i+1}))dt. \quad (16)$$

Similar to Definition 2.3 in [12], we can introduce the following concept of a weak solution for problem (1).

Definition 2. Say that $u \in H_0^1$ is a weak solution for problem (1) if (16) holds for any $v \in H_0^1$.

Now, we consider the functionals $I_\lambda, \Phi, \Psi : H_0^1 \longrightarrow \mathbf{R}$ defined by putting

$$\Phi(u) := \frac{1}{2}\|u\|^2, \quad (17)$$

$$\Psi(u) := \frac{1}{\lambda}\sum_{i=1}^{m}(\alpha_{i-1}-\alpha_i, \mathrm{u}(t_i)) + \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} b_i(t)F_i(u(t)-u(t_{i+1}))dt, \quad (18)$$

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u). \quad (19)$$

Since $b_i \in \mathscr{L}]s_i, t_{i+1}]$, $F_i \in \mathscr{F}$ and $\alpha_i \in \mathbf{R}^N$, Φ and Ψ are two continuously Gâteaux differentiable with

$$\Phi'(u)v = \int_0^T (\dot{u}(t), \dot{v}(t))dt, \quad (20)$$

$$\Psi'(u)v = \frac{1}{\lambda}\sum_{i=1}^{m}(\alpha_{i-1}-\alpha_i, v(t_i)) + \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}(b_i(t)\nabla F_i(u(t)-u(t_{i+1})), v(t)-v(t_{i+1}))dt, \quad (21)$$

for every $u, v \in H_0^1$. Hence,

$$I_\lambda'(u)v = \Phi'(u)v - \lambda\Psi'(u)v. \quad (22)$$

Namely, $I_\lambda \in C^1(H_0^1, \mathbf{R})$. Definition 2, together with (20)–(22), shows that a critical point of I_λ corresponds to a weak solution for (1).

3. Main Result and Its Proof

From now on, we refer to M as the range of i, unless specifically stated. Let $\|b_i\|_1$ be the usual norm in $L^1]s_i, t_{i+1}]$. The following theorem is our main result.

Theorem 3. *Suppose that*

(F) There exist $r_i > 0$ and $\mu_i > 2$ such that for $|\xi| \geq r_i$,

$$0 < \mu_i F_i(\xi) \leq (\nabla F_i(\xi), \xi), \quad (23)$$

hold. Then, for every $b_i \in \mathscr{L}]s_i, t_{i+1}] \setminus \{0\}$ and for every

$$\lambda \in \left]0, \sup_{c>0}\frac{(c^2/2k^2) - c\sum_{i=1}^{m}|\alpha_{i-1}-\alpha_i|}{\sum_{i=1}^{m}\|b_i\|_1 \max_{|\xi|\leq 2c} F_i(\xi)}\right[, \quad (24)$$

(1) admits at least one nontrivial weak solution.

Proof. We complete the proof in four steps.

Step 1. Clearly, H_0^1 is a real Banach space. Φ, Ψ are two continuously Gâteaux differentiable functional. In addition, (17), (18), and $F_i \in \mathscr{F}$ imply that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$.

Step 2. I_λ satisfies the (PS)-condition. That is, every $\{u_n\}$ such that $I_\lambda(u_n)$ is bounded and $I_\lambda'(u_n) \longrightarrow 0$ as $n \longrightarrow +\infty$ contains a convergent subsequence.

By (11), we have

$$\left|\sum_{i=1}^{m}(\alpha_{i-1}-\alpha_i, u(t_i))\right| \leq \sum_{i=1}^{m}|\alpha_{i-1}-\alpha_i|\|u\|_\infty \leq \sum_{i=1}^{m}|\alpha_{i-1}-\alpha_i|\gamma\|u\|. \quad (25)$$

