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Research Article

Stability Theory for Nullity and Deficiency of Linear Relations

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Let $\mathscr{A}$ and $\mathscr{B}$ be two closed linear relations acting between two Banach spaces X and Y, and let λ be a complex number. We study the stability of the nullity and deficiency of $\mathscr{A}$ when it is perturbed by $\lambda\mathscr{B}$. In particular, we show the existence of a constant $\rho > 0$ for which both the nullity and deficiency of $\mathscr{A}$ remain stable under perturbation by $\lambda\mathscr{B}$ for all λ inside the disk $|\lambda| < \rho$.

1. Introduction

For purposes of introduction, we shall consider bounded linear operators A and B with domain X and range in Y. As usual, let $N(A)$ and $R(A)$ denote the null space and range of A respectively. The dimensions of $N(A)$ and $Y/R(A)$ are called the nullity and the deficiency of A respectively and denoted by $\alpha(A)$ and $\beta(A)$. It is well known that $\alpha(A)$ and $\beta(A)$ have some kind of stability when A is subjected to some kind of perturbation (see for example [1]). More precisely, $\alpha(A)$ and $\beta(A)$ are unchanged when A is perturbed by some bounded linear operator B under certain prescribed conditions. This stability can be described in the form

$$\alpha(A-B)-\beta(A-B)=\alpha(A)-\beta(A). \tag{1}$$

Another convenient way of describing this stability is to put it in the form

$$\begin{aligned}\alpha(A-B)&=\alpha(A),\\ \beta(A-B)&=\beta(A).\end{aligned} \tag{2}$$

The stability concept described here is very useful in studying eigenvalue problems of the form $Ax=\lambda Bx$ and $A^*y=\lambda B^*y$, where A^* denotes the adjoint operator.

This paper deals with the stability theory for nullity and deficiency of linear relations, and it can be seen as a generalization of the classical theory for the corresponding quantities for linear operators. The theory and exposition developed here goes along the lines of the classical texts on the perturbation theory for linear operators (see for example [1, 2]), but in a more general setting. Some stability theorems for multivalued linear operators or what we refer to here as linear relations have been considered in [3] and more recently in [4]. In either of these cases, the perturbing multivalued linear operator $\mathscr{B}$ does not vary with the varying λ as the case we consider here.

2. Preliminaries

2.1. Relations on Sets. In this section, we introduce some notation and consider some basic concepts concerning relations on sets. Let U and V be two nonempty sets. By a *relation* $\mathscr{T}$ from U to V, we mean a mapping whose domain $D(\mathscr{T})$ is a nonempty subset of U and, taking values in $2^V \setminus \varnothing$, the collection of all nonempty subsets of V. Such a mapping $\mathscr{T}$ is also referred to as a *multivalued* operator or at times as a *set-valued function*. If $\mathscr{T}$ maps the elements of its domain to singletons, then $\mathscr{T}$ is said to be a *single valued* mapping or operator. Let $\mathscr{T}$ be a relation from U to V, and let $\mathscr{T}(u)$ denote the image of an element $u\in U$ under $\mathscr{T}$. If we define $\mathscr{T}(u)=\varnothing$ for $u\in U$ and $u\notin D(\mathscr{T})$, then the domain $D(\mathscr{T})$ of $\mathscr{T}$ is given by

$$D(\mathscr{T})=\{u\in U : \mathscr{T}(u)\neq\varnothing\}. \tag{3}$$

Denote by $R(U,V)$ the class of all relations from U to V. If $\mathcal{T}$ belongs to $R(U,V)$, the graph of $\mathcal{T}$, which we denote by $G(\mathcal{T})$, is the subset of $U\times V$ defined by

$$G(\mathcal{T})=\{(u,v)\in U\times V : u\in D(\mathcal{T}), v\in \mathcal{T}(u)\}. \quad (4)$$

A relation $\mathcal{T}\in R(U,V)$ is uniquely determined by its graph, and conversely, any nonempty subset of $U\times V$ uniquely determines a relation $\mathcal{T}\in R(U,V)$.

For a relation $\mathcal{T}\in R(U,V)$, we define its inverse $\mathcal{T}^{-1}$ as the relation from V to U whose graph $G(\mathcal{T}^{-1})$ is given by

$$G(\mathcal{T}^{-1})=\{(v,u)\in V\times U : (u,v)\in G(\mathcal{T})\}. \quad (5)$$

Let $\mathcal{T}\in R(U,V)$. Given a subset M of U, we define the image of M, $\mathcal{T}(M)$ to be

$$\mathcal{T}(M)=\bigcup\{\mathcal{T}(m): m\in M\cap D(\mathcal{T})\}. \quad (6)$$

With this notation we define the range of $\mathcal{T}$ by

$$R(\mathcal{T}):=\mathcal{T}(U). \quad (7)$$

Let N be a nonempty subset of V. The definition of $\mathcal{T}^{-1}$ given in (5) above implies that

$$\mathcal{T}^{-1}(N)=\{u\in D(\mathcal{T}): N\cap\mathcal{T}(u)\neq\varnothing\}. \quad (8)$$

If in particular $v\in R(\mathcal{T})$, then

$$\mathcal{T}^{-1}(v)=\{u\in D(\mathcal{T}): v\in\mathcal{T}(u)\}. \quad (9)$$

For a detailed study of relations, we refer to [3, 5–8], and [9].

2.2. Linear Relations. Let X and Y be linear spaces over a field $\mathbb{K}=\mathbb{R}$ (or $\mathbb{C}$), and let $\mathcal{T}\in R(X,Y)$. We say that $\mathcal{T}$ is a *linear relation* or a *multivalued linear operator* if for all $x,z\in D(\mathcal{T})$ and any nonzero scalar α we have

(1) $T(x)+\mathcal{T}(z)=\mathcal{T}(x+z)$

(2) $\alpha\mathcal{T}(x)=\mathcal{T}(\alpha x)$

The equalities in items (1) and (2) above are understood to be set equalities. These two conditions indirectly imply that the domain of a linear relation is a linear subspace. The class of linear relations in $R(X,Y)$ will be denoted by $L R(X,Y)$. If $X=Y$, then we denote $LR(X,X)$ by $LR(X)$. We say that $\mathcal{T}$ is a linear relation in X if $\mathcal{T}\in LR(X)$. We shall use the term *operator* to refer to a single valued linear operator while a multivalue linear operator will be generally referred to as a *linear relation*.

If X and Y are normed linear spaces, we say that $\mathcal{T}\in LR(X,Y)$ is closed if its graph $G(\mathcal{T})$ is a closed subspace of $X\times Y$. The collection of all such $\mathcal{T}$ will be denoted by $CLR(X,Y)$.

We conclude this section with the following theorems which are taken from [3].

Theorem 1. *Let $\mathcal{T}\in R(X,Y)$. The following properties are equivalent.*

(i) $\mathcal{T}$ is a linear relation

(ii) $G(\mathcal{T})$ is a linear subspace of $X\times Y$

(iii) $\mathcal{T}^{-1}$ is a linear relation

(iv) $G(\mathcal{T}^{-1})$ is a linear subspace of $Y\times X$

Corollary 2. *Let $\mathcal{T}\in R(X,Y)$.*

(i) Then, $\mathcal{T}$ is a linear relation if and only if

$$\mathcal{T}(\alpha x_1+\beta x_2)=\alpha\mathcal{T}(x_1)+\beta\mathcal{T}(x_2) \quad (10)$$

holds for all $x_1,x_2\in D(\mathcal{T})$ and some nonzero scalars α and β

(ii) If $\mathcal{T}$ is a linear relation, then $\mathcal{T}(0)$ and $\mathcal{T}^{-1}(0)$ are linear subspaces

For a linear relation $\mathcal{T}$, the subspace $\mathcal{T}^{-1}(0)$ is called the *null space* (or *kernel*) of $\mathcal{T}$ and is denoted by $N(\mathcal{T})$.

Theorem 3. *Let $\mathcal{T}$ be a linear relation in a linear space X, and let $x\in D(\mathcal{T})$. Then, $y\in\mathcal{T}(x)$ if and only if*

$$\mathcal{T}(x)=\mathcal{T}(0)+y. \quad (11)$$

Theorem 3 shows that $\mathcal{T}$ is single valued if and only if $\mathcal{T}(0)=\{0\}$.

Theorem 4. *Let $\mathcal{T}\in R(X,Y)$. Then, $\mathcal{T}$ is a linear relation if and only if for all $x_1,x_2\in D(\mathcal{T})$ and all scalars α and β*

$$\alpha\mathcal{T}(x_1)+\beta\mathcal{T}(x_2)\subset\mathcal{T}(\alpha x_1+\beta x_2). \quad (12)$$

Theorem 5. *Let $\mathcal{T}\in LR(X,Y)$. Then,*

(a) $\mathcal{T}(M+N)=\mathcal{T}M=\mathcal{T}N$ for $M\subset X$ and $N\subset D(\mathcal{T})$

(b) $\mathcal{T}\mathcal{T}^{-1}(M)=M\cap R(\mathcal{T})+\mathcal{T}(0)$ for $M\subset Y$

(c) $\mathcal{T}^{-1}\mathcal{T}(M)=M\cap D(\mathcal{T})+\mathcal{T}^{-1}(0)$ for $M\subset X$

2.3. Normed Linear Relations. Let X be a normed linear space. By B_X, we shall mean the set

$$B_X:=\{x\in X : |x|\leq 1\}. \quad (13)$$

For a closed linear subspace E of X, we denote by Q_E the natural quotient map with domain X and null space E. For $\mathcal{T}\in LR(X,Y)$, we shall denote $Q_{\overline{\mathcal{T}(0)}}$ by $Q_{\mathcal{T}}$. It is well known that for $\mathcal{T}\in LR(X,Y)$, the operator $Q_{\mathcal{T}}\mathcal{T}$ is single valued (see [3]).

For $\mathcal{T} \in LR(X, Y)$, we set $\|\mathcal{T}(x)\| = \|Q_{\mathcal{T}}\mathcal{T}(x)\|$ for $x \in D(\mathcal{T})$ and $\|\mathcal{T}\| = \|Q_{\mathcal{T}}\mathcal{T}\|$. Note that these notions do not define a norm since nonzero relations can have zero norm.

Lemma 6. *Let $\mathcal{A}, \mathcal{B} \in CLR(X, Y)$ be such that $D(\mathcal{B}) \supset D(\mathcal{A})$ and $\mathcal{B}(0) \subset \mathcal{A}(0)$. If $x_1, x_2 \in D(\mathcal{A})$ are such that $\mathcal{A}(x_1) \cap \mathcal{B}(x_2) \neq \emptyset$, then $\mathcal{A}(x_1) - \mathcal{B}(x_2) \subset \mathcal{A}(0)$.*

Proof. Let $z \in \mathcal{A}(x_1) \cap \mathcal{B}(x_2)$. Since $Q_{\mathcal{A}}$ and $Q_{\mathcal{B}}$ are single valued, we see that

$$Q_{\mathcal{A}}(\mathcal{A}(x_1) - \mathcal{B}(x_2)) = Q_{\mathcal{A}}\mathcal{A}(x_1) - Q_{\mathcal{A}}\mathcal{B}(x_2) = \tilde{z} - \tilde{z} = \widehat{0}. \tag{14}$$

Hence, $\mathcal{A}(x_1) - \mathcal{B}(x_2) \in \mathcal{A}(0)$. □

The following lemma is proved in [3].

Lemma 7. *The following properties are equivalent for a linear relation $\mathcal{A}$.*

(i) $\mathcal{A}$ is closed

(ii) $Q_{\mathcal{A}}\mathcal{A}$ is closed and $\mathcal{A}(0)$ is closed

Lemma 8.

(a) Let $\mathcal{T} \in LR(X, Y)$ be bounded. Then, $\|\mathcal{T}x\| \leq \|\mathcal{T}\|\|x\|$

(b) For $\mathcal{S}, \mathcal{T} \in LR(X, Y)$ with $D(\mathcal{S}) \subset D(\mathcal{T})$ and $\mathcal{T}(0) \subset \mathcal{S}(0)$, we have

$$\|\mathcal{S}(x) + \mathcal{T}(x)\| \geq \|\mathcal{S}(x)\| - \|\mathcal{T}(x)\|. \tag{15}$$

Proof.

(a) From ([3], II.1.6), we have $\|\mathcal{T}\| = \sup_{x \in B_{D(\mathcal{T})}} \|\mathcal{T}(x)\|$ so that

$$\begin{gathered} \|\mathcal{T}\| = \sup_{x \in D(\mathcal{T})} \left\| \frac{1}{\|x\|}\mathcal{T}(x) \right\|, \\ \|\mathcal{T}\| \geq \left\| \frac{1}{\|x\|}\mathcal{T}(x) \right\|, \quad x \in D(\mathcal{T}). \end{gathered} \tag{16}$$

The inequality $\|\mathcal{T}\|\|x\| \geq \|\mathcal{T}(x)\|$, for all $x \in D(\mathcal{T})$, then follows from ([3], II.1.5).

