Research Article

An Extension of the Carathéodory Differentiability to Set-Valued Maps

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This paper uses the generalization of the Hukuhara difference for compact convex set to extend the classical notions of Carathéodory differentiability to multifunctions (set-valued maps). Using the Hukuhara difference and affine multifunctions as a local approximation, we introduce the notion of CH-differentiability for multifunctions. Finally, we tackle the study of the relation among the Fréchet differentiability, Hukuhara differentiability, and CH-differentiability.

1. Introduction

Given X and Y nonempty sets, if for each element x belonging to X is assigned, due to a certain law denoted here by $F : X \rightrightarrows Y$, a set $F(x)$ contained in Y , we say that F is a multifunction (we also use the term multivalued function or set-valued map). Among the many and various examples and uses of multifunctions, we can find from those simpler provided by elementary algebra and trigonometry to those appearing, for example, in applications of control theory involving optimal control problems and structural properties of control systems such as stability, controllability/reachability ([1–4]), and perturbation theory for ordinary differential equations in the real Euclidean space $\mathbb{R}^n$ ([5, 6]); of particular interest is also studying the differentiability of multifunctions applied to perturbed optimization problems as observed in [7], where it is considered the problem “minimize $f(w, x)$ as $x \in A(w)$ ” being $f : W \times X \rightarrow \mathbb{R} \cup \{+\infty\}$ an arbitrary function, $A : W \rightrightarrows X$ a multifunction with $0 \in \text{dom } A$ and W, X Banach spaces. The concept of differentiability for multifunction has been discussed by many authors ([8–13]). In [11], it extends the classical notion of Fréchet differentiability to multifunctions, using affine multifunctions and Hausdorff distance.

Other concepts of differentiability can be found in the literature of classical analysis, as the Carathéodory derivative proposed in [14], the strength of Carathéodory’s formulation

relies on the concept of continuity, and its proof uses strongly the properties of continuous functions. Another advantage about this formulation is that it does not require the difference quotient present in the Fréchet formulation, which is the key to generalize it to a function of several variables. In [15], it explored some of the advantages of Carathéodory’s characterization over Fréchet’s characterization proposed in [16].

In this work, motivated by the Carathéodory derivative proposed in [14] and generalization of Hukuhara difference proposed ([9, 17, 18]), an extension of the classical notion of Carathéodory differentiability to multifunctions defined on finite-dimensional normed spaces is approached. For this purpose, we assume finite-dimensional normed vector spaces X and Y and consider the family of affine multifunctions (see [11] for details of affine multifunctions and the Fréchet differentiability of multifunctions):

$$\text{MAFF}_0(X, Y) = \{A : X \rightarrow \text{bcc}Y \mid A \text{ is affine and } A(0) = \{0\}\}, \quad (1)$$

where $\text{bcc}Y$ is the collection of all bounded closed convex subsets of Y , provided with a suitable metric. Then, we associate the differentiability (Carathéodory) of the multifunc-

tions to the existence of continuous functions

$$\phi_1, \phi_2 : X \longrightarrow \text{MAFF}_0(X, Y). \quad (2)$$

This paper is organized as follows. In Section 2, we present some definitions and basic results of generalized Hukuhara difference and affine multifunction. Section 3 presents our extension of Carathéodory differentiability to multifunctions (see Definition 4), named here as CH-differentiability (Carathéodory Hukuhara differentiability), which is based on the set in (1) and the generalized Hukuhara difference, [9]. Also, we show an example of CH-differentiable multifunctions (see Example 4) and study some properties of the CH-differentiability such as uniqueness, continuity, sum, and composition. Finally, in Section 4, we provided a relation involving the Fréchet differentiability, generalized Hukuhara differentiability, and CH-differentiability (Theorems 7, 8, 9, 10).

2. Preliminaries

Let X and Y be finite-dimensional normed vector spaces over the set of real numbers $\mathbb{R}$.

A multifunction $F : X \rightrightarrows Y$ from X into Y assigns to each $x \in X$ a (possibly empty) subset $F(x) \subset Y$.

The sets

$$\begin{aligned} \text{dom } F &= \{x \in X : F(x) \neq \emptyset\}, \\ \text{gr } F &= \{(x, y) \in X \times Y : y \in F(x)\}, \end{aligned} \quad (3)$$

are called the effective domain and the graph of the multifunction F , respectively.

By $\text{bcc}Y$, we denote the collection of all bounded closed convex subsets of Y . The collection $\text{bcc}Y$ equipped with the Minkowsky addition:

$$A + B = \{a + b : a \in A, b \in B\} \quad (A, B \in \text{bcc}Y) \quad (4)$$

and multiplication by nonnegative real numbers:

$$\alpha A = \{\alpha a : a \in A\} \quad (A \in \text{bcc}Y, \alpha \geq 0) \quad (5)$$

is a semilinear space. The function

$$d_H(A, B) = \inf \{\alpha \geq 0 : A \subset B + \alpha B_Y, B \subset A + \alpha B_Y\} \quad (6)$$

where B_Y is the unit ball in Y , is called the Hausdorff distance, and determines a structure of a metric space on $\text{bcc}Y$.