Let $\mu := \min\{\mu_i : i \in M\}$, then $\mu > 2$. From (F) and (25), we deduce that

$$\mu I_\lambda(u_n) - I_\lambda'(u_n)u_n = \left(\frac{\mu}{2}-1\right)\|u_n\|^2 - (\mu-1)\sum_{i=1}^{m}(\alpha_{i-1}-\alpha_i, u_n(t_i)) - \lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} b_i(t)(\mu F_i(u_n(t)-u_n(t_{i+1})) - (\nabla F_i(u_n(t)-u_n(t_{i+1})), u_n(t)-u_n(t_{i+1})))dt \geq \left(\frac{\mu}{2}-1\right)\|u_n\|^2 - (\mu-1)\sum_{i=1}^{m}|\alpha_{i-1}-\alpha_i|\gamma\|u_n\|, \quad (26)$$

which implies that u_n is bounded in H_0^1. Since H_0^1 is a reflexive Banach space, passing to a subsequence if necessary, we may assume that there is a $u_0 \in H_0^1$ such that $u_n u_0$ in H_0^1. Then, $\{u_n\}$ converges uniformly to u_0 on $[0, T]$ and $u_n \longrightarrow u_0$ in $L^2(]0, T[\,; \mathbf{R}^N)$. In addition,

$$\left\langle I'_\lambda(u_m) - I'_\lambda(u_n), u_m - u_n \right\rangle = \|u_m - u_n\|^2 - \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} b_i(t)(\nabla F_i(u_m(t) - u_m(t_{i+1})) - \nabla F_i(u_n(t) - u_n(t_{i+1})), u_m(t) - u_m(t_{i+1}) - u_n(t) + u_n(t_{i+1}))dt, \quad (27)$$

$$\begin{aligned} &|u_m(t) - u_m(t_{i+1}) - u_n(t) + u_n(t_{i+1})| \\ &\le |u_m(t) - u_n(t)| + |u_m(t_{i+1}) - u_n(t_{i+1})| \\ &\le 2\|u_m - u_n\|_\infty \longrightarrow 0, \text{as } m, n \longrightarrow \infty. \end{aligned} \quad (28)$$

Hence, the second term on the right hand of (27) converges to 0 as $m, n \longrightarrow \infty$ because of (28) and $F_i \in C^1(\mathbf{R}^\mathbf{N})$. Moreover, $I'_\lambda(u_n) \longrightarrow 0$ as $n \longrightarrow +\infty$ implies

$$\left|\left\langle I'_\lambda(u_m) - I'_\lambda(u_n), u_m - u_n \right\rangle\right| \le \|I'_\lambda(u_m) - I'_\lambda(u_n)\| \|u_m - u_n\| \longrightarrow 0, \quad (29)$$

as $m, n \longrightarrow \infty$. (27)–(29) show that $\|u_m - u_n\| \longrightarrow 0$ as m, $n \longrightarrow \infty$. By the completeness of H_0^1, we see that $\{u_n\}$ contains a convergent subsequence in H_0^1.

Step 3. I_λ is unbounded from below.

By (F), there exist $\alpha_i, \beta_i > 0$ such that

$$F_i(\xi) \ge \alpha_i |\xi|^{\mu_i} - \beta_i. \quad (30)$$

for every $\xi \in \mathbf{R}^N$.

Denote $\alpha := \min\{\alpha_i\}$, $\beta := \max\{\beta_i\}$, $\mu = \min\{\mu_i\}$. Hence,

$$\mathrm{F}_i(\xi) \ge \alpha|\xi|^\mu - \beta. \quad (31)$$

for every $\xi \in \mathbf{R}^N$.

Let $k > 0$ and $v_0 \in H_0^1$ with $\|v_0\| = 1$ and $v_0(t)$ is not a constant for a.e. $[0, t_1]$. In view of (31), we have

$$\int_{s_i}^{t_{j+1}} F_i(kv_0 - kv_0(t_{i+1}))dt \ge \alpha k^\mu \int_{s_i}^{t_{j+1}} |v_0 - v_0(t_{i+1})|^\mu dt - \beta(t_{j+1} - s_i). \quad (32)$$