(b) Since $\mathcal{T}(0) \subset \mathcal{S}(0)$, we see that $(\mathcal{S} + \mathcal{T})(0) = \mathcal{S}(0) + \mathcal{T}(0) = \mathcal{S}(0)$ since $\mathcal{S}(0)$ is a subspace (linear subset). For $x \in D(\mathcal{S})$, let $s \in \mathcal{S}(x)$ and let $t \in \mathcal{T}(x)$. Then, $s + t \in (\mathcal{S} + \mathcal{T})(x) = \mathcal{S}(x) + \mathcal{T}(x)$, and so by ([3], II.1.4), we get

$$\begin{aligned} \|\mathcal{S}(x) + \mathcal{T}(x)\| &= \operatorname{dist}(s + t, (\mathcal{S} + \mathcal{T})(0)) \\ &= \operatorname{dist}(s + t, \mathcal{S}(0)) \\ &\geq \operatorname{dist}(s, \mathcal{S}(0)) - \operatorname{dist}(t, (\mathcal{S}(0)) \\ &\geq \operatorname{dist}(s, \mathcal{S}(0)) - \operatorname{dist}(t, (\mathcal{T}(0)) \\ &= \|\mathcal{S}(x)\| - \|\mathcal{T}(x)\|. \end{aligned} \tag{17}$$

□

Let$X$ be a normed space. By X', we denote the norm dual of X, that is, the space of all continuous linear functionals x' defined on X, with norm

$$\|x'\| = \inf\left\{\lambda : \left|\left[x, x'\right]\right| \leq \lambda\|x\|, \quad \text{for all } x \in X\right\}, \tag{18}$$

where $[x, x'] := x'(x)$ denotes the action of $x' \in X'$ on $x \in X$. If $M \subset X$ and $N \subset X'$, we write $M^{\perp}$ and $N^{\top}$ to mean

$$\begin{gathered} M^{\perp} := \left\{x' \in X' : \left[x, x'\right] = 0, \quad \text{for all } x \in M\right\}, \\ N^{\top} := \left\{x \in X : \left[x, x'\right] = 0, \quad \text{for all } x' \in N\right\}. \end{gathered} \tag{19}$$

Let $\mathcal{T}$ be a linear relation with $D(\mathcal{T}) \subset X$ and $R(\mathcal{T}) \subset Y$. We define the adjoint $\mathcal{T}'$ of $\mathcal{T}$ by

$$G\left(\mathcal{T}'\right) := G\left(-\mathcal{T}^{-1}\right)^{\perp} \subset Y' \times X', \tag{20}$$

where

$$\left[(y, x), \left(y', x'\right)\right] = \left[x, x'\right] + \left[y, y'\right]. \tag{21}$$

This means that

$$\left(y', x'\right) \in G\left(\mathcal{T}'\right) \text{ if and only if } \left[y, y'\right] - \left[x, x'\right] = 0, \quad \text{for all } (x, y) \in G(\mathcal{T}). \tag{22}$$

From (22), we see that $y'(y) = x'(x)$ for all $y \in \mathcal{T}(x)$, $x \in D(\mathcal{T})$. Hence,

$$x' \in \mathcal{T}'\left(y'\right), \quad \text{if and only if } y'\mathcal{T}(x) = x'(x), \text{for all } x \in D(\mathcal{T}). \tag{23}$$

This means that x' is an extension of $y'\mathcal{T}(x)$, and therefore, the adjoint $\mathcal{T}'$ can be characterized as follows:

$$G\left(\mathcal{T}'\right) = \left\{\left(y', x'\right) \in Y' \times X' \text{ such that } x' \text{ is an extension of } y'\mathcal{T}.\right. \tag{24}$$

Please note that $\mathcal{T}' \in CLR(Y', X')$ (see [3], III.1.2).

Lemma 9 ([3], III.1.4). *Let $\mathcal{T}$ be a closed linear relation. Then,*

(1) $N(\mathcal{T}') = R(\mathcal{T})^{\perp}$

(2) $\mathcal{T}'(0) = D(\mathcal{T})^{\perp}$

(3) $N(\mathcal{T}) = R(\mathcal{T}')^{\top}$

(4) $\mathcal{T}(0) = D(\mathcal{T}')^{\top}$

Remark 10. If $\mathcal{T}$ and $\mathcal{S}$ are closed linear relations with $D(\mathcal{T}) \subset D(\mathcal{S})$ and $\mathcal{S}(0) \subset \mathcal{T}(0)$, then $\mathcal{S}'(0) \subset \mathcal{T}'(0)$ by Lemma 9(2).

3. Lower Bound of a Closed Linear relation

Consider a closed linear relation $\mathcal{A}$ on a Banach space X, and let $N(\mathcal{A})$ denote the null space of $\mathcal{A}$ which is closed since $\mathcal{A}$ is closed. Since $N(\mathcal{A}) \subset D(\mathcal{A})$, a coset $\tilde{x} \in \tilde{X} = X/N(\mathcal{A})$ which contains a point of $x \in D(\mathcal{A})$ consists entirely of points of $D(\mathcal{A})$. To see that this is the case, let $\tilde{x} \in \tilde{X}$ and let $x, y \in \tilde{x}$ with $x \in D(\mathcal{A})$. Then, $y - x \in N(\mathcal{A}) \subset D(\mathcal{A})$ and the linearity of $D(\mathcal{A})$ implies that $y = x + (y - x) \in D(\mathcal{A})$. Let $\tilde{D}$ denote the collection of all such cosets $\tilde{x}$. On setting,

$$A(\tilde{x}) := Q_{\mathcal{A}}\mathcal{A}(x), \quad \text{for } \tilde{x} \in \tilde{D}, \tag{25}$$

we define a linear operator $A : \tilde{X} \longrightarrow \widehat{X}$, where $\widehat{X} := X/\mathcal{A}(0)$. To see that (25) is well defined, let $x, y \in \tilde{x}$. Then, $x - y \in N(\mathcal{A})$, and therefore,

$$0 \in \mathcal{A}(0) = \mathcal{A}(x - y) = \mathcal{A}(x) - \mathcal{A}(y). \tag{26}$$

We see from (26) that $\mathcal{A}(x) \cap \mathcal{A}(y) \neq \varnothing$. So, let $u \in \mathcal{A}(x) \cap \mathcal{A}(y)$. Then,

$$\mathcal{A}(x) = \mathcal{A}(0) + u = \mathcal{A}(y), \tag{27}$$

so that $Q_{\mathcal{A}}\mathcal{A}(x) = Q_{\mathcal{A}}\mathcal{A}(y)$. We have

$$D(A) = \tilde{D}, R(A) = R(Q_{\mathcal{A}}\mathcal{A}), N(A) = \{\tilde{0}\}. \tag{28}$$

Remark 11. Since $\mathcal{A}(0) \subset R(\mathcal{A})$, we also have that a coset $\widehat{x} \in \widehat{X}$ that contains a point of $R(\mathcal{A})$ consists entirely of element of $R(\mathcal{A})$. To see that this is the case, let $\widehat{x}$ be a coset in $\widehat{X}$ and let $u, v \in \widehat{x}$ with $u \in R(\mathcal{A})$. Then, $v - u \in \mathcal{A}(0) \subset R(\mathcal{A})$. The linearity of $R(\mathcal{A})$ implies that $v = u + (v - u) \in R(\mathcal{A})$.

Lemma 12. *The linear operator A defined by (25) is closed.*

Proof. Let $\{\tilde{x}_n\}$ be a sequence in $\tilde{D}$ such that $\tilde{x}_n \longrightarrow \tilde{x} \in \tilde{X}$, and let $\{Q_{\mathcal{A}}\mathcal{A}(x_n)\}$ be a sequence in $R(A)$ such that $Q_{\mathcal{A}}\mathcal{A}(x_n) \longrightarrow \widehat{y} \in \widehat{X}$. Let $x_n \in \tilde{x}_n$ and $x \in \tilde{x}$. Since $\tilde{x}_n \longrightarrow \tilde{x}$, we see that $\operatorname{dist}(x_n - x, N(\mathcal{A})) \longrightarrow 0$. This means that $x_n - x$ converges to some element of $N(\mathcal{A})$, say,

$$x_n - x \longrightarrow u \in N(\mathcal{A}). \tag{29}$$

From (29), we see that $x_n \longrightarrow x + u = w \in \tilde{x}$.

Since $Q_{\mathcal{A}}\mathcal{A}(x_n) \longrightarrow \widehat{y} \in \widehat{X}$, that is, $\widehat{z}_n \longrightarrow \widehat{y}$, we see that $\operatorname{dist}(z_n - y, \mathcal{A}(0)) \longrightarrow 0$ as $n \longrightarrow \infty$ and so $z_n \longrightarrow y + v = z \in \widehat{y}$ for some $v \in \mathcal{A}(0)$ (where $z_n \in \mathcal{A}(x_n)$ for each $n \in \mathbb{N}$). The closedness of $\mathcal{A}$ implies that $w \in D(\mathcal{A})$ and $z \in \mathcal{A}(w)$. Hence, $\tilde{x} \in \tilde{D}$ and $A(\tilde{x}) = Q_{\mathcal{A}}\mathcal{A}(x) = \widehat{y}$, showing that A is closed. □

We see that A^{-1} is single valued since $A^{-1}(\widehat{0}) = \{\tilde{0}\}$. We now introduce the quantity $\gamma(\mathcal{A})$ called the lower bound of the linear relation $\mathcal{A}$. By definition,

$$\gamma(\mathcal{A}) = \frac{1}{\|A^{-1}\|}, \tag{30}$$

with the understanding that $\gamma(\mathcal{A}) = 0$ if A^{-1} is unbounded and that $\gamma(\mathcal{A}) = \infty$ if $A^{-1} = 0$. It follows from (30) that

$$\gamma(\mathcal{A}) = \sup\{\gamma \in \mathbb{R} : \|\mathcal{A}(x)\| \geq \gamma\|\tilde{x}\| = \operatorname{dist}(x, N(\mathcal{A})), \quad \forall x \in D(\mathcal{A})\}. \tag{31}$$

Note that $\gamma(\mathcal{A}) = \infty$ if and only if $\mathcal{A}(x) = \mathcal{A}(0)$ for all $x \in D(\mathcal{A})$. In order for (31) to hold even for this case, one should stipulate that $\infty \times 0 = 0$. Obviously, $\gamma(\mathcal{A}) = \gamma(A)$.

Please note that characterization (31) implies that if $\gamma(\mathcal{A}) = 0$ then the domain of $\mathcal{A}$ cannot consist of the zero element alone.

The fact that $\gamma(\mathcal{A}) = \infty$ if and only if $\mathcal{A}(x) = \mathcal{A}(0)$ for all $x \in D(\mathcal{A})$ leads to Lemma 13 (see also [3], Proposition II.2.2).

Lemma 13. *For $\mathcal{A} \in CLR(X, Y)$, we have*

$$\gamma(\mathcal{A}) = \begin{cases} \infty, & \text{if } D(\mathcal{A}) \subset N(\mathcal{A}), \\ \inf\left\{\dfrac{\|\mathcal{A}(x)\|}{\|\tilde{x}\|} : x \in D(\mathcal{A}) \& x \notin N(\mathcal{A}), \right. & \text{otherwise}. \end{cases} \tag{32}$$

Remark 14. A bounded linear operator T is closed if and only if $D(T)$ is closed.

Proof. Suppose that $u_n \longrightarrow u$ with $u_n \in D(T)$. The boundedness of T implies that $T(u_n)$ is a Cauchy sequence and therefore converges, say $T(u_n) \longrightarrow v$. The closedness of T implies that $u \in D(T)$ and $T(u) = v$. This shows that $D(T)$ is closed.

If $\mathcal{S}$ is a closed linear relation from X to Y, the graph of $\mathcal{S}$, $G(\mathcal{S})$ is a closed subset of $X \times Y$. Sometimes, it is convenient to regard it as a subset of $Y \times X$. More precisely, let $G'(\mathcal{S})$ be the linear subset of $Y \times X$ consisting of all pairs of the form (v, u), where $u \in D(\mathcal{S})$ and $v \in \mathcal{S}(u)$. We shall call $G'(\mathcal{S})$ the inverse graph of $\mathcal{S}$. As in the case of the graph $G(\mathcal{S})$, $G'(\mathcal{S})$ is closed if and only $\mathcal{S}^{-1}$ is closed. Clearly, $G(\mathcal{S}) = G'(\mathcal{S}^{-1})$. Thus, $\mathcal{S}^{-1}$ is closed if and only $\mathcal{S}$ is closed.