Remark 1. It is well known that the Minkowsky addition is associative and commutative with neutral element $\{0\}$. If $\alpha = -1$, the scalar multiplication gives the opposite $-A = (-1)A = \{-a \mid a \in A\}$ but, in general,

$$A + (-A) \neq \{0\}, \quad (7)$$

i.e., the opposite of A is not the inverse of A unless $A = \{a\}$ is a singleton. The first implication of this fact is that, in gen-

eral, the additive simplification is not valid, that is $(A + C = B + C) \nRightarrow A = B$ or $(A + B) - B \neq A$.

Remark 2. The partial solution for (7) is the Hukuhara difference, this difference has been introduced as a set C for which $A \ominus B = C \Leftrightarrow A = B + C$, and the important property of $\ominus$ is $A \ominus A = \{0\}$ and $(A + B) \ominus B = A$, $\forall A, B \in \text{bcc}X$. The Hukuhara difference is unique, but it does not always exist (a necessary condition for $A \ominus B$ exist is that A contains a translation $\{c\} + B$ of B).

A generalization of the Hukuhara difference proposed in [17] aims to overcome this situation.

Definition 1 (see [9]). The generalized Hukuhara difference of two sets $A, B \in \text{bcc}Y$ (gH-difference for short) is defined as follows:

$$A \ominus_g B = C \Leftrightarrow \begin{cases} (a) & A = B + C; \\ \text{or} (b) & B = A + (-1)C. \end{cases} \quad (8)$$

Proposition 1 (see [18]). *If $C = A \ominus_g B$ exists, it is unique and if also $A \ominus B$ exists then $A \ominus_g B = A \ominus B$.*

Remark 3 (see [18]). A necessary condition for $A \ominus_g B$ to exist is that either A contains a translation of B (as for $A \ominus B$) or B contains a translation of A . In fact, for any given $c \in C$, we get $B + \{c\} \subseteq A$ from (i) or $A + \{-c\} \subseteq B$ from (ii).

Remark 4 (see [18]). If $A \ominus_g B$ exists, then $B \ominus_g A$ exist and $B \ominus_g A = -(A \ominus_g B)$.

Remark 5 (see Proposition 4 in [9]). The generalized Hukuhara difference exists for any two compact intervals.

The generalized Hukuhara difference has the following properties:

Proposition 2 (see [18]). *Let $A, B \in \text{bcc}Y$ then*

- (1) *If the gH-difference exists, it is unique and it is a generalization of the usual Hukuhara difference, since $A \ominus_g B = A \ominus B$ whenever $A \ominus B$ exists*
- (2) $A \ominus_g A = \{0\}$,
- (3) *if $A \ominus_g B$ exists in the sense (a), then $B \ominus_g A$ exists in the sense (b) and vice versa*
- (4) $(A + B) \ominus_g B = A$
- (5) $\{0\} \ominus_g (A \ominus_g B) = (-B) \ominus_g (-A)$,
- (6) *We have $(A \ominus_g B) = (B \ominus_g A) = C$ if and only if $C = \{0\}$ and $A = B$*

From now on, in this paper, we will consider the generalized Hukuhara difference, $\ominus_g$.

2.1. Affine Multifunction and Fréchet Differentiability. Let X and Y be finite-dimensional normed vector spaces over the set of real numbers $\mathbb{R}$.

A multifunction $H : X \rightarrow \text{bcc } Y$ is said to be convex if

$$\alpha H(x_1) + (1 - \alpha)H(x_2) \subset H(\alpha x_1 + (1 - \alpha)x_2), \quad (9)$$

for all $x_1, x_2 \in \text{dom } H$ and $\alpha \in [0, 1]$;
affine if

$$\alpha H(x_1) + (1 - \alpha)H(x_2) = H(\alpha x_1 + (1 - \alpha)x_2), \quad (10)$$

for all $x_1, x_2 \in \text{dom } H$ and $\alpha \in [0, 1]$.

Proposition 3 (see [11]). *An affine multifunction $F : X \rightarrow \text{bcc } Y$ is uniformly continuous on $\text{dom } F$ in the Hausdorff sense.*

Definition 2. A multifunction $F : X \rightarrow \text{bcc } Y$ is said to be Fréchet differentiable at a point $x_0 \in \text{int}(\text{dom } F)$, if there exists an affine multifunction $\mathcal{A}(x_0 | \cdot) : X \rightarrow \text{bcc } Y$ such that $0 \in \text{int}(\text{dom } \mathcal{A}(x_0 | \cdot))$ and

$$\lim_{h \rightarrow 0} \frac{d_H(F(x_0 + h), \mathcal{A}(x_0 | h))}{\|h\|} = 0. \quad (11)$$

This definition is equivalent to the following: for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $h \in \delta B_X$, we have

$$\begin{aligned} F(x_0 + h) &\subset \mathcal{A}(x_0 | h) + \varepsilon \|h\| B_Y, \\ \mathcal{A}(x_0 | h) &\subset F(x_0 + h) + \varepsilon \|h\| B_Y, \end{aligned} \quad (12)$$

where B_X and B_Y are unit balls in X and Y , respectively. So immediately, we have that $\mathcal{A}(x_0 | 0) = F(x_0)$.