Denote

$$K_i := \int_{s_i}^{t_{j+1}} |v_0 - v_0(t_{i+1})|^\mu \, dt. \quad (33)$$

Then,

$$0 \le K_i \le \int_{s_i}^{t_{j+1}} 2^\mu \|v_0\|_\infty^\mu \, dt \le \int_{s_i}^{t_{j+1}} (2\gamma)^\mu \|v_0\|^\mu dt = (2\gamma)^\mu (t_{i+1} - s_i) \quad (34)$$

Hence, by (17)–(19) and (32)–(34), we have

$$\begin{aligned} I_\lambda(kv_0) &= \frac{1}{2}\|kv_0\|^2 - \sum_{i=1}^{m}(\alpha_{i-1} - \alpha_i, kv_0(t_i)) + \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} b_i(t) F_i(kv_0(t) - kv_0(t_{i+1}))dt \\ &\le \frac{1}{2}k^2 + k\gamma \sum_{i=1}^{m}|\alpha_{i-1} - \alpha_i| - \alpha \sum_{i=0}^{m}\|b_i\|_1 (t_{i+1} - s_i)(2k\gamma)^\mu + \beta T \\ &= -\alpha \sum_{i=0}^{m}\|b_i\|_1 (t_{i+1} - s_i)(2\gamma)^\mu k^\mu + \frac{1}{2}k^2 + \gamma \sum_{i=1}^{m}|\alpha_{i-1} - \alpha_i| k + \beta T, \end{aligned} \quad (35)$$

which shows that $I_\lambda(kv_0) \longrightarrow -\infty$ as $k \longrightarrow \infty$ because the coefficient of k^μ is negative.

Step 4. $\lambda \in]0, \sup_{c>0}((c^2/2k^2)c^2/2k^2 - c\sum_{i=1}^{m} | \alpha_{i-1} - \alpha_i |)/\sum_{i=1}^{m}\|b_i\|_1 \max_{|\xi|\le 2c} F_i(\xi)[$.

Let c be such that

$$0 < \lambda < \frac{c^2/2k^2 - c\sum_{i=1}^{m}|\alpha_{i-1} - \alpha_i|}{\sum_{i=1}^{m}\|b_i\|_1 \max_{|\xi|\le 2c} F_i(\xi)}, \quad (36)$$

and put

$$r = \frac{c^2}{2k^2}. \quad (37)$$

By (18), one has

$$\begin{aligned} &\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) \le \frac{c}{\lambda}\sum_{i=1}^{m}|\alpha_{i-1} - \alpha_i| + \sum_{i=1}^{m}\|b_i\|_1 \max_{|\xi|\le 2c} F_i(\xi) < +\infty, \\ &0 < \lambda < \frac{r}{c/\lambda\sum_{i=1}^{m}|\alpha_{i-1} - \alpha_i| + \sum_{i=1}^{m}\|b_i\|_1 \max_{|\xi|\le 2c} F_i(\xi)} \le \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}. \end{aligned} \quad (38)$$

Now, we apply Theorem 1 to conclude that (1) admits at least one nontrivial weak solution. □

4. Example

Let us consider the following problem:

$$\begin{cases} -\ddot{u} = (t + 0.1)(u - u(t_{i+1}))|u - u(t_{i+1})|^2, & t \in (s_i, t_{i+1}], i = 0, 1, \\ \dot{u}(t) = (0.1, 0.11, 0.12), & t \in (t_i, s_i], i = 0, 1, \\ \dot{u}(s_i^+) = \dot{u}(s_i^-), & i = 0, 1, \\ u(0) = 0 = u(1), & \dot{u}(0) = (0, 0, 0), \end{cases} \quad (39)$$

where $0 = s_0 < t_1 = 0.1 < 0.9 = s_1 < t_2 = 1$.

In this case, $N = 3$, $T = \lambda = 1$, $b_i(t) = t + 0.1, t \in (s_i, t_{i+1}]$, $F_i(u) = (1/4)|u|^4, i = 0, 1$, $k = \sqrt{2(1 + (1/\pi^2))}$. Choose $c = 0.5$, one can verify that all the conditions in Theorem 3 are satisfied and (39) admits at least one nontrivial weak solution.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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