Lemma 15. *If $\mathscr{A}$ is a closed linear relation in a Banach space X, then $R(\mathscr{A})$ is closed if and only if $\gamma(\mathscr{A}) > 0$.*

Proof. By definition, $\gamma(\mathscr{A}) > 0$ if and only if A^{-1} is bounded (where A is the operator defined in (25)), and this is true if and only if $D(A^{-1}) = R(A) = R(Q_{\mathscr{A}}\mathscr{A})$ is closed (we use the fact that A^{-1} is closed because A is closed, and then apply Remark (14)).

Now, assume that $\gamma(\mathscr{A}) > 0$ and let $\{y_n\}$ be a convergent sequence in $R(\mathscr{A})$ with

$$y_n \longrightarrow y. \tag{33}$$

Since $Q_{\mathscr{A}}$ is a bounded linear operator, the sequence$\{Q_{\mathscr{A}}(y_n)\}$ is a Cauchy sequence in $R(Q_{\mathscr{A}}\mathscr{A})$ and therefore converges to a point $\widehat{z} \in R(Q_{\mathscr{A}}\mathscr{A}) \subset \widehat{X} = X/\mathscr{A}(0)$ since $R(Q_{\mathscr{A}}\mathscr{A})$ is closed. We see that $\text{dist}(y_n - z, \mathscr{A}(0)) \longrightarrow 0$ as $n \longrightarrow \infty$ so that $y_n - z \longrightarrow v$ for some $v \in \mathscr{A}(0)$, that is,

$$y_n \longrightarrow z + v \in \widehat{z}. \tag{34}$$

Since $\mathscr{A}(0) \subset R(\mathscr{A})$, a coset $\widehat{x} \in \widehat{X}$ that contains a point of $R(\mathscr{A})$ consists entirely of element of $R(\mathscr{A})$. To see that this is the case, let $\widehat{x}$ be a coset in $\widehat{X}$ and let $u, v \in \widehat{x}$ with $u \in R(\mathscr{A})$. Then, $v - u \in \mathscr{A}(0) \subset R(\mathscr{A})$. The linearity of $R(\mathscr{A})$ implies that $v = u + (v - u) \in R(\mathscr{A})$.

We see from (33) and (34) that $y \in \widehat{z}$ and that $y \in R(\mathscr{A})$ since $z \in R(\mathscr{A})$ and $y \in \widehat{z}$. This shows that $R(\mathscr{A})$ is closed.

On the other hand, assume that $R(\mathscr{A})$ is closed. Since A^{-1} is closed (since A is closed), it is enough, by the closed graph theorem, to show that $D(A^{-1}) = R(A) = R(Q_{\mathscr{A}}\mathscr{A})$ is closed. So, assume that $\{\widehat{z}_n\}$ is a sequence in $R(Q_{\mathscr{A}}\mathscr{A})$ such that $\widehat{z}_n \longrightarrow \widehat{z} \in \widehat{X}$. Then, $\text{dist}(z_n - z, \mathscr{A}(0)) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, there exists an element $w \in \mathscr{A}(0)$ such that $z_n \longrightarrow z + w \in \widehat{z}$. The closedness of $R(\mathscr{A})$ implies that $z + w \in R(\mathscr{A})$ so that $\widehat{z} \in R(Q_{\mathscr{A}}\mathscr{A})$.

Please see ([3], III.5.3) for another proof of Lemma 15.

For the definition of continuity and openness of a linear relation $\mathscr{T}$ mentioned in Lemmas 16 and 17, please refer to [3].

Lemma 16 ([3], II.3.2, III.1.3, III.1.5, III.4.6). *Let $\mathscr{S}, \mathscr{T} \in LR(X, Y)$. Then,*

(a) *$\mathscr{T}$ is continuous if and only if $\|T\| < \infty$*

(b) *$(\lambda\mathscr{T})' = \lambda\mathscr{T}'$ (for $\lambda \neq 0$)*

(c) *$\mathscr{T}$ is open if and only if $\gamma(\mathscr{T}) > 0$*

(d) *If $D(\mathscr{S}) \supset D(\mathscr{T})$ and $\|\mathscr{S}\| < \infty$ then $(\mathscr{T} + \mathscr{S})' = \mathscr{T}' + \mathscr{S}'$*

Lemma 17 ([3], III.4.6).

(a) *$\mathscr{T}$ is continuous if and only if $D(\mathscr{T}') = \mathscr{T}(0)^{\perp}$*

(b) *$\mathscr{T}$ is open if and only if $R(\mathscr{T}') = N(\mathscr{T})^{\perp}$*

(c) *If $\mathscr{T}$ is continuous, then $\|\mathscr{T}'\| = \|\mathscr{T}\|$*

(d) *If $\mathscr{T}$ is open, then $\gamma(\mathscr{T}) = \gamma(\mathscr{T}')$*

4. The Gap between Closed Linear Manifolds and Their Dimensions

Let Z be a Banach space, and let L be a closed subspaces of Z. We denote by S_L the unit sphere of L, that is, $S_L := \{u \in L : \|u\| = 1\}$. For any two closed linear manifolds M and N of Z with $M \neq \{0\}$, define the gap between M and N, denoted by $\delta(M, N)$ to be

$$\delta(M, N) := \sup_{u \in S_M} \text{dist}(u, N), \tag{35}$$

and set $\delta(M, N) = 0$ if $M = \{0\}$. $\delta(M, N)$ can also be characterized as the smallest number δ for which

$$\text{dist}\,(u, N) \leq \delta\|u\|, \quad \text{for all } u \in M. \tag{36}$$

It can be seen from the definition that $0 \leq \delta(M, N) \leq 1$.

See [1] for Lemma 18.

Lemma 18. *Let M and N be linear manifolds in a Banach space Z. If* $\dim M > \dim N$, *then there exists an $x \in M$ such that*

$$dist(x, N) = \|x\| > 0. \tag{37}$$

Lemma 18 can be expressed in the language of the quotient space as follows.

Lemma 19. *Let M and N be linear manifolds in a Banach space Z. If* $\dim M > \dim N$, *then there exists an $x \in M$ such that*

$$\|\tilde{x}\| = \|x\| > 0, \quad \tilde{x} \in \tilde{X} := X/N \ (N \text{ is closed since } \dim N < \infty). \tag{38}$$

Lemma 20 is a direct consequence of the preceding one.

Lemma 20. *If $\|\tilde{x}\| < \|x\|$ for every none zero $x \in M$, where $\tilde{x} \in \tilde{X} = X/N$, then* $\dim M \leq \dim N$.

See ([1], Page 200) and [2] for Lemmas 21 and 22 respectively.

Lemma 21. *Let M and N be closed linear manifolds of a Banach space Z. If $\delta(M, N) < 1$, then* $\dim M \leq \dim N$.

Lemma 22. *Let x be an element of a normed linear space X, and let M and N be closed linear subspaces of X. Consider the quotient space $\tilde{X} := X/N$, and let $\tilde{x}$ denote the quotient class of x. For any $\varepsilon > 0$, there exists $x_0 \in \tilde{x}$ such that*

$$dist(x_0, M) \geq (1 - \varepsilon)\left(\frac{1 - \delta(M, N)}{1 + \delta(M, N)}\right)\|x_0\|. \tag{39}$$

5. The Quantity $\nu(\mathscr{A} : \mathscr{B})$

Let X and Y be two linear spaces and let $\mathscr{A}, \mathscr{B} \in LR(X, Y)$ with $\mathscr{B}(0) \subset \mathscr{A}(0)$. For $n \in \mathbb{N}$, let M_n and N_n be the linear manifolds of X and M'_n and N'_n be the linear manifolds of Y' defined inductively as follows:

$$M_0 = X, M_n = \mathscr{B}^{-1}(\mathscr{A}(M_{n-1})), \quad \text{for } n = 1, 2, \cdots, \tag{40}$$

$$N_1 = \mathscr{A}^{-1}(0), N_n = \mathscr{A}^{-1}(\mathscr{B}(N_{n-1})), \quad \text{for } n = 2, 3, \cdots, \tag{41}$$

$$M'_0 = Y', M'_n = \mathscr{B}'^{-1}\left(\mathscr{A}'\left(M'_{n-1}\right)\right), \quad \text{for } n = 1, 2, \cdots, \tag{42}$$

$$N'_1 = \mathscr{A}'^{-1}(0), N'_n = \mathscr{A}'^{-1}\left(\mathscr{B}'\left(N'_{n-1}\right)\right), \quad \text{for } n = 2, 3, \cdots. \tag{43}$$

If $M_k \supset M_{k+1}$, then $\mathscr{A}(M_k) \supset \mathscr{A}(M_{k+1})$, and therefore,

$$M_{k+1} = \mathscr{B}^{-1}(\mathscr{A}(M_k)) \supset \mathscr{B}^{-1}(\mathscr{A}(M_{k+1})) = M_{k+2}. \tag{44}$$

Since $M_0 = X \supset D(\mathscr{B}) \supset M_1$, we conclude by induction that

$$M_0 \supset M_1 \supset M_2 \supset \cdots \supset N(\mathscr{B}). \tag{45}$$

Similarly,

$$N_1 \subset N_2 \subset N_3 \subset \cdots \subset D(\mathscr{A}). \tag{46}$$

Note that

$$N_1 = N(\mathscr{A}). \tag{47}$$

Lemma 23. *Let n be a positive integer. The following first n conditions are equivalent to one another, and they in turn imply that condition (κ) holds.*

(1) $N_1 \subset M_n$
(2) $N_2 \subset M_{n-1}$
$\vdots$
(n) $N_n \subset M_1$,
(κ) $\mathscr{A}(N_{k+1}) \cap B(N_k) \neq \varnothing, N_k \subset D(\mathscr{B})$, for $k = 1, 2, \cdots, n$.

Proof. First, we prove the equivalence of the conditions (1) to (n). For each $r = 1, 2, \cdots, n-1$, (r) implies $(r+1)$. In fact if $N_r \subset M_{n-r+1}$, then (44), (45), and (47) imply that $N_{r+1} = \mathscr{A}^{-1}(\mathscr{B}(N_r)) \subset \mathscr{A}^{-1}(\mathscr{B}(M_{n-r+1})) \subset \mathscr{A}^{-1}(\mathscr{A}(M_{n-r}) + \mathscr{B}(0)) \subset \mathscr{A}^{-1}(\mathscr{A}(M_{n-r}) + \mathscr{A}(0)) = \mathscr{A}^{-1}[\mathscr{A}(M_{n-r}) + \mathscr{A}(N(\mathscr{A}))] = \mathscr{A}^{-1}[\mathscr{A}(M_{n-r}) + N(\mathscr{A})] \subset M_{n-r} + N(\mathscr{A}) + \mathscr{A}^{-1}(0) = M_{n-r} + \mathscr{A}^{-1}(0) = M_{n-r} + N_1 \subset M_{n-r} + N_r \subset M_{n-r} + M_{n-r+1} = M_{n-r}$.

Conversely, $(r+1)$ implies r. In fact, if $N_{r+1} \subset M_{n-r}$, then

$$N_r \subset N_{r+1} \subset M_{n-r} = \mathscr{B}^{-1}(\mathscr{A}(M_{n-r-1})), \tag{48}$$

so that each $x \in N_r$ has the property that there exists a $z \in \mathscr{B}(x)$ such that $z \in \mathscr{A}(y)$ for some $y \in M_{n-r-1}$. Then, $y \in \mathscr{A}^{-1}(\mathscr{B}(N_r)) = N_{r+1} \subset M_{n-r}$ and so $x \in \mathscr{B}^{-1}(\mathscr{A}(M_{n-r})) = M_{n-r+1}$ This proves that $N_r \subset M_{n-r+1}$.

Next, we prove that (n) implies (κ). So, suppose that (n) is satisfied. Then, $N_k \subset N_n \subset M_1 = \mathscr{B}^{-1}(\mathscr{A}X) \subset D(\mathscr{B})$ for $k < n$, so that for each $x \in N_k$, there exists a $z \in \mathscr{B}(x)$ such that $z \in \mathscr{A}(y)$ for some $y \in X$. Then, $y \in \mathscr{A}^{-1}(\mathscr{B}(N_k)) = N_{k+1}$ and so $\mathscr{A}(N_{k+1}) \cap \mathscr{B}(N_k) \neq \varnothing$. □

If $N_1 \subset M_n$, then $N_1 \subset M_{n'}$, for all $n' < n$ since M_n is a nonincreasing sequence. We denote by $\nu(\mathscr{A} : \mathscr{B})$ the smallest number n for which the condition $N_1 \subset M_n$ (or any one of the other equivalent conditions) is not satisfied. We set $\nu(\mathscr{A} : \mathscr{B}) = \infty$ if there is no such n. This is the case if for example $\mathscr{A}^{-1}(0) \subset \mathscr{B}^{-1}(0)$.