3. CH-Differentiability and Some Properties

In this section, we use the Hukuhara difference and the affine multifunction to build the concept of differentiability of a multifunction.

Let X and Y be finite-dimensional normed vector space over real $\mathbb{R}$, define the space:

$$\text{MAFF}_0(X, Y) = \{A : X \rightarrow \text{bcc } Y \mid A \text{ is affine and } A(0) = \{0\}\} \quad (13)$$

Example 1. Let $\rho : X \rightarrow \mathbb{R}$ be a linear function in X^* , such that $\rho(0) = 0$. Choose the function $F : X \rightarrow \text{bcc } Y$ defined by $F(x) = \rho(x)B$, for $B \in \text{bcc } Y$. Then, $F \in \text{MAFF}_0(X, \text{bcc } Y)$.

In effect, $F(\alpha x + (1 - \alpha)y) = (\alpha \rho(x) + (1 - \alpha)\rho(y))B = \alpha F(x) + (1 - \alpha)F(y)$.

Proposition 4. For $F, G \in \text{MAFF}_0(X, Y)$,

$$\tilde{d}(F, G) = \sup_{\|x\| \leq 1} \{d_H(F(x), G(x))\}, \quad (14)$$

is a metric on $\text{MAFF}_0(X, Y)$, where d_H is the Hausdorff metric.

Proof.

$$\tilde{d}(F, F) = \sup_{\|x\| \leq 1} \{d_H(F(x), F(x))\} = \sup_{\|x\| \leq 1} \{0\} = 0. \quad (15)$$

(1) If $\tilde{d}(F, G) = 0$, then for all $x \in X$ such that $\|x\| \leq 1$, we have $d_H(F(x), G(x)) = 0$, thus $F(x) = G(x)$ for $x \in X$ such that $\|x\| \leq 1$. On the other hand, since F is affine and $F(0) = \{0\}$, for $\|x\| > 1$, we have

$$\begin{aligned} F(x) &= \|x\| \cdot \frac{1}{\|x\|} F(x) = \|x\| \cdot \left(\frac{1}{\|x\|} F(x) + \left(1 - \frac{1}{\|x\|}\right) \{0\} \right) \\ &= \|x\| \cdot F\left(\frac{1}{\|x\|} \cdot x + \left(1 - \frac{1}{\|x\|}\right) \cdot 0\right) = \|x\| \cdot F\left(\frac{x}{\|x\|}\right) \\ &= \|x\| \cdot G\left(\frac{x}{\|x\|}\right) = G(x). \end{aligned} \quad (16)$$

Therefore, $F = G$.

$$\begin{aligned} \tilde{d}(F, G) &= \sup_{\|x\| \leq 1} \{d_H(F(x), G(x))\} = \sup_{\|x\| \leq 1} \{d_H(G(x), F(x))\} \\ &= \tilde{d}(G, F). \end{aligned} \quad (17)$$

(2) If F, G , and H are multifunctions, then for each $x \in X$ such that $\|x\| \leq 1$, we have

$$\begin{aligned} d_H(F(x), H(x)) &\leq d_H(F(x), G(x)) + d_H(G(x), H(x)) \\ &\leq \tilde{d}(F, G) + \tilde{d}(G, H). \end{aligned} \quad (18)$$

Therefore, $\tilde{d}(F, H) \leq \tilde{d}(F, G) + \tilde{d}(G, H)$.

Definition 3. Let X, Y be finite-dimensional normed vector spaces over the set of real numbers $\mathbb{R}$. A function $\phi : X \rightarrow \text{MAFF}_0(X, \text{bcc } Y)$ is said to be continuous at x_0 , if given $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\|x - x_0\| < \delta$, then $\tilde{d}(\phi(x), \phi(x_0)) < \varepsilon$.

Example 2. Let x_0 be a fixed point and x in X . By the Hahn-Banach theorem (see [19], page 3.), there is a linear function $\rho_x : X \rightarrow \mathbb{R}$ in X^* , such that

$$\rho_x(0) = 0, \rho_x(x - x_0) = \|x - x_0\|_X^2, \text{ and } \|\rho_x\|_{X^*} = \|x - x_0\|_X. \quad (19)$$

Let be fixed $B \in \text{bcc}Y$. By Example 1, the functions $F_x : X \rightrightarrows \text{bcc}Y$ defined by $F_x(h) = \rho_x(h)B$ are in $\text{MAFF}_0(X, \text{bcc}Y)$. Now, the function

$$\begin{aligned} \phi : X &\longrightarrow \text{MAFF}_0(X, \text{bcc}Y), \\ x &\longrightarrow F_x, \end{aligned} \quad (20)$$

is continuous at x_0 .

In effect, $\phi(x_0) = 0$ and

$$\begin{aligned} \tilde{d}(\phi(x), \phi(x_0)) &= \sup_{\|h\| \leq 1} \{d_H(F_x(h), F_{x_0}(h))\} \\ &= \sup_{\|h\| \leq 1} \{d_H(\rho_x(h)B, 0)\} \leq \|\rho_x\|_{X^*} = \|x - x_0\|_X. \end{aligned} \quad (21)$$

Theorem 1. If $\phi, \varphi : X \longrightarrow \text{MAFF}_0(X, Y)$ are continuous at the point x_0 . Then, $\eta(x) = \phi(x) \odot_g \varphi(x)$ is continuous at x_0 .

Proof. Since ϕ and φ are continuous at x_0 , given $\varepsilon > 0$ and $\alpha \in [0, 1]$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that, if $\|x - x_0\| < \delta = \min(\delta_1, \delta_2)$ then $\tilde{d}(\phi(x), \phi(x_0)) < \alpha\varepsilon$ and $\tilde{d}(\varphi(x), \varphi(x_0)) < (1 - \alpha)\varepsilon$. Thus, for $\|h\| < \delta$ and the definition of affine multifunction, we have

$$\begin{aligned} \phi(x_0 + h)h &\subseteq \phi(x_0)h + \alpha\varepsilon\|h\|B_Y, \\ \varphi(x_0 + h)h &\subseteq \varphi(x_0)h + (1 - \alpha)\varepsilon\|h\|B_Y, \\ \phi(x_0)h &\subseteq \phi(x_0 + h)h + \alpha\varepsilon\|h\|B_Y, \\ \varphi(x_0)h &\subseteq \varphi(x_0 + h)h + (1 - \alpha)\varepsilon\|h\|B_Y. \end{aligned} \quad (22)$$

Now, if $x \in \eta(x_0 + h)(h)$, from Definition 1, there is $a \in \phi(x_0 + h)(h)$ and $b \in \varphi(x_0 + h)(h)$, such that $x = a - b$; thus, from (1) and (7), we have

$$\begin{aligned} a &= a_1 + \alpha\varepsilon\|h\|a_2, \\ b &= b_1 + (1 - \alpha)\varepsilon\|h\|b_2. \end{aligned} \quad (23)$$

where $a_1 \in \phi(x_0)(h)$, $b_1 \in \varphi(x_0)(h)$, and $a_2, b_2 \in B_Y$. So

$$\begin{aligned} x &= a - b = a_1 + \alpha\varepsilon\|h\|a_2 - b_1 - (1 - \alpha)\varepsilon\|h\|b_2 \\ &= (a_1 - b_1) + \varepsilon\|h\|(\alpha a_2 + (1 - \alpha)(-b_2)) = x_1 + \varepsilon\|h\|x_2. \end{aligned} \quad (24)$$

Then, $x_1 = a_1 - b_1 \in \phi(x_0)(h) \odot_g \varphi(x_0)(h)$, this is, $x_1 \in \eta(x_0)(h)$ and $x_2 = \alpha a_2 + (1 - \alpha)(-b_2) \in B_Y$, since $a_2, -b_2 \in B_Y$ and B_Y is convex. Therefore, $x \in \eta(x_0)(h) + \varepsilon\|h\|B_Y$. In consequence, $\eta(x_0 + h) \subseteq \eta(x_0)(h) + \varepsilon\|h\|B_Y$. Analogously, $\eta(x_0) \subseteq \eta(x_0 + h)(h) + \varepsilon\|h\|B_Y$. Thus, η is continuous.

Now, motivated by the Carathéodory's Definition in [14], Definition 1, and Proposition 2, we define the following:

Definition 4. A multifunction $F : X \longrightarrow \text{bcc}Y$ is CH-differentiable (Carathéodory and Hukuhara differentiable) at a point $x_0 \in \text{int}(\text{Dom } F)$, if there are functions $\phi_1, \phi_2 : V_{x_0} \longrightarrow \text{MAFF}_0(X, \text{bcc}Y)$ continuous at a point $x_0 \in V_{x_0}$, where V_{x_0} is a neighborhood of the point x_0 , such that $\phi_1(x_0) = \phi_2(x_0)$ and

$$\begin{aligned} F(x_0 + h) \odot_g F(x_0) &= \phi_1(x_0 + h)(h), \\ F(x_0) \odot_g F(x_0 + h) &= \phi_2(x_0 + h)(h), \end{aligned} \quad (25)$$

$$\forall h \in X \text{ such that } x_0 + h \in V_{x_0}.$$

Remark 6. Given $F : X \rightrightarrows \text{bcc}Y$. It is natural to ask the following question. Are the functions ϕ 's in Definition 4 unique, the answer is no, even it does not fit for functions (see [15]). Nevertheless, we have the uniqueness of the differential.

Example 3. The constant function $F : X \rightrightarrows \text{bcc}Y$ defined by $F(x) = B$, for $B \in \text{bcc}Y$ is CH-differentiable at X . In effect, $F(x_0 + h) \odot_g F(x_0) = B \odot_g B = \{0\} = \phi(x_0 + h)(h)$, where $\phi(x_0 + h)(h) = \{0\} \forall h \in X$.

Example 4. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable function at $x_0 \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. The function $F : \mathbb{R}^n \rightrightarrows \text{bcc}\mathbb{R}^m$ defined by $F(x) = f(x)\{b\}$, is CH-differentiable at x_0 . In effect, $F(x_0 + h) \odot_g F(x_0) = f(x_0 + h)\{b\} \odot_g f(x_0)\{b\} = (f(x_0 + h) - f(x_0))\{b\} = \psi(x_0 + h)(h)\{b\}$ where ψ exists because f is differentiable, thus, Carathéodory differentiable. Now, consider $\phi : V_{x_0} \longrightarrow \text{MAFF}_0(X, \text{bcc}Y)$ defined by $\phi(x)(z) = \psi(x)(z)\{b\}$. Then, $F(x_0 + h) \odot_g F(x_0) = \phi(x_0 + h)(h)$.