Lemma 24. *Let X and Y be Banach spaces and let $\mathscr{A}, \mathscr{B} \in CLR(X, Y)$ with $D(\mathscr{A}) = D(\mathscr{B}) = X$. Then,*

$$\begin{aligned} &M'_n \subset (\mathscr{B}(N_n))^{\perp}, \\ &N'_n \subset (\mathscr{A}(M_{n-1}))^{\perp}, \quad \textit{for } n = 1, 2, \cdots \end{aligned} \tag{49}$$

Proof. First, we show that (49) holds for $n = 1$. To begin with, let $y' \in M'_1$ and let $x \in D(\mathscr{B}) \cap N_1$. Then, by definition, $y' \in \mathscr{B}'^{-1}[\mathscr{A}'(Y')]$ and $x \in \mathscr{A}^{-1}(0) \cap D(\mathscr{B})$. Hence, there exists an element $x' \in \mathscr{A}'(Y') \cap R(\mathscr{B}')$ such that $(y', x') \in G(\mathscr{B}')$. Since $x' \in A'(Y')$, there exists an element $f' \in D(\mathscr{A}') \subset Y'$ such that $(f', x') \in G(\mathscr{A}')$. Since $(x, 0) \in G(\mathscr{A})$, (22) implies that $f'(0) = x'(x)$ so that $x'(x) = 0$. So, for $y \in \mathscr{B}(x)$, $y'(y) = x'(x) = 0$, showing the $y' \in [\mathscr{B}(N_1)]^{\perp}$.

The second inclusion follows from (see Lemma 9(1)).

$$N'_1 = N\left(\mathscr{A}'\right) = R(\mathscr{A})^{\perp} = [\mathscr{A}(M_0)]^{\perp} \tag{50}$$

We shall therefore assume that (49) has been proved for $n = k$ and prove it for $n = k+1$. So, let $g' \in M'_{k+1}$ and let $z \in D(\mathscr{B}) \cap N_{k+1}$. Then, $g' \in \mathscr{B}'^{-1}[\mathscr{A}'(M'_k)]$ and $z \in \mathscr{A}^{-1}[\mathscr{B}(N_k)] \cap D(\mathscr{B})$. Hence, there exists an element $h' \in \mathscr{A}'(M'_k)$ such that $(g', h') \in G(\mathscr{B}')$. Since $h' \in \mathscr{A}'(M'_k)$, it follows that there exists an element $l' \in M'_k$ such that $(l', h') \in G(\mathscr{A}')$. The fact that $z \in N_{k+1}$ means that there is an element $w \in \mathscr{B}(N_k)$ such that $(z, w) \in G(\mathscr{A})$. This means that $l'(w) = h'(z)$ and $h'(z) = 0$ since $l \in [\mathscr{B}(N_k)]^{\perp}$. So, for $u \in \mathscr{B}(z)$, $g'(u) = h'(z) = 0$ meaning that $g' \in [\mathscr{B}(N_{k+1})]^{\perp}$ and that $M'_{k+1} \subset [\mathscr{B}(N_{k+1})]^{\perp}$. This proves the first inclusion in (49). The second inclusion can be proved in a similar way. □

Lemma 25. *Let $\mathscr{A} \in CLR(X, Y)$. For every $f' \in N(\mathscr{A})^{\perp}$, there exists $g' \in Y'$ such that $g'(y) = f'(x)$, for all $y \in \mathscr{A}(x)$, and all $x \in D(\mathscr{A})$.*

Proof. Define a linear functional g' on Y' by setting $g'(y) = f'(x)$ for all $y \in \mathcal{A}(x)$ and all $x \in D(\mathcal{A})$. Then, g' is defined on $R(\mathcal{A})$ and is bounded. To show that g' is indeed bounded, we first note that for $y \in \mathcal{A}(x)$,

$$|g'(y)| = |f'(x)| \le \|f\|\|x\|, \tag{51}$$

and consider the quotient space $\tilde{X} := X/N(\mathcal{A})$. Let $x_1 \in \tilde{x}$. Then, $x - x_1 = u$ for some $u \in N(\mathcal{A})$ so that $f(x) = f(x_1)$. This equality means that $\|x\|$ in (51) can be replaced with $\|x_1\|$, for any $x_1 \in \tilde{x}$ without changing the inequality. This therefore means that

$$\begin{aligned} |g'(y)| &\le \|f'\|\|\tilde{x}\| \le \|f'\|\gamma(\mathcal{A})^{-1}\|\mathcal{A}x\| \\ &= \|f'\|\gamma(\mathcal{A})^{-1}\|Q_{\mathcal{A}}y\| \\ &\le \|f'\|\gamma(\mathcal{A})^{-1}\|Q_{\mathcal{A}}\|\|y\|, \end{aligned} \tag{52}$$

that is, g' is bounded on $R(\mathcal{A})$. The Hahn-Banach extension theorem implies that g' can be extended to the whole of Y' without changing its bound. □

Remark 26. Lemma 25 above implies that $N(\mathcal{A})^{\perp} \subset R(\mathcal{A}')$ and that $N(\mathcal{A})^{\perp} = R(\mathcal{A}')$ by Lemma 9(3).

Lemma 27. *Let $\mathcal{A}, \mathcal{B} \in CLR(X, Y)$ with $D(\mathcal{A}) = D(\mathcal{B}) = X$, $R(\mathcal{A})$ closed and $\mathcal{B}$ bounded. If $\mathcal{B}(0) \subset \mathcal{A}(0)$, then*

$$M_1' = [\mathcal{B}(N_1)]^{\perp} \tag{53}$$

$$\nu\left(\mathcal{A}' : \mathcal{B}'\right) = \nu(\mathcal{A} : \mathcal{B}) \tag{54}$$

Proof. Let $f' \in [\mathcal{B}(N_1)]^{\perp} = (\mathcal{B}(\mathcal{A}^{-1}(0)))^{\perp}$. Since $\mathcal{B}(0) \subset \mathcal{B}(N_1)$, Lemma 16 (*a*) together with Lemma 17 (a) imply that $f' \in D(\mathcal{B}')$. So, let $g' \in \mathcal{B}'(f')$, that is, $(f', g') \in G(\mathcal{B}')$. This means that for $x \in N_1$ and $y \in \mathcal{B}(x)$, $g'(x) = f'(y) = 0$, which shows that $g' \in N_1^{\perp} = N(\mathcal{A})^{\perp}$ and therefore $g' \in R(\mathcal{A}')$ and so $g' \in R(\mathcal{A}')$ by Remark 26. It follows that $f' \in \mathcal{B}'^{-1}[\mathcal{A}'(Y')] = M_1'$. This shows that $[\mathcal{B}(N_1)]^{\perp} \subset M_1'$. Equality (53) then is followed by (49). To prove the second equality, let $\nu = \nu(\mathcal{A} : \mathcal{B})$. Then, $N_1 \subset M_n$ for all $n < \nu$. Since $M_n = \mathcal{B}^{-1}[\mathcal{A}(M_{n-1})]$, we see that

$$\mathcal{B}(N_1) \subset \mathcal{B}(M_n) \subset \mathcal{A}(M_{n-1}) + \mathcal{B}(0) \subset \mathcal{A}(M_{n-1}) + \mathcal{A}(0) = \mathcal{A}(M_{n-1}), \tag{55}$$

where the last equality follows from the fact that $\mathcal{A}(0) \subset \mathcal{A}(M_{n-1})$ and $\mathcal{A}(M_{n-1})$ is a linear space. We see from (55) that $[\mathcal{A}(M_{n-1})]^{\perp} \subset [\mathcal{B}(N_1)]^{\perp}$. It then follows from (49) and (53) that $N_n' \subset M_1'$. This means that $\nu' = \nu(\mathcal{A}' : \mathcal{B}') > n$ and that $\nu' \ge \nu$.

To prove the opposite inequality, let $n < \nu'$. Then, we have $N_1' \subset M_n'$. If follows from Lemmas 9(1), (47), and (49) that $[\mathcal{A}(X)]^{\perp} \subset [\mathcal{B}(N_n)]^{\perp}$. Since $R(\mathcal{A}) = \mathcal{A}(X)$ is closed, this implies that $\mathcal{B}(N_n) \subset A(X)$. Since $D(\mathcal{B}) = X$, we see that $N_n \subset N_n + \mathcal{B}(0) \subset \mathcal{B}^{-1}[\mathcal{A}(X)] = M_1$. This shows that $\nu > n$ and therefore $\nu \ge \nu'$. □

6. Nullity and Deficiency

In this section, we study the behaviour of the nullity and deficiency for linear relations under some perturbations. For $\mathcal{A} \in LR(X, Y)$, the nullity $\alpha(\mathcal{A})$ and the deficiency $\beta(\mathcal{A})$ are defined by

$$\begin{aligned} \alpha(\mathcal{A}) &:= \dim\ N(\mathcal{A}), \\ \beta(\mathcal{A}) &:= \dim\ Y/R(\mathcal{A}). \end{aligned} \tag{56}$$

Lemma 28 ([3], III.7.2). *Let $\mathcal{T}$ be a closed linear relation with $\gamma(\mathcal{T}) > 0$. Then $\alpha(\mathcal{T}') = \beta(\mathcal{T})$.*

Let X and Y be Banach spaces, and let $\mathcal{A}$ be a closed linear relation with $D(\mathcal{A}) \subset X$ and $R(\mathcal{A}) \subset Y$. Let $n \in \{\mathbb{N} \cup \infty\}$ be such that for any $\varepsilon > 0$ there exists an n-dimensional closed linear subset N_{ε} of $N(\mathcal{A})$ such that

$$\|\mathcal{A}(x)\| \le \varepsilon\|x\|, \quad \text{for all } x \in N_{\varepsilon}, \tag{57}$$

while this is not true if n is replaced by a larger number. In such a case, we set $\alpha'(\mathcal{A}) := n$ and define $\beta'(\mathcal{A})$ to be

$$\beta'(\mathcal{A}) := \alpha'\left(\mathcal{A}'\right). \tag{58}$$

Lemmas 29 and 30 show that $\alpha'(\mathcal{A})$ is defined for every closed linear relation $\mathcal{A}$.

Lemma 29. *Assume that for every $\varepsilon > 0$ and any closed linear subset $\mathcal{M}$ of X of finite codimension; there is an $x \in \mathcal{M} \cap D(\mathcal{A})$ such that $\|x\| = 1$ and $\|\mathcal{A}(x)\| \le \varepsilon$, then $\alpha'(\mathcal{A}) = \infty$.*

Proof. We have to show that for each $\varepsilon > 0$, there exists an infinite dimensional closed linear subset $N_{\varepsilon} \subset D(\mathcal{A})$ with the property (57). First, we construct two sequences $x_n \in D(\mathcal{A})$ and $f_n \in X'$ such that

$$\begin{aligned} &\|x_n\| = 1, \|f_n\| = 1, f_n(x_n) = 1, f_k(x_n) = 0, \quad k = 1, 2, \cdots, n-1, \\ &\|\mathcal{A}(x_n)\| \le 3^{-n}\varepsilon, \quad n \in \mathbb{N}. \end{aligned} \tag{59}$$

For $n = 1$, the result holds by ([1], III-Corollary 1.24). Suppose that x_n, f_k have been constructed for $k = 1, 2, \cdots, n-1$. Then, x_n and f_n can be constructed in he following way. Let $M \subset X$ be the collection of all $x \in X$ such that $f_k(x) = 0, k = 1, 2, \cdots, n-1$. Since M is a closed linear subset of X with finite codimension (dim $M^{\perp} \le n-1$ and use codim M=dim $M^{\perp}$), there is an $x_n \in M \cap D(\mathcal{A})$ such that $\|x_n\| = 1$ and $\|A(x_n)\| \le 3^{-n}\varepsilon$. For this x_n, there exists an $f_n \in X'$ such that $\|f_n\| = 1$ and $f_n(x_n) = 1$ (see [1], III-Corollary 1.24). It follows from (59) that the x_n are linearly independent so that $M_{\varepsilon}' := \text{span}\,\{x_1, x_2, \cdots\}$ is infinite dimensional. Each $x \in M_{\varepsilon}'$ has the form

$$x = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n, \quad (60)$$

for some positive integer n. Hence, for $k = 1, 2, \cdots, n$,

$$f_k(x) = \xi_1 f_k x_1 + \xi_2 f_k(x_2) + \cdots + \xi_{k-1} f_k(x_{k-1}) + \xi_k. \quad (61)$$