Theorem 2. Let X, Y be two Banach spaces and $F : X \rightrightarrows \text{bcc}Y$ be a CH-differentiable at x_0 . Suppose that, there are functions ϕ and φ satisfying the CH-differentiability conditions. Then, $\phi(x_0) = \varphi(x_0)$.

Proof. Since $F(x_0 + h) \odot_g F(x_0) = \phi(x_0 + h)(h)$ and $F(x_0 + h) \odot_g F(x_0) = \varphi(x_0 + h)(h)$, for all $h \in V_{x_0}$. Now define the function

$$\eta : V_{x_0} \longrightarrow \text{MAFF}_0(X, \text{bcc}Y), \quad (26)$$

by $\eta(x) = \phi(x) \odot_g \varphi(x)$. Observe that $\eta(x_0 + h)(h) = \{0\}$. Thus

$$d_H(\eta(x_0)(h), \{0\}) = d_H(\eta(x_0)(h), \eta(x_0 + h)(h)). \quad (27)$$

Now, since ϕ and φ are continuous at x_0 . We have η is continuous at x_0 . Whence, given $\varepsilon > 0$, there is $\delta > 0$ such that if $\|h\| < \delta$, then

$$d_H\left(\eta(x_0)\left(\frac{h}{\|h\|}\right), \eta(x_0 + h)\left(\frac{h}{\|h\|}\right)\right) \leq \tilde{d}(\eta(x_0), \eta(x_0 + h)) < \varepsilon. \quad (28)$$

Now, for each $z \in X$ with $\|z\| = 1$ there is $N \in \mathbb{N}$, such that for $n > N$, there exists $h_n \in X$, with $\|h_n\| < \delta$ and $h_n/\|h_n\| \rightarrow z$ (just take the sequence on the line segment joining 0 to z). Hence, we have

$$\lim_{n \rightarrow +\infty} d_H\left(\eta(x_0)\left(\frac{h_n}{\|h_n\|}\right), \{0\}\right) = 0. \quad (29)$$

Then,

$$\eta(x_0)\left(\lim_{n \rightarrow +\infty} \frac{h_n}{\|h_n\|}\right) = \lim_{n \rightarrow +\infty} \eta(x_0)\left(\frac{h_n}{\|h_n\|}\right) = \{0\}. \quad (30)$$

Therefore,

$$\eta(x_0)(z) = \{0\} \text{ and } \phi(x_0) = \varphi(x_0).$$

Lemma 1. Let $F, G : X \rightarrow bccY$ be two multifunctions and $x_0 \in X$, if $\lim_{x \rightarrow x_0} F(x) = F(x_0)$ and $\lim_{x \rightarrow x_0} G(x) = G(x_0)$. Then

$$\lim_{x \rightarrow x_0} F(x) + G(x) = F(x_0) + G(x_0). \quad (31)$$

Proof. In effect, if $\lim_{x \rightarrow x_0} F(x) = F(x_0)$ and $\lim_{x \rightarrow x_0} G(x) = G(x_0)$, then given $\varepsilon > 0$ and $\alpha \in [0, 1]$, there exists $\delta > 0$, such that if $d(x, x_0) < \delta$. Then, $d(F(x), F(x_0)) < \alpha\varepsilon$ and $d(G(x), G(x_0)) < (1 - \alpha)\varepsilon$. Hence, we obtain that

$$\begin{aligned} F(x) &\subseteq F(x_0) + \alpha\varepsilon B_Y, \\ G(x) &\subseteq G(x_0) + (1 - \alpha)\varepsilon B_Y. \end{aligned} \quad (32)$$

It follows that if $z \in F(x) + G(x)$ then $z = a + b$, with $a \in F(x)$ and $b \in G(x)$, thus, $a = a_1 + \alpha\varepsilon a_2$ and $b = b_1 + (1 - \alpha)\varepsilon b_2$, with $a_1 \in F(x_0)$, $b_1 \in G(x_0)$, and $a_2, b_2 \in B_Y$, so $z = a_1 + b_2 + \varepsilon(\alpha a_2 + (1 - \alpha)b_2)$. As $(\alpha a_2 + (1 - \alpha)b_2) \in B_Y$, we have $z \in F(x_0) + G(x_0) + \varepsilon B_Y$. Thus, $F(x) + G(x) \subseteq F(x_0) + G(x_0) + \varepsilon B_Y$. Using $F(x_0) \subseteq F(x) + \alpha\varepsilon B_Y$ and $G(x_0) \subseteq G(x) + (1 - \alpha)\varepsilon B_Y$, similarly, $F(x_0) + G(x_0) \subseteq F(x) + G(x) + \varepsilon B_Y$. Therefore

$$d_H(F(x) + G(x), F(x_0) + G(x_0)) < \varepsilon. \quad (33)$$

Theorem 3. If $F : X \rightarrow bccY$ is a CH-differentiable multifunction at $x_0 \in X$. Then, F is Hausdorff continuous at x_0 .

Proof. Since $F(x_0 + h) \odot_g F(x_0) = \phi(x_0 + h)(h)$, we have, from the above proposition,

$$\begin{aligned} \lim_{h \rightarrow 0} F(x_0 + h) &= \lim_{h \rightarrow 0} F(x_0) + \phi(x_0 + h)(h) = F(x_0) + \phi(x_0)(0) \\ &= F(x_0). \end{aligned} \quad (34)$$

Theorem 4. Let $F, G : X \rightarrow bccY$ be two CH-differentiable multifunctions at x_0 . Then, $\alpha F(x) + \beta G(x)$, with $\alpha, \beta \in \mathbb{R}$, is CH-differentiable at x_0 .

Proof. We have, $F(x_0 + h) \odot_g F(x_0) = \phi(x_0 + h)(h)$ and $G(x_0 + h) \odot_g G(x_0) = \varphi(x_0 + h)(h)$, thus

$$\begin{aligned} \alpha F(x_0 + h) + \beta G(x_0 + h) &= \alpha(F(x_0) + \phi(x_0 + h)(h)) + \beta(G(x_0) \\ &\quad + \varphi(x_0 + h)(h)) \\ &= \alpha F(x_0) + \beta G(x_0) + (\alpha\phi + \beta\varphi)(x_0 + h)(h). \end{aligned} \quad (35)$$

Hence, since $(\alpha\phi + \beta\varphi)$ is continuous, the result follows.

Theorem 5. If $f : X \rightarrow X$ is a differentiable function at x_0 and $F : X \rightarrow bccY$ is a CH-differentiable multifunction at $f(x_0)$, then $F \circ f : X \rightarrow bccY$, is CH-differentiable at x_0 .

Proof. Observe that, $F(f(x_0) + h) \odot_g F(f(x_0)) = \phi(f(x_0) + h)(h)$ and $f(x_0 + h) - f(x_0) = \psi(x_0 + h)(h)$, thus

$$\begin{aligned} F \circ f(x_0 + h) \odot_g F \circ f(x_0) &= F(f(x_0 + h)) \odot_g F(f(x_0)) \\ &= F(f(x_0) + \psi(x_0 + h)(h)) \odot_g F(f(x_0)) \\ &= \phi(f(x_0) + \psi(x_0 + h)(h))(\psi(x_0 + h)(h)) \\ &= \phi(f(x_0) + \psi(x_0 + h)(h)) \circ \psi(x_0 + h)(h). \end{aligned} \quad (36)$$

Theorem 6. If $F \in \text{MAFF}_0(X, bccY)$, then F is a CH-differentiable multifunction at x_0 .

Proof. Let $F \in \text{MAFF}_0(X, bccY)$ and $\alpha \in (0, 1)$. Choose $h = (1 - \alpha/\alpha)w$ small, thus

$$\begin{aligned} F(x_0 + h) &= F\left(x_0 + \frac{1 - \alpha}{\alpha}w\right) = F\left(\frac{1}{\alpha}(\alpha x_0 + (1 - \alpha)w)\right) \\ &= \frac{1}{\alpha}(\alpha F(x_0) + (1 - \alpha)F(w)) = F(x_0) + \frac{1 - \alpha}{\alpha}F(w) \\ &= F(x_0) + F\left(\frac{1 - \alpha}{\alpha}w\right) = F(x_0) + F(h). \end{aligned} \quad (37)$$

Now, choose $\phi : V_{x_0} \longrightarrow MAFF_0(X, bccY)$, defined by $\phi(h) = F$.

4. CH-Differentiability vs. Fréchet Differentiability and Generalized Hukuhara Differentiability

The following result is useful for studying the relationship between CH-differentiability and Fréchet differentiability.

Lemma 4. Let A be in $bccY$ and $\phi \in MAFF_0(X, bccY)$.

Consider $\varphi : X \longrightarrow bccY$ defined by

$$\varphi(x) = A + \phi(x). \quad (38)$$

Then, φ is an affine multifunction.

Proof. Let $x_1, x_2 \in X$ and $\alpha \in [0, 1]$, then

$$\phi(\alpha x_1 + (1 - \alpha)x_2) = \alpha\phi(x_1) + (1 - \alpha)\phi(x_2) \quad (39)$$

with $\phi(0) = \{0\}$. On the other hand, since $A = \alpha A + (1 - \alpha)A$. We have

$$\begin{aligned} \phi(\alpha x_1 + (1 - \alpha)x_2) &= A + \phi(\alpha x_1 + (1 - \alpha)x_2) \\ &= \alpha A + (1 - \alpha)A + \alpha\phi(x_1) + (1 - \alpha)\phi(x_2) \\ &= \alpha(A + \phi(x_1)) + (1 - \alpha)(A + \phi(x_2)) \\ &= \alpha\varphi(x_1) + (1 - \alpha)\varphi(x_2). \end{aligned} \quad (40)$$

Theorem 7. If $F : X \longrightarrow bccY$ is a CH-differentiable multifunction at x_0 , then F is Fréchet differentiable at x_0 .