We show that the coefficients ξ_k satisfy the inequality

$$|\xi_k| \le 2^{k-1}\|x\|, k = 1, 2, \cdots, n. \quad (62)$$

For $k = 1$, this is clear from (59) and (61). If we assume that (62) has been proved for $k < j$, we see from (61) that

$$\begin{aligned} |\xi_j| &\le \left|f_j(x)\right| + |\xi_1|\left|f_j(x_1)\right| + \cdots + |\xi_{j-1}|\left|f_j(x_{j-1})\right| \\ &\le \|x\| + |\xi_1| + |\xi_2| + \cdots + |\xi_{j-1}| \\ &\le \|x\| + \|x\| + 2\|x\| + \cdots + 2^{j-2}\|x\| \\ &= \|x\|\left[2 + 2\left(1 + 2 + 2^2 + \cdots + 2^{j-1}\right)\right] = 2^{j-1}\|x\|. \end{aligned} \quad (63)$$

It follows from (59), (61), and (62) that

$$\begin{aligned} \|\mathscr{A}(x)\| &\le |\xi_1|\|\mathscr{A}x_1\| + \cdots + |\xi_n|\|\mathscr{A}x_n\| \\ &\le \left(\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots + \frac{2^{n-1}}{3^n}\right)\varepsilon\|x\| \le \varepsilon\|x\|. \end{aligned} \quad (64)$$

Let $u \in \bar{\mathscr{M}}'_\varepsilon$, and let $\{u_n\}$ be a sequence in $\mathscr{M}'_\varepsilon$ such that $u_n \longrightarrow u$. The boundedness of $Q_{\mathscr{A}}\mathscr{A}$ on M'_ε implies that $\{Q_{\mathscr{A}}\mathscr{A}(x_n)\}$ is a Cauchy sequence in $\tilde{\mathscr{Y}} := Y/\mathscr{A}(0)$ and therefore converges, say $Q_{\mathscr{A}}\mathscr{A}(x_n) \longrightarrow \tilde{v} \in \tilde{\mathscr{Y}}$. This means that $\text{dist}(x_n - v, \mathscr{A}(0)) \longrightarrow 0$ as $n \longrightarrow \infty$, that is, $x_n - v \longrightarrow z \in \mathscr{A}(0)$ for some $z \in \mathscr{A}(0)$. In other words, $x_n \longrightarrow v + z = w \in \tilde{v}$. The closedeness of $\mathscr{A}$ implies that $x \in D(\mathscr{A})$ and $w \in \mathscr{A}(x)$. Hence, $Q_{\mathscr{A}}\mathscr{A}$ is defined and bounded on the closure of M'_ε with the same bound. □

Lemma 30. *If $\mathscr{A}$ is a closed linear relation with closed range(that is, $\gamma(\mathscr{A}) > 0$), then $\alpha'(\mathscr{A}) = \alpha(\mathscr{A})$ and $\beta'(\mathscr{A}) = \beta(\mathscr{A})$.*

Proof. By Lemma 16, $\gamma(\mathscr{A}) > 0$ implies $\gamma(\mathscr{A}') > 0$ while Lemma 28 implies that $\alpha(\mathscr{A}') = \beta(\mathscr{A})$. In view of (58), it is enough to show that $\alpha'(\mathscr{A}) = \alpha(\mathscr{A})$. It is clear that $\alpha'(\mathscr{A}) \ge \alpha(\mathscr{A})$. Now suppose that there exists a closed linear manifold N_ε with $\dim N_\varepsilon > \alpha(\mathscr{A}) = \dim N(\mathscr{A})$ and with property (57). Pick $x \in N_\varepsilon$ such that $\|\tilde{x}\| = \|x\| = 1$ where $\tilde{x} \in \tilde{X} := X/N(\mathscr{A})$ (this is possible by ([2], Lemma 241). For this x, $\|\mathscr{A}(x)\| \ge \gamma(\mathscr{A})$ on the one hand and $\|\mathscr{A}(x)\| \le \varepsilon$ on the other hand, leading to the inequality $\gamma(\mathscr{A}) \le \varepsilon$. In other words, there is no N_ε with $\dim N_\varepsilon > \alpha(\mathscr{A}) = \dim N(\mathscr{A})$ for $\varepsilon < \gamma(\mathscr{A})$. This proves that $\alpha'(\mathscr{A}) \le \alpha(\mathscr{A})$ and that $\alpha'(\mathscr{A}) = \alpha(\mathscr{A})$. The second equality follows from (58) and Lemma 28. □

Lemma 31. *Let $\mathscr{T} \in CLR(X)$ with nonclosed range (that is, $\gamma(\mathscr{T}) = 0$), then*

$$\alpha'(\mathscr{T}) = \infty. \quad (65)$$

Proof. Let M be any closed linear manifold of X with finite codimension, and let $Q_{\mathscr{T}}$ be denoted by Q. Consider the mapping $T : X/M \longrightarrow Q\mathscr{T}(X)/Q\mathscr{T}(M)$ defined by setting $T(\tilde{x}) = \widetilde{Q\mathscr{T}(x)}$. Then, T is clearly well defined and linear. It is well defined since

$$T\left(\widetilde{x+v}\right) = \widetilde{Q\mathscr{T}(x+v)} = \widetilde{Q\mathscr{T}(x) + Q\mathscr{T}(v)} = \widetilde{Q\mathscr{T}(x)} = T\tilde{x}, \quad (66)$$

for any $v \in M$. It follows that $Q\mathscr{T}(X)/Q\mathscr{T}(M)$ is a finite dimensional space since M has finite codimension. ([1], III-Lemma 1.9) implies that $Q\mathscr{T}(X)$ is a closed subset of $\widehat{Y} := Y/\mathscr{T}(0)$ if $Q\mathscr{T}(M)$ is a closed subspace of the same space. This would mean that $\mathscr{T}(X)$ is a closed subset of Y. To see why this is true, let $\{y_n\}$ be a convergent sequence in $\mathscr{T}(X)$ with $y_n \longrightarrow y \in Y$. Then, $\{Qy_n\}$ is a Cauchy sequence in $\widehat{Y}$ and therefore converges to some point $\widehat{z} \in Q\mathscr{T}(X)$. In other words, $y_n - z \longrightarrow w \in \mathscr{T}(0)$, so that $y_n \longrightarrow z + w \in \widehat{z}$. The uniqueness of the limit implies that $y = z + w \in \widehat{z}$ and that $y \in R(\mathscr{T})$ since $z \in R(\mathscr{T})$ and every coset that contains and element of $R(\mathscr{T})$ consists entirely of elements of $R(\mathscr{T})$. Next, we show that if $\mathscr{T}(M)$ is closed then $Q\mathscr{T}(M)$ is closed. So, assume that $\mathscr{T}(M)$ is closed and let $\{\widehat{z}\}$ be a sequence in $Q\mathscr{T}(M)$ that converges to an element $\widehat{z} \in \widehat{Y}$. Then, $z_n - z \longrightarrow v \in \mathscr{T}(0)$ and so $z_n \longrightarrow z + v \in \widehat{z}$. The closedness of $\mathscr{T}(M)$ implies that $z + v \in \mathscr{T}(M)$ and that $\widehat{z} \in Q\mathscr{T}(M)$.

The contradiction that $\mathscr{T}(X)$ is both open and closed means that $Q\mathscr{T}(M)$ is not closed and that $\mathscr{T}(M)$ is not closed and therefore $\gamma(\mathscr{T}_M) = 0$. Hence, there exists, for any $\varepsilon > 0$, an $x \in M \cap D(\mathscr{T})$ such that $\|x\| = 1$ and $\|\mathscr{T}(x)\| \le \varepsilon\|\tilde{x}\| \le \varepsilon\|x\| = \varepsilon$, where $\tilde{x} \in \tilde{X} = X/N(\mathscr{T})$. This shows that the conditions of Lemma 29 are satisfied and therefore $\alpha'(\mathscr{T}) = \infty$. □

Theorem 32. *Let X and Y be Banach spaces, and let $\mathscr{A}$ be a closed linear relation with $D(\mathscr{A}) \subset X$, having closed range $R(\mathscr{A}) \subset Y$, and with $\alpha(\mathscr{A})$ finite. Let $\mathscr{B}$ be a closed bounded linear relation such that $D(\mathscr{B}) \supset D(\mathscr{A})$, $\mathscr{B}(0) \subset \mathscr{A}(0)$, and*

$$\|\mathscr{B}\| < \gamma(\mathscr{A}). \quad (67)$$

Then, the linear relation $\mathscr{A} + \mathscr{B}$ is closed and has closed range. Moreover,

$$\alpha(\mathscr{A} + \mathscr{B}) \le \alpha(\mathscr{A}), \beta(\mathscr{A} + \mathscr{B}) \le \beta(\mathscr{A}). \quad (68)$$

Proof. Let $\{x_n\}$ be a sequence in $D(\mathscr{A})$ such that $x_n \longrightarrow x \in X$, and let $\{y_n\}$ be a sequence in $R(\mathscr{A} + \mathscr{B})$ such that $y_n \longrightarrow y \in Y$, where $y_n = u_n + v_n$ with $u_n \in \mathscr{A}(x_n)$ and $v_n \in \mathscr{B}(x_n)$ for each $n \in \mathbb{N}$. In other words,

$$u_n + v_n \longrightarrow y. \quad (69)$$

Note that (67) implies that $\{Q_{\mathscr{B}}\mathscr{B}(x_n)\}$ is a Cauchy sequence in $\tilde{Y} := Y/\mathscr{B}(0)$ and therefore converges to a point of $\tilde{Y}$, say $Q_{\mathscr{B}}\mathscr{B}(x_n) \longrightarrow \tilde{v} \in \tilde{Y}$. Hence, $\text{dist}(v_n - v, \mathscr{B}(0)) \longrightarrow 0$ as $n \longrightarrow 0$, that is, $v_n - v \longrightarrow z$ for some $z \in \mathscr{B}(0)$. Hence, $v_n \longrightarrow v + z \in \tilde{v}$. The closedness of $\mathscr{B}$ implies that $x \in D(\mathscr{B})$ and $v + z \in \mathscr{B}(x)$. Hence, $y = y - v - z + (v + z) \in \mathscr{A}(x) + \mathscr{B}(x)$ and so $\mathscr{A} + \mathscr{B}$ is closed.

To complete the proof, it is enough to show that

$$\alpha'(\mathscr{A} + \mathscr{B}) \le \alpha(\mathscr{A}), \beta'(\mathscr{A} + \mathscr{B}) \le \beta'(\mathscr{A}), \tag{70}$$

and then apply Lemma 31 to conclude that $\mathscr{A} + \mathscr{B}$ has closed range and Lemma 30 to establish the inequalities in the theorem since $\alpha'(\mathscr{A} + \mathscr{B}) \ge \alpha(\mathscr{A} + \mathscr{B})$ by definition and $\beta'(\mathscr{A} + \mathscr{B}) \ge \alpha(\mathscr{A} + \mathscr{B})$ by (58) and Lemma 28.

To prove (70), suppose that for a given $\varepsilon > 0$ there exists a closed linear manifold $N_\varepsilon \subset D(\mathscr{A} + \mathscr{B}) = D(\mathscr{A})$ such that

$$\|(\mathscr{A} + \mathscr{B})(x)\| \le \varepsilon\|x\|, \quad \text{for every } x \in N_\varepsilon. \tag{71}$$

It then follows from (71) and Lemma 8 that

$$\begin{aligned}(\|\mathscr{B}\| + \varepsilon)\|x\| &\ge \|\mathscr{B}(x)\| + \|(\mathscr{A} + \mathscr{B})(x)\| \ge \|\mathscr{B}(x)\| \\ &+ (\|\mathscr{A}x\| - \|\mathscr{B}(x)\|) \ge \|\mathscr{A}(x)\| \ge \gamma(\mathscr{A})\|\tilde{x}\|,\end{aligned} \tag{72}$$

where $\tilde{x} \in \tilde{X} := X/N(\mathscr{A})$. If we pick ε such that $0 < \varepsilon < \gamma(\mathscr{A}) - \|\mathscr{B}\|$, we see from (72) that $\|\tilde{x}\| < \|x\|$ for all nonzero $x \in D(\mathscr{A})$. It therefore follows from Lemma 20 that

$$\dim\ N_\varepsilon \le \dim\ N(\mathscr{A}) = \alpha(\mathscr{A}), \tag{73}$$

which means that $\alpha'(\mathscr{A} + \mathscr{B}) \le \alpha(\mathscr{A})$.

To prove the second inequality, we note that Lemma 16 together with Lemma 17 implies that $\|\mathscr{B}'\| = \|\mathscr{B}\|$, $\gamma(\mathscr{A}') = \gamma(\mathscr{A})$, and $(\mathscr{A} + \mathscr{B})' = \mathscr{A}' + \mathscr{B}'$. It therefore follows that $\|\mathscr{B}'\| \le \gamma(\mathscr{A}')$. Applying what has been proved above to the pair $\mathscr{A}', \mathscr{B}'$, we see that

$$\beta'(\mathscr{A} + \mathscr{B}) = \alpha'\left((\mathscr{A} + \mathscr{B})'\right) = \alpha'\left(\mathscr{A}' + \mathscr{B}'\right) \le \alpha\left(\mathscr{A}'\right) = \beta(\mathscr{A}), \tag{74}$$

where the last equality follows from Lemma 28. □

Lemma 33. *Let X and Y be Banach spaces and let $\mathscr{T}$ be a closed linear relation with $D(\mathscr{T}) \subset X$ and $R(\mathscr{T}) \subset Y$. Set*

$$\|x\|_{D(\mathscr{T})} := \|x\| + \|\mathscr{T}(x)\|, \quad x \in D(\mathscr{T}). \tag{75}$$

Then, $D(\mathscr{T})$ becomes a Banach space if $\|\cdot\|_{D(\mathscr{T})}$ is chosen as the norm.

Proof. That $\|\cdot\|_{D(\mathscr{T})}$ defines a norm on $D(\mathscr{T})$ is clear. To prove completeness, assume that $\{x_n\}$ is a Cauchy sequence in $D(\mathscr{T})$. Then, $\{x_n\}$ and $\{Q_{\mathscr{T}}\mathscr{T}(x_n)\}$ are Cauchy sequences in X and $\tilde{Y} = Y\mathscr{T}(0)$, respectively, and therefore converge, say $x_n \longrightarrow x \in X$ and $Q_{\mathscr{T}}\mathscr{T}(x_n) \longrightarrow \tilde{u} \in \tilde{Y}$. Let $u_n \in \mathscr{T}(x_n)$ for each $n \in \mathbb{N}$. Then, $\tilde{u}_n \longrightarrow \tilde{u}$ and so $dist\,(u_n - u, \mathscr{T}(0)) \longrightarrow 0$ as $n \longrightarrow \infty$, that is, $u_n - u \longrightarrow v \in \mathscr{T}(0)$. We therefore see that $u_n \longrightarrow u + v = s \in \tilde{u}$. The closedness of $\mathscr{T}$ implies that $x \in D(\mathscr{T})$ and that $s \in \mathscr{T}(x)$. Now,

$$\begin{aligned}\|x_n - x\|_{D(\mathscr{T})} &= \|x_n - x\| + \|Q_{\mathscr{T}}\mathscr{T}(x_n - x)\| \\ &= \|x_n - x\| + \|Q_{\mathscr{T}}u_n - Q_{\mathscr{T}}s)\| \\ &= \|x_n - x\| + \|\tilde{u}_n - \tilde{u}\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned} \tag{76}$$

This shows that $D(\mathscr{T})$ is complete. □

Let X and Y be Banach spaces, and let $\mathscr{A}, \mathscr{B} \in CLR(X, Y)$ be such that $D(\mathscr{A}) \subset D(\mathscr{B})$ and $\mathscr{B}(0) \subset \mathscr{A}(0)$. In Theorem 34, we write $\|\mathscr{B}(x)\|_{\mathscr{A}}$ to mean the quantity $\|Q_{\mathscr{A}}\mathscr{B}(x)\|$. The quantities $\|\mathscr{A}(x)\|_{\mathscr{A}}$ and $\|\mathscr{B}(x)\|_{\mathscr{B}}$ are defined in a similar way

Theorem 34. *Let X and Y be Banach spaces, and let $\mathscr{A}$ be a closed linear relation with $D(\mathscr{A}) \subset X$ and with closed range $R(\mathscr{A}) \subset Y$. Let $\mathscr{B}$ be a closed linear relation such that $D(\mathscr{A}) \subset D(\mathscr{B}) \subset X$, $R(\mathscr{B}) \subset Y$, $\mathscr{B}(0) \subset \mathscr{A}(0)$, and*

$$\|\mathscr{B}(x)\|_{\mathscr{B}} \le \sigma\|x\| + \tau\|\mathscr{A}(x)\|_{\mathscr{A}}, \quad \forall x \in D(\mathscr{A}), \tag{77}$$

where σ and τ are nonnegative constants such that

$$\sigma + \tau\gamma(\mathscr{A}) < \gamma(\mathscr{A}). \tag{78}$$

Then, the linear relation $\mathscr{A} + \mathscr{B}$ is closed and has closed range. If $\alpha(\mathscr{A}) < \infty$, then

$$\begin{aligned}\alpha(\mathscr{A} + \mathscr{B}) &\le \alpha(\mathscr{A}), \\ \beta(\mathscr{A} + \mathscr{B}) &\le \beta(\mathscr{A}).\end{aligned} \tag{79}$$

Proof. Let $\{x_n\}$ be a sequence in $D(\mathscr{A})$ such that $x_n \longrightarrow x \in X$, and let $\{y_n\}$ be a sequence in $R(\mathscr{A} + \mathscr{B})$ such that $y_n \longrightarrow y \in Y$, where $y_n = u_n + v_n$ with $u_n \in \mathscr{A}(x_n)$ and $v_n \in \mathscr{B}(x_n)$ for each $n \in \mathbb{N}$. Note that (77) implies that

$$\|\mathscr{A}(x)\|_{\mathscr{A}} - \|\mathscr{B}(x)\|_{\mathscr{B}} \ge (1 - \tau)\|\mathscr{A}(x)\|_{\mathscr{A}} - \sigma\|x\|. \tag{80}$$

Since $\|\mathscr{B}(x)\|_{\mathscr{B}} = \|Q_{\mathscr{B}}\mathscr{B}(x)\|_{\mathscr{B}} \ge \|Q_{\mathscr{A}}\mathscr{B}(x)\|_{\mathscr{A}}$, we see that

$$\|Q_{\mathscr{A}}\mathscr{A}(x)\|_{\mathscr{A}} - \|Q_{\mathscr{A}}\mathscr{B}(x)\|_{\mathscr{A}} \ge (1 - \tau)\|\mathscr{A}(x)\| - \sigma\|x\| \tag{81}$$

and that

$$\|Q_{\mathscr{A}}\mathscr{A}(x) + Q_{\mathscr{A}}\mathscr{B}(x)\|_{\mathscr{A}} \ge (1 - \tau)\|Q_{\mathscr{A}}\mathscr{A}(x)\| - \sigma\|x\|. \tag{82}$$

Inequality (82) and the linearity of $Q_{\mathscr{A}}$ imply that

$$\|Q_{\mathscr{A}}(u_n + v_n)\|_{\mathscr{A}} \ge (1 - \tau)\|Q_{\mathscr{A}}u_n\| - \sigma\|x_n\|, \tag{83}$$

so that

$$\|y_n\| = \|u_n + v_n\| \geq (1-\tau)\|Q_{\mathscr{A}} u_n\| - \sigma\|x_n\|. \tag{84}$$

It therefore follows that for $m, n \in \mathbb{N}$,

$$\|y_n - y_m\| \geq (1-\tau)\|Q_{\mathscr{A}} u_n - Q_{\mathscr{A}} u_m)\| - \sigma\|x_n - x_m\|. \tag{85}$$

Since $1-\tau > 0$ by (78) and both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, it follows by (85) that $\{Q_{\mathscr{A}} u_n\}$ is a Cauchy sequence and therefore converges, say

$$\tilde{u}_n \longrightarrow \tilde{u}, \tag{86}$$

where we denote $Q_{\mathscr{A}} u_n$ by $\tilde{u}_n$ in $Y/\mathscr{A}(0)$. The convergence in (86) implies that dist $(u_n - u, \mathscr{A}(0)) \longrightarrow 0$ as $n \longrightarrow \infty$. This means that $u_n - u$ converges to an element of $\overline{\mathscr{A}(0)} = \mathscr{A}(0)$, say $u_n - u \longrightarrow z \in \mathscr{A}(0)$. This means that $u_n \longrightarrow z - u = s$. The closedness of $\mathscr{A}$ implies that $x \in D(\mathscr{A})$ and $s \in \mathscr{A}(x)$. Since $u_n \longrightarrow s$, we see that $Q_{\mathscr{A}} u_n = Q_{\mathscr{A}} s$. Applying (77) to $x_n - x$, we see that $Q_{\mathscr{B}}\mathscr{B}(x_n) \longrightarrow Q_{\mathscr{B}}\mathscr{B}(x)$, that is, dist $(v_n - v, \mathscr{B}(0)) \longrightarrow 0$ as $n \longrightarrow \infty$, $v \in \mathscr{B}(x)$. This shows that $v_n - v$ converges to an element say w of $\mathscr{B}(0)$, that is, $v_n \longrightarrow w - v = r \in \mathscr{B}(x)$ since $\mathscr{B}(x) = \mathscr{B}(0) + v$. Hence, $y = s + r \in (\mathscr{A} + \mathscr{B})(x)$, showing that $\mathscr{A} + \mathscr{B}$ is closed.

We introduce a norm on $D(\mathscr{A})$ by

$$\|x\|_{\check{D}} := (\sigma+\varepsilon)\|x\| + (\tau + \varepsilon\|\mathscr{A}(x)\| \geq \varepsilon\|x\|, \tag{87}$$

for some arbitrary but fixed positive constant ε. Note that the space $D(\mathscr{A})$ becomes a Banach space by Lemma 33, which we denote by $\check{D}$. We now regard $\mathscr{A}$ and $\mathscr{B}$ as linear relations with $D(\mathscr{A}) = D(\mathscr{B}) = \check{D}$ and denote them by $\check{\mathscr{A}}$ and $\check{\mathscr{B}}$ respectively. Since $\|x\|_{\check{D}} = (\sigma+\varepsilon)\|x\| + (\tau+\varepsilon)\|\mathscr{A}(x)\| > \sigma\|x\| + \tau\|\mathscr{A}(x)\| \geq \|B(x)\|$ for every $x \in \check{D}$ and $\|\check{\mathscr{B}}\| := \sup_{x \in B_{\check{D}}} \|\check{\mathscr{B}}(x)\|$, we see that $\|\check{\mathscr{B}}\| \leq 1$. From $\|\mathscr{A}(x)\| \leq (\tau = \varepsilon)^{-1}\|x\|_{\check{D}}$ and the definition of $\|\check{\mathscr{A}}\|$, we also see that $\|\check{\mathscr{A}}\| \leq (\tau+\varepsilon)^{-1}$.

It is clear that $R(\check{\mathscr{A}}) = R(\mathscr{A})$ is closed and that

$$\begin{gathered} \alpha\left(\check{\mathscr{A}}\right) = \alpha(\mathscr{A}), \beta\left(\check{\mathscr{A}}\right) = \beta(\mathscr{A}), \\ \alpha\left(\check{\mathscr{A}} + \check{\mathscr{B}}\right) = \alpha(\mathscr{A}+\mathscr{B}), \beta\left(\check{\mathscr{A}} + \check{\mathscr{B}}\right) = \beta(\mathscr{A}+\mathscr{B}). \end{gathered} \tag{88}$$

Please note that $\gamma(\check{\mathscr{A}}) = \gamma(\mathscr{A})$ if $\gamma(\mathscr{A}) = \infty$. In order to relate $\gamma(\check{\mathscr{A}})$ to $\gamma(\mathscr{A})$ in the other case, we recall that in this case,

$$\begin{aligned} \gamma\left(\check{\mathscr{A}}\right) &= \inf\left\{\frac{\|\check{\mathscr{A}}(x)\|}{\|\tilde{x}\|_{\check{D}}} : x \in \check{D}, x \notin N\left(\check{\mathscr{A}}\right)\right\} \\ &= \inf\left\{\frac{\|\mathscr{A}(x)\|}{\|\tilde{x}\|_{\check{D}}} : x \in \check{D}, x \notin N\left(\check{\mathscr{A}}\right)\right\}, \end{aligned} \tag{89}$$

where $\tilde{x} \in \tilde{X} := X/N(\mathscr{A})$.

But

$$\begin{aligned} \|\tilde{x}\|_{\check{D}} &= \inf_{z \in N(\mathscr{A})} \|x - z\|_{\check{D}} = \inf_{z \in N(\mathscr{A})} [(\sigma+\varepsilon)\|x-z\| + (\tau+\varepsilon)\|\mathscr{A}(x-z)\|] \\ &= (\sigma+\varepsilon)\|\tilde{x}\| + (\tau+\varepsilon)\|\mathscr{A}(x)\|, \end{aligned} \tag{90}$$

where we have used the linearity of the natural quotient map and the fact that $\mathscr{A}(z) = \mathscr{A}(0)$.