Proof. Using the CH-differentiability of F at x_0 , there exist $\phi_1, \phi_2 : V_{x_0} \longrightarrow MAFF_0(X, bccY)$ continuous functions at $x_0 \in V_{x_0}$, where V_{x_0} is a neighborhood at x_0 , such that $\phi_1(x_0) = \phi_2(x_0)$ and

$$\begin{aligned} F(x_0 + h) \odot_g F(x_0) &= \phi_1(x_0 + h)(h), \\ F(x_0) \odot_g F(x_0 + h) &= \phi_2(x_0 + h)(h), \end{aligned} \quad (41)$$

$\forall h \in X$ such that $x_0 + h \in V_{x_0}$. On the other hand, since $\phi_1(x_0)$ is an affine multifunction, using Lemma 4.1,

$$A(h) = F(x_0) + \phi_1(x_0)(h), \quad (42)$$

is an affine multifunction. We affirm that A is Fréchet differentiable. In effect, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|h\| < \delta$, then

$$d_H(\phi_1(x_0 + h)(z), \phi_1(x_0)(z)) \leq \tilde{d}(\phi_1(x_0 + h), \phi_1(x_0)) < \varepsilon, \quad (43)$$

for all $z \in X$, such that $\|z\| \leq 1$. In particular, for $z = h/\|h\|$, we have

$$\begin{aligned} d_H\left(\frac{1}{\|h\|}\phi_1(x_0 + h)(h), \frac{1}{\|h\|}\phi_1(x_0)(h)\right) \\ = d_H\left(\phi_1(x_0 + h)\left(\frac{h}{\|h\|}\right), \phi_1(x_0)\left(\frac{h}{\|h\|}\right)\right) < \varepsilon. \end{aligned} \quad (44)$$

This is equivalent to

$$\begin{aligned} \phi_1(x_0 + h)(h) &\subseteq \phi_1(x_0)(h) + \varepsilon\|h\|B_Y, \\ \phi_1(x_0)(h) &\subseteq \phi_1(x_0 + h)(h) + \varepsilon\|h\|B_Y. \end{aligned} \quad (45)$$

Now, since $F(x_0 + h) \odot F(x_0) = \phi_1(x_0 + h)(h)$, we have

$$\begin{aligned} F(x_0 + h) &= F(x_0) + \phi_1(x_0 + h)(h) \subseteq F(x_0) + \phi_1(x_0)(h) \\ &\quad + \varepsilon\|h\|B_Y \subseteq A(h) + \varepsilon\|h\|B_Y. \end{aligned} \quad (46)$$

Analogously,

$$\begin{aligned} A(h) &= F(x_0) + \phi_1(x_0)(h) \subseteq F(x_0) + \phi_1(x_0 + h)(h) \\ &\quad + \varepsilon\|h\|B_Y \subseteq F(x_0 + h) + \varepsilon\|h\|B_Y. \end{aligned} \quad (47)$$

The reciprocal of the above theorem is not always true, because the Hukuhara difference does not exist (see Remark 2). But when this difference exists, we have a relationship between Fréchet's differentiability and CH-differentiability.

Theorem 8. If the Hukuhara difference belongs to $bccY$ and $F : X \longrightarrow bccY$ is a Fréchet differentiable multifunction at x_0 , then F is CH-differentiable at x_0 .

Proof. Let A be the Fréchet differential of F at x_0 and $x \in X$. By the Hahn-Banach theorem, there exists a lineal function $\rho_x : X \longrightarrow \mathbb{R}$ in X^* , such that

$$\rho_x(0) = 0, \rho_x(x - x_0) = \|x - x_0\|_X^2, \text{ and } \|\rho_x\|_{X^*} = \|x - x_0\|_X. \quad (48)$$

Then, we can define the function $\phi : X \longrightarrow MAFF_0(X, bccY)$ by

$$\phi_1(x) = \begin{cases} \frac{1}{\|x - x_0\|_X^2} [\rho_x(\cdot)((f(x) \odot_g f(x_0))!_g A(x - x_0))] + A(\cdot), & \text{if } x \neq x_0, \\ A(\cdot), & \text{if } x = x_0. \end{cases} \quad (49)$$

This function is continuous at x_0 (see Example 2). And

$$\begin{aligned}\phi_1(x)(x-x_0) &= \frac{1}{\|x-x_0\|_X^2} [\rho_x(x-x_0)((f(x) \odot_g f(x_0)) \odot_g A(x-x_0))] + A(x-x_0) \\ &= \frac{1}{\|x-x_0\|_X^2} [\|x-x_0\|_X^2 ((f(x) \odot_g f(x_0)) \odot_g A(x-x_0))] + A(x-x_0) \\ &= ((f(x) \odot_g f(x_0)) \odot_g A(x-x_0)) + A(x-x_0) = f(x) \odot_g f(x_0).\end{aligned}\quad (50)$$

Analogously, for

$$\phi_2(x) = \begin{cases} \frac{1}{\|x-x_0\|_X^2} [\rho_x(\cdot)((f(x_0) \odot_g f(x)) \odot_g A(x-x_0))] + A(\cdot), & \text{if } x \neq x_0, \\ A(\cdot), & \text{if } x = x_0. \end{cases}\quad (51)$$

we have that ϕ_2 is continuous and $\phi_2(x)(x-x_0) = f(x_0) \odot_g f(x)$.