Hence,

$$\begin{aligned} \gamma\left(\check{\mathscr{A}}\right) &= \inf\left\{\frac{\|A(x)\|}{(\sigma+\varepsilon)\|\tilde{x}\| + (\tau+\varepsilon)\|\mathscr{A}(x)\|} : x \in D(\mathscr{A}), x \notin N(\mathscr{A})\right\} \\ &= \frac{\gamma(\mathscr{A})}{(\sigma+\varepsilon) + (\tau+\varepsilon)\gamma(\mathscr{A})}, \end{aligned} \tag{91}$$

where we have used the fact that $f(t) = t/(\alpha + t)$ is an increasing function for any constant α.

In view of (78), we can make $\gamma(\check{\mathscr{A}}) > 1$ by choosing ε small enough. Since $\|\check{\mathscr{B}}\| \leq 1$, we can apply Theorem 32 to the pair $\check{\mathscr{A}}, \check{\mathscr{B}}$ with the result that $R(\check{\mathscr{A}} + \check{\mathscr{B}}) = R(\mathscr{A} + \mathscr{B})$ is closed and (68) holds with $\mathscr{A}, \mathscr{B}$ replaced with $\check{\mathscr{A}}, \check{\mathscr{B}}$. The result then follows (88). ☐

7. Stability Theorems

Consider an eigenvalue problem of the form

$$Ax = \lambda B, \tag{92}$$

where A and B are linear operators from X to Y and the associated problem

$$A^* f' = \lambda B^* f', \tag{93}$$

where the adjoints A^* and B^* exist. The null space $N(A - \lambda B)$ of the linear operator $A - \lambda B$ is the solution set of the eigenvalue problem (92). Similarly, $N(A^* - \lambda B^*) = R(A - \lambda B)^{\perp}$ is the solution set of the eigenvalue problem (93). In studying the above eigenvalue problems, one therefore gets interested in the behaviour of $N(A - \lambda B)$ and $N(A^* - \lambda B^*)$.

In the setting of linear relations, the eigenvalue problems (92) and (93) can be formulated as

$$\mathscr{A}(x) \cap \lambda\mathscr{B}(x) \neq \varnothing, \tag{94}$$

$$\mathscr{A}'\left(x'\right) \cap \lambda\mathscr{B}'\left(x'\right) \neq \varnothing, \tag{95}$$

where $\mathscr{A}, \mathscr{B} \in LR(X, Y)$. Conditions (94) and (93) are equivalent to

$$(\mathscr{A} - \lambda\mathscr{B})(x) = (\mathscr{A} - \lambda\mathscr{B})(0), \tag{96}$$

$$\left(\mathscr{A}' - \lambda\mathscr{B}'\right)\left(x'\right) = \left(\mathscr{A}' - \lambda\mathscr{B}'\right)(0), \tag{97}$$

respectively.

As before, the solution sets of (96) and (97) are $N(\mathscr{A} - \lambda\mathscr{B})$ and $N(\mathscr{A}' - \lambda\mathscr{B}') = R(\mathscr{A} - \lambda\mathscr{B})^{\perp}$, respectively. In this last section, we study the stability of the dimensions of the null spaces of $\mathscr{A} - \lambda\mathscr{B}$ and $\mathscr{A}' - \lambda\mathscr{B}'$ as λ varies in some specified subset of the complex plane.

Theorem 35. *Let X and Y be Banach spaces and let $\mathscr{A}, \mathscr{B} \in CLR(X, Y)$ be such that $\mathscr{A}$ has closed range, $D(\mathscr{B}) \supset D(\mathscr{A})$, $\mathscr{B}(0) \subset \mathscr{A}(0)$, and*

$$\|\mathscr{B}(x)\| \le \sigma\|x\| + \tau\|\mathscr{A}(x)\|, \quad \textit{for every } x \in D(\mathscr{A}), \tag{98}$$

where σ and τ are nonnegative constants. Then $\mathscr{A} - \lambda\mathscr{B}$ is closed for $|\lambda| < \gamma(\mathscr{A})/(\sigma + \tau\gamma(\mathscr{A}))$, and if $R(\mathscr{A}) \setminus \mathscr{A}(0) \ne \varnothing$, then $\gamma(\mathscr{A} - \lambda\mathscr{B}) < \infty$ for $|\lambda| \ge \gamma(\mathscr{A})/(\sigma + \tau\gamma(\mathscr{A}))$.

Proof. It follows from Theorem 34 that $\mathscr{A} - \lambda\mathscr{B}$ is closed if $|\lambda| < \gamma(\mathscr{A})/(\sigma + \tau\gamma(\mathscr{A}))$.

If $\gamma(\mathscr{A} - \lambda\mathscr{B}) = \infty$, then $(\mathscr{A} - \lambda\mathscr{B})(x) = (\mathscr{A} - \lambda\mathscr{B})(0) = \mathscr{A}(0)$. The fact that $(\mathscr{A} - \lambda\mathscr{B})(x) = (\mathscr{A} - \lambda\mathscr{B})(0)$ for every $x \in D(\mathscr{A} - \lambda\mathscr{B}) = D(\mathscr{A})$ implies that $\mathscr{A}(x) \cap \lambda\mathscr{B}(x) \ne \varnothing$ for every $x \in D(\mathscr{A})$. Since $\mathscr{B}(0) \subset \mathscr{A}(0)$, it follows that $\|\mathscr{A}(x)\| \le \|\lambda\mathscr{B}(x)\|$ for every $x \in D(\mathscr{A})$ and therefore

$$\|\mathscr{A}(x)\| \le |\lambda|\|\mathscr{B}(x)\| \le |\lambda|(\sigma\|x\| + \tau\|\mathscr{A}(x)\|), \tag{99}$$

so that

$$(1 - |\lambda||\tau)\|\mathscr{A}(x)\| \le \sigma|\lambda|\|x\|. \tag{100}$$

Since $R(\mathscr{A}) \ne \mathscr{A}(0)$, we see that there exists at least one $\tilde{x}$ in $\tilde{X} = X/N(\mathscr{A})$ with $\tilde{x} \ne 0$. Inequality (100) implies that

$$\gamma(\mathscr{A})\|\tilde{x}\| \le \|\mathscr{A}(x)\| \le \sigma|\lambda|\|x\|/(1 - |\lambda|\tau). \tag{101}$$

Since x can vary freely in $\tilde{x}$, we conclude that $\gamma(\mathscr{A}) \le \sigma|\lambda|/(1 - |\lambda|\tau)$ and that $|\lambda| \ge \gamma(\mathscr{A})/(\sigma + \tau\gamma(\mathscr{A}))$. □

Theorem 36. *Let X and Y be Banach spaces, and let $\mathscr{A}, \mathscr{B} \in CLR(X, Y)$ be such that $\mathscr{A}$ has closed range, $D(\mathscr{B}) \supset D(\mathscr{A})$, $\mathscr{B}(0) \subset \mathscr{A}(0)$, and*

$$\|\mathscr{B}(x)\| \le \sigma\|x\| + \tau\|\mathscr{A}(x)\|, \quad \textit{for every } x \in D(\mathscr{A}), \tag{102}$$

where σ and τ are nonnegative constants. If $\nu(\mathscr{A} : \mathscr{B}) = \infty$, then

$$\delta(N(\mathscr{A}), N(\mathscr{A} - \lambda\mathscr{B})) \le \frac{\sigma|\lambda|}{\gamma(\mathscr{A}) - |\lambda|(\sigma + \tau\gamma(\mathscr{A}))}. \tag{103}$$

Proof. Let N_k be as defined in (41) and consider a sequence z_k with the following properties:

$$\begin{gathered} z_k \in N_k, \mathscr{A}(z_{k+1}) \cap \mathscr{B}(z_k) \ne \varnothing, \\ \xi\|z_{k+1}\| \le \|\mathscr{A}(z_{k+1})\|, \quad k = 1, 2, \cdots, \end{gathered} \tag{104}$$

where ξ is a positive constant. We show that for each $z \in N(\mathscr{A})$ and $\xi < \gamma(\mathscr{A})$, there is a sequence z_k that satisfies (104) such that $z = z_1$. We set $z = z_1$ and construct z_k by induction. Suppose $z_1, z_2, \cdots x_k$ have been constructed with properties (104). Since $z_k \in N_k \subset M_1 = \mathscr{B}^{-1}(\mathscr{A}(X))$, there exists a $z_{k+1} \in D(\mathscr{A})$ such that $\mathscr{A}(z_{k+1}) \cap \mathscr{B}(z_k) \ne \varnothing$. Since $\gamma(\mathscr{A})\|\tilde{z}_{k+1}\| \le \|\mathscr{A}(z_{k+1})\|$ and z_{k+1} can be replaced by any other element of $\tilde{z}_{k+1}$, we can choose z_{k+1} such that $\xi\|z_{k+1}\| \le \|\mathscr{A}(z_{k+1})\|$. Since $\mathscr{A}(z_{k+1}) \cap \mathscr{B}(z_k) \ne \varnothing$, we see that $z_{k+1} \in \mathscr{A}^{-1}(\mathscr{B}(N_n)) = N_{k+1}$. This completes the induction process.

Since $\mathscr{A}(z_{k+1}) \cap \mathscr{B}(z_k) \ne \varnothing$ and $\mathscr{A}(0) \supset \mathscr{B}(0)$, we see that

$$\|\mathscr{A}(z_{k+1})\| \le \|\mathscr{B}(z_k)\| \le \sigma\|z_k\| + \tau\|A(z_k)\|. \tag{105}$$

For $k = 1$, (105) gives $\|\mathscr{A}(z_2)\| \le \|\mathscr{B}(z_1)\| \le \sigma\|z_1\|$ since $z_1 \in N(\mathscr{A})$. For $k \ge 2$, (104) implies that

$$\begin{aligned} \|\mathscr{A}(z_{k+1})\| &\le \|\mathscr{B}(z_k)\| \le \sigma\|z_k\| + \tau\|\mathscr{A}(z_k)\| \\ &\le \left(\sigma\xi^{-1} + \tau\right)\|\mathscr{A}(z_k)\| \le \left(\sigma\xi^{-1} + \tau\right)^2\|\mathscr{A}(z_{k-1})\| \\ &\le \cdots \le \left(\sigma\xi^{-1} + \tau\right)^{k-1}\|\mathscr{A}(z_2)\| \\ &= \xi^{-(k-1)}(\sigma + \xi\tau)^{k-1}\|\mathscr{A}(z_2)\| \\ &\le \sigma\xi^{-(k-1)}(\sigma + \xi\tau)^{k-1}\|z_1\|. \end{aligned} \tag{106}$$

We also see from (104) and (106) that

$$\|z_{k+1}\| \le \sigma\xi^{-k}(\sigma + \xi\tau)^{k-1}\|z_1\|, \quad k = 1, 2, \cdots \tag{107}$$

The bounds in (106) and (107) imply that the series

$$\begin{aligned} u(\lambda) &= \sum_{k=1}^{\infty} \lambda^{k-1} z_k, \\ \lambda(\mathscr{A}) &= \sum_{k=1}^{\infty} \lambda^k Q_{\mathscr{A}}\mathscr{A}(z_{k+1}), \\ \lambda(\mathscr{B}) &= \sum_{k=1}^{\infty} \lambda^{k-1} Q_{\mathscr{B}}\mathscr{B}(z_k), \\ \lambda(\mathscr{B}_{\mathscr{A}}) &= \sum_{k=1}^{\infty} \lambda^{k-1} Q_{\mathscr{A}}\mathscr{B}(z_k) \end{aligned} \tag{108}$$

are absolutely convergent for $|\lambda| < \xi/(\sigma + \xi\tau)$. The convergence of the last series follows from the fact that $\|Q_{\mathscr{A}}\mathscr{B}(z_k)\| \le \|Q_{\mathscr{B}}\mathscr{B}(z_k)\|$ since $\mathscr{B}(0) \subset \mathscr{A}(0)$.