Now, we remember the generalized Hukuhara differentiability.

Definition 5: (see [9]). Let $x_0 \in \mathbb{R}$ and $F : \mathbb{R} \rightarrow \text{bcc}Y$ be a multifunction, then the gH-derivative of the multifunction F at x_0 is defined as

$$\partial_F(x_0) = \lim_{h \rightarrow 0} \frac{F(x_0+h) \odot_g F(x_0)}{h}. \quad (52)$$

If $\partial_F(x_0) \in \text{bcc}Y$ satisfying (52) exists, we say that F is generalized Hukuhara differentiable (gH-differentiable for short) at x_0 .

Theorem 9. Let $F : \mathbb{R} \rightarrow \text{cc}Y$ be a multifunction, if F is CH-differentiable at $x_0 \in \mathbb{R}$. Then, F is generalized Hukuhara differentiable at $x_0 \in \mathbb{R}$.

Proof. Using the CH-differentiability of F , there exist multifunctions $\phi_1, \phi_2 : V_{x_0} \rightarrow \text{MAFF}_0(\mathbb{R}, \text{bcc}Y)$ continuous at $x \in V_{x_0}$ (neighborhood at x_0), such that $\phi_1(x_0) = \phi_2(x_0)$ and

$$\begin{aligned}F(x_0+h) \odot_g F(x_0) &= \phi_1(x_0+h)(h), \\ F(x_0) \odot_g F(x_0+h) &= \phi_2(x_0+h)(h),\end{aligned}\quad (53)$$

for all $h \in \mathbb{R}$ such that $x_0+h \in V_{x_0}$. Hence,

$$\lim_{h \rightarrow 0} \frac{F(x_0+h) \odot_g F(x_0)}{h} = \lim_{h \rightarrow 0} \frac{\phi_1(x_0+h)(h)}{h} = \lim_{h \rightarrow 0} \phi_1(x_0+h)(1). \quad (54)$$

Now, from the continuity of ϕ_1 , given $\varepsilon > 0$ there exists $\delta > 0$ such that, if $|h| < \delta$ then $\tilde{d}(\phi(x_0+h), \phi(x_0)) < \varepsilon$, in particular

$$d_H(\phi(x_0+h)(1), \phi(x_0)(1)) < \varepsilon, \quad (55)$$

thus

$$\lim_{h \rightarrow 0} \frac{F(x_0+h) \odot_g F(x_0)}{h} = \phi_1(x_0)(1). \quad (56)$$

Remark 7. Throughout this article, we confine ourselves to multifunctions $F : X \rightarrow \text{bcc}Y$, as the generalized Hukuhara difference not always exist, to study the inverse of the Theorem (4.5), consider the case $\text{bcc}\mathbb{R} = \mathbb{I}$ (where $\mathbb{I}$ denote the set of closed bounded interval of the real line), and the gH-difference exist for any element in $\mathbb{I}$. These multifunctions are called interval-valued functions and are studied in detail in [9].

In the case $\mathbb{I}$, the Hausdorff distance has the following properties:

Proposition 5 (see [9]). For $A, B \in \mathbb{I}$, we have

$$\begin{aligned}d_H(kA, kB) &= |k|d_H(A, B), \forall k \in \mathbb{R}, \\ d_H(A, B) &= d_H(A \odot_g B, \{0\}).\end{aligned}\quad (57)$$

Theorem 10. Let $F : \mathbb{R} \rightarrow \mathbb{I}$ be a multifunction, if F is generalized Hukuhara differentiable at $x_0 \in \mathbb{R}$, then F is CH-differentiable at $x_0 \in \mathbb{R}$.

Proof. Using the generalized Hukuhara differentiability of F , there is $\partial_F(x_0) \in \mathbb{I}$ satisfying (52). We define

$$\mathcal{A}_{x_0} : \mathbb{R} \rightarrow \mathbb{I}, \quad (58)$$

given by $\mathcal{A}_{x_0}(h) = F(x_0) + h\partial_F(x_0)$. By Lemma 4.1, the multifunction (58) is affine, moreover $\mathcal{A}_{x_0}(0) = F(x_0)$. On the other hand,

$$\lim_{h \rightarrow 0} \frac{d_H(F(x_0+h), \mathcal{A}_{x_0}(h))}{|h|} = \lim_{h \rightarrow 0} \frac{d_H(F(x_0+h), F(x_0) + h\partial_F(x_0))}{|h|}. \quad (59)$$

Thus, by Proposition 5, we have

$$\lim_{h \rightarrow 0} \frac{d_H(F(x_0+h), \mathcal{A}_{x_0}(h))}{|h|} = \lim_{h \rightarrow 0} d_H\left(\frac{F(x_0+h) \odot_g F(x_0)}{h}, \partial_F(x_0)\right) = 0, \quad (60)$$

since $\lim_{h \rightarrow 0} (F(x_0+h) \odot_g F(x_0))/h = \partial_F(x_0)$. Therefore, F is Fréchet differentiable, and by Theorem 8, is CH-differentiable.

Data Availability

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication.

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