Let $u_n(\lambda)$, $\lambda_n(\mathscr{A})$, $\lambda_n(\mathscr{B})$, and $\lambda_n(\mathscr{B}_{\mathscr{A}})$ denote the sequences of the partial sums of the above series in that order. Then, for each n, $u_n(\lambda) \in D(\mathscr{A})$ and $\lambda_n(\mathscr{A}) \in \tilde{Y} := Y/\mathscr{A}(0)$. Furthermore, $u_n(\lambda) \longrightarrow u(\lambda)$ and $\lambda_n(\mathscr{A}) \longrightarrow \lambda(\mathscr{A})$. Since $Q_{\mathscr{A}}\mathscr{A}$ is closed by Lemma 7, we see that $u(\lambda) \in D(Q_{\mathscr{A}}\mathscr{A}) = D(\mathscr{A})$ and that

$$Q_{\mathscr{A}}\mathscr{A}(u(\lambda)) = \lambda(\mathscr{A}) = \sum_{k=1}^{\infty} \lambda^k Q_{\mathscr{A}}\mathscr{A}(z_{k+1}). \tag{109}$$

Since $\mathscr{A}(z_{k+1}) \cap \mathscr{B}(z_k) \neq \varnothing$, a similar argument shows that

$$\begin{aligned} Q_{\mathscr{A}}\mathscr{B}(u(\lambda)) = \lambda(\mathscr{B}_{\mathscr{A}}) &= \sum_{k=1}^{\infty} \lambda^{k-1} Q_{\mathscr{A}}\mathscr{B}(z_k) \\ &= \sum_{k=1}^{\infty} \lambda^k Q_{\mathscr{A}}\mathscr{A}(z_{k+1}) = \lambda(\mathscr{A}). \end{aligned} \tag{110}$$

One also obtains the equality $Q_{\mathscr{B}}\mathscr{B}(u(\lambda)) = \lambda(\mathscr{B}) = \sum_{k=1}^{\infty} \lambda^k Q_{\mathscr{B}}\mathscr{B}(z_k)$ using the closedness of $\mathscr{B}$.

From (109) and (110), we see that

$$Q_{\mathscr{A}}[\mathscr{A}(u(\lambda)) - \lambda\mathscr{B}(u(\lambda))] = \tilde{0}, \tag{111}$$

and so $u(\lambda) \in N(\mathscr{A} - \lambda\mathscr{B})$.

Furthermore,

$$\|u(\lambda) - z_1\| \leq \sum_{k=1}^{\infty} \boxtimes |\lambda|^{k-1} \|z_k\| \leq \left(\frac{\sigma|\lambda|}{\xi - |\lambda|(\sigma + \tau\xi)}\right) \|z_1\|. \tag{112}$$

Since there is such a $u(\lambda) \in N(\mathscr{A} - \lambda\mathscr{B})$ for every $z - z_1 \in N(\mathscr{A})$, we conclude that

$$\delta(N(\mathscr{A}), N(\mathscr{A} - \lambda\mathscr{B})) \leq \frac{\sigma|\lambda|}{\gamma(\mathscr{A}) - |\lambda|(\sigma + \tau\gamma(\mathscr{A}))}. \tag{113}$$

□

We observe that if $\alpha(\mathscr{A}) < \infty$ then Theorem 34 can be used to conclude that $\mathscr{A} - \lambda\mathscr{B}$ has closed range if $|\lambda| < \gamma(\mathscr{A})/(\sigma + \tau\gamma(\mathscr{A}))$. However, this conclusion is not possible if no restriction is imposed on $\alpha(\mathscr{A})$. This case is considered in Lemma 37.

Lemma 37. *Let $\mathscr{A}$ and $\mathscr{B}$ be as in Theorem 36 with $\nu(\mathscr{A} : \mathscr{B}) = \infty$. Then, $\mathscr{A} - \lambda\mathscr{B}$ has closed range for $|\lambda| < \gamma(\mathscr{A})/(3\sigma + \tau\gamma(\mathscr{A}))$.*

Proof. In the present case, let $x \in X$ and set $y = x - u$ for any $u \in N(\mathscr{A} - \lambda\mathscr{B})$. Lemma 22 implies that for any $\varepsilon > 0$,

$$\|\tilde{y}\| = \operatorname{dist}(y, N(\mathscr{A})) \geq \frac{1 - \delta(N(\mathscr{A}), N(\mathscr{A} - \lambda\mathscr{B}))}{1 + \delta(N(\mathscr{A}), N(\mathscr{A} - \lambda\mathscr{B}))}(1 - \varepsilon)\|y\|. \tag{114}$$

Suppose that $x \in D(\mathscr{A}) = D(\mathscr{A} - \lambda\mathscr{B})$, and let $\delta := \delta(N(\mathscr{A}) : N(\mathscr{A} - \lambda\mathscr{B}))$. Since $(\mathscr{A} - \lambda\mathscr{B})(u) = (\mathscr{A} - \lambda\mathscr{B})(0) = \mathscr{A}(0)$, we see that,

$$\begin{aligned} \|(\mathscr{A} - \lambda\mathscr{B})(x)\| &= \|(\mathscr{A} - \lambda\mathscr{B})(y)\| \geq \|\mathscr{A}(y)\| - |\lambda|\|\mathscr{B}(y)\| \\ &\geq \|\mathscr{A}(y)\| - |\lambda|(\sigma\|y\| + \tau\|\mathscr{A}(y)\| \\ &= (1 - \tau|\lambda|)\|\mathscr{A}(y)\| - \sigma|\lambda|\|y\| \\ &\geq (1 - \tau|\lambda|)\gamma(\mathscr{A})\|\tilde{y}\| - \sigma|\lambda|\|y\| \\ &\geq (1 - \tau|\lambda|)\gamma(\mathscr{A})\left(\frac{1-\delta}{1+\delta}\right)(1 - \varepsilon)\|y\| - \sigma|\lambda|\|y\| (\text{by} (115)) \\ &\geq [\gamma(\mathscr{A}) - (2\sigma + \tau\gamma(\mathscr{A}))|\lambda|](1 - \varepsilon)\|y\| - \sigma|\lambda|\|y\| (\text{by} (104)) \\ &= [(\gamma(\mathscr{A}) - (2\sigma + \tau\gamma(\mathscr{A}))|\lambda|)(1 - \varepsilon) - \sigma|\lambda|]\|y\|. \end{aligned} \tag{115}$$

Let $\widehat{X}$ denote the quotient space $X/N(\mathscr{A} - \lambda\mathscr{B})$. Since $x - y = u \in N(\mathscr{A} - \lambda\mathscr{B})$, we see that $\|y\| \geq \|\widehat{y}\| = \|\widehat{x}\|$, and therefore, (115) implies that

$$\|(\mathscr{A} - \lambda\mathscr{B})(x)\| \geq [(\gamma(\mathscr{A}) - (2\sigma + \tau\gamma(\mathscr{A}))|\lambda|)(1 - \varepsilon) - \sigma|\lambda|]\|\widehat{x}\|. \tag{116}$$

Letting $\varepsilon \longrightarrow 0$ in (116) leads to the inequality

$$\|(\mathscr{A} - \lambda\mathscr{B})(x)\| \geq [(\gamma(\mathscr{A}) - (2\sigma + \tau\gamma(\mathscr{A}))|\lambda|) - \sigma|\lambda|]\|\widehat{x}\|, \tag{117}$$

from which we conclude that

$$\gamma(\mathscr{A} - \lambda\mathscr{B}) \geq (\gamma(\mathscr{A}) - (3\sigma + \tau\gamma(\mathscr{A}))|\lambda|). \tag{118}$$

It therefore follows that $\gamma(\mathscr{A} - \lambda\mathscr{B}) > 0$, and therefore, $R(\mathscr{A} - \lambda\mathscr{B})$ is closed if $|\lambda| < \gamma(\mathscr{A})/(3\sigma + \tau\gamma(\mathscr{A}))$. □

Finally, we establish the stability of both the nullity and deficiency of $\mathscr{A} - \lambda\mathscr{B}$ for λ inside the disk $|\lambda| < \rho$ for some constant ρ.

Theorem 38. *Let X and Y be Banach spaces, and let $\mathscr{A}, \mathscr{B} \in CLR(X, Y)$ be such that $\mathscr{A}$ has closed range, $D(\mathscr{B}) \supset D(\mathscr{A})$, $\mathscr{B}(0) \subset \mathscr{A}(0)$, and*

$$\|\mathscr{B}(x)\| \leq \sigma\|x\| + \tau\|\mathscr{A}(x)\|, \quad \text{for every } x \in D(\mathscr{A}), \tag{119}$$

where σ and τ are nonnegative constants. If $\nu(\mathscr{A} : \mathscr{B}) = \infty$, then $\alpha(\mathscr{A} - \lambda\mathscr{B})$ and $\beta(\mathscr{A} - \lambda\mathscr{B})$ are constants for all λ for which $|\lambda| < \gamma(\mathscr{A})/(3\sigma + \tau\gamma(\mathscr{A}))$.

Proof. Let $u \in N(\mathscr{A} - \lambda\mathscr{B})$. Then, $\mathscr{A}(u) \cap \lambda\mathscr{B}(u) \neq \varnothing$ and we see from (101) that

$$\|\tilde{u}\| \leq \sigma|\lambda| \frac{\|u\|}{(1 - |\lambda|\tau)\gamma(\mathscr{A})}. \tag{120}$$

Since $\|\tilde{u}\| = \operatorname{dist}(u, N(\mathscr{A}))$, we see from characterization (36) that

$$\delta(N(\mathscr{A} - \lambda\mathscr{B}), N(\mathscr{A})) \leq \frac{\sigma|\lambda|}{(1 - |\lambda|\tau)\gamma(\mathscr{A})}. \tag{121}$$

Since $\sigma|\lambda|/((1 - |\lambda|\tau)\gamma(\mathscr{A})) < 1$ if $|\lambda| < \gamma(\mathscr{A})/(\sigma + \tau\gamma(\mathscr{A}))$, Lemma 21 implies that

$$\alpha(\mathscr{A} - \lambda\mathscr{B}) \leq \alpha(\mathscr{A}), \quad \text{for } |\lambda| < \frac{\gamma(\mathscr{A})}{\sigma + \tau\gamma(\mathscr{A})}. \tag{122}$$

The reverse inequality follows from Theorem 36 by noting that the right-hand side of (103) is less than one if $|\lambda| < \gamma(\mathscr{A})/(2\sigma + \tau\gamma(\mathscr{A}))$. We therefore conclude by Lemma 21 that $\alpha(\mathscr{A}) \leq \alpha(\mathscr{A} - \lambda\mathscr{B})$ if $|\lambda| < \gamma(\mathscr{A})/(2\sigma + \tau\gamma(\mathscr{A}))$. Combined with (122), we conclude that

$$\alpha(\mathscr{A}) = \alpha(\mathscr{A} - \lambda\mathscr{B}), \quad \text{for } |\lambda| < \frac{\gamma(\mathscr{A})}{2\sigma + \tau\gamma(\mathscr{A})}. \tag{123}$$

To show that $\beta(\mathscr{A} - \lambda) = \beta(\mathscr{A})$, we make use of the linear relations $\breve{\mathscr{A}}$ and $\breve{\mathscr{B}}$ as defined in the proof of Theorem 34. Since $\breve{\mathscr{A}}$ is bounded, Lemmas 16 (c), 17 (d), and 15 imply that $R(\breve{\mathscr{A}}')$ has closed range. Since $\breve{\mathscr{B}}(0)' \subset \breve{\mathscr{A}}(0)'$ by Remark 10 and $\nu(\breve{\mathscr{A}}' : \breve{\mathscr{B}}') = \infty$ by Lemma 27, all the assumptions of Theorem 38 are satisfied by the pair $\breve{\mathscr{A}}'$ and $\breve{\mathscr{B}}'$. Since $\|\breve{\mathscr{B}}'\| = \|\breve{\mathscr{B}}\| < 1$ by Lemmas 16 (a) and 17 (c), it follows from (123) that

$$\alpha\left(\breve{\mathscr{A}}' - \lambda\breve{\mathscr{B}}'\right) = \alpha\left(\breve{\mathscr{A}}'\right), \quad \text{for } |\lambda| < \frac{\gamma\left(\breve{\mathscr{A}}'\right)}{2\left\|\breve{\mathscr{B}}'\right\|}. \tag{124}$$

Since $(\breve{\mathscr{A}} - \lambda\breve{\mathscr{B}})' = (\breve{\mathscr{A}}' - \lambda\breve{\mathscr{B}}')$ by Lemma 16 (b) and (d) and $(\breve{\mathscr{A}} - \lambda\breve{\mathscr{B}})$ has closed range (since $\mathscr{A} - \lambda\mathscr{B}$ has closed range), it follows from (88), Lemma 28, and (124) that

$$\begin{aligned} \beta(\mathscr{A} - \lambda\mathscr{B}) &= \beta\left(\breve{\mathscr{A}} - \lambda\breve{\mathscr{B}}\right) = \alpha\left(\breve{\mathscr{A}}' - \lambda\breve{\mathscr{B}}'\right) \\ &= \alpha\left(\breve{\mathscr{A}}'\right) = \beta\left(\breve{\mathscr{B}}\right) = \beta(\mathscr{B}). \end{aligned} \tag{125}$$

□

Theorem 38 remains true if we replace the requirement $\nu(\mathscr{A} : \mathscr{B}) = \infty$ with $\mathscr{B}^{-1}(0) \subset \mathscr{A}^{-1}(0)$.

Data Availability

The data in terms of references used in this manuscript can be publicly accessed.

Conflicts of Interest

The authors declare that there is no conflict of interest.

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