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Research Article

Controlled Continuous $*$-K-g-Frames for Hilbert C^*-Modules

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Frame theory has a great revolution for recent years. This theory has been extended from Hilbert spaces to Hilbert C^*-modules. In this paper, we define and study the new concept of controlled continuous $*$-K-g-frames for Hilbert C^*-modules and we establish some properties.

1. Introduction and Preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [1] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [2] by Daubechies et al., frame theory began to be widely used, particularly in the more specialized context of wavelet frame and Gabor frame [3, 4]. Frames have been used in signal processing, image processing, data compression, and sampling theory. The concept of a generalization of frame to a family indexed by some locally compact space endowed with a Radon measure was proposed by Kaiser [5] and independently by Ali et al. [6]. These frames are known as continuous frames. Gabardo and Han in [7] called them frames associated with measurable spaces, Askari-Hemmat et al. in [8] called them generalized frames, and in mathematical physics, they are known as coherent states.

In 2012, Gavruta [9] introduced the notion of K-frame in Hilbert space to study the atomic systems with respect to a bounded linear operator K. Moreover, the following two recent papers [10, 11] are closely linked to Gravuta's paper and represent a generalization of it by considering also unbounded operators.

Controlled frames in Hilbert spaces have been introduced by Balazs et al. [12] to improve the numerical efficiency of iterative algorithms for inverting the frame operator.

Frank and Larson [13] were the first one that extend frames to Hilbert C^*-modules.

Controlled frames in Hilbert C^*-modules was introduced by Kouchi and Rahimi [14], where the authors showed that they share many useful properties with their corresponding notions in a Hilbert space. Finally, we note that controlled K-g-frames in Hilbet spaces have been introduced by Hua and Huang [15]. The theory of continuous frames has been generalized in Hilbert C^*-modules. For more details, see [16–22].

In this paper, we introduce the notion of a controlled continuous $*$-K-g-frame for Hilbert C^*-modules.

In the following, we briefly recall the definitions and basic properties of C^*-algebras and Hilbert $\mathscr{A}$-modules. Our references for C^*-algebras are [23, 24]. For a C^*-algebra $\mathscr{A}$, if $a \in \mathscr{A}$ is positive, we write $a \geq 0$ and $\mathscr{A}^+$ denotes the set of all positive elements of $\mathscr{A}$.

Definition 1. Let $\mathscr{A}$ be a unital C^* -algebra and $\mathscr{H}$ be a left $\mathscr{A}$ -module, such that the linear structures of $\mathscr{A}$ and $\mathscr{H}$ are compatible. $\mathscr{H}$ is a pre-Hilbert $\mathscr{A}$-module if $\mathscr{H}$ is equipped with an $\mathscr{A}$-valued inner product $\langle .,. \rangle_{\mathscr{A}} : \mathscr{H} \times \mathscr{H} \rightarrow \mathscr{A}$, such that it is sesquilinear, positive definite, and respects the module action. In the other words,

(i) $\langle x, x \rangle_{\mathscr{A}} \geq 0$ for all $x \in \mathscr{H}$ and $\langle x, x \rangle_{\mathscr{A}} = 0$ if and only if $x = 0$

(ii) $\langle ax+y,z\rangle_{\mathscr{A}} = a\langle x,z\rangle_{\mathscr{A}} + \langle y,z\rangle_{\mathscr{A}}$ for all $a \in \mathscr{A}$ and $x, y, z \in \mathscr{H}$

(iii) $\langle x,y\rangle_{\mathscr{A}} = \langle y,x\rangle_{\mathscr{A}}^*$ for all $x, y \in \mathscr{H}$

For $x \in \mathscr{H}$, we define$\|x\| = \|\langle x,x\rangle_{\mathscr{A}}\|^{1/2}$. If $\mathscr{H}$ is complete with $\|.\|$, it is called a Hilbert $\mathscr{A}$-module or a Hilbert C^*-module over $\mathscr{A}$. For every a in C^*-algebra $\mathscr{A}$, we have $|a| = (a^*a)^{1/2}$ and the $\mathscr{A}$-valued norm on $\mathscr{H}$ is defined by$|x| = \langle x,x\rangle_{\mathscr{A}}^{1/2}$ for $x \in \mathscr{H}$.

Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert $\mathscr{A}$-modules. A map $T : \mathscr{H} \rightarrow \mathscr{K}$ is said to be adjointable if there exists a map $T^* : \mathscr{K} \rightarrow \mathscr{H}$ such that $\langle Tx,y\rangle_{\mathscr{A}} = \langle x,T^*y\rangle_{\mathscr{A}}$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$.

We reserve the notation $End_{\mathscr{A}}^*(\mathscr{H},\mathscr{K})$ for the set of all adjointable operators from $\mathscr{H}$ to $\mathscr{K}$, and $End_{\mathscr{A}}^*(\mathscr{H},\mathscr{H})$ is abbreviated to $End_{\mathscr{A}}^*(\mathscr{H})$.

The following lemmas will be used to prove our mains results.

Lemma 2 [25]. *Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert $\mathscr{A}$ -modules and $T \in End_{\mathscr{A}}^*(\mathscr{H},\mathscr{K})$. Then, the following statements are equivalent:*

(i) T is surjective

(ii) T^* is bounded below with respect to the norm; i.e., there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$, $x \in \mathscr{K}$

(iii) T^* is bounded below with respect to the inner product; i.e., there is $m' > 0$ such that

$$\langle T^*x, T^*x\rangle_{\mathscr{A}} \geq m'\langle x,x\rangle_{\mathscr{A}}, x \in \mathscr{K} \tag{1}$$

Lemma 3. *Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert $\mathscr{A}$-modules and $T \in End_{\mathscr{A}}^*(\mathscr{H},\mathscr{K})$. Then, the following statements are equivalent:*

(i) The operator T is bounded and $\mathscr{A}$-linear

(ii) There exist $0 \leq k$ such that $\langle Tx,Tx\rangle_{\mathscr{A}} \leq k\langle x,x\rangle_{\mathscr{A}}$ for all $x \in \mathscr{H}$

Lemma 4 [26]. *Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert $\mathscr{A}$ -modules and $T \in End^*(\mathscr{H},\mathscr{K})$. Then,*

(i) if T is injective and T has a closed range, then the adjointable map T^*T is invertible and

$$\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2 \tag{2}$$

(ii) if T is surjective, then the adjointable map TT^* is invertible and

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2 \tag{3}$$

For the following theorem, $R(T)$ denote the range of the operator T.

Theorem 5 [27]. *Let $\mathscr{H}$ be a Hilbert $\mathscr{A}$ -module over a C^*-algebra $\mathscr{A}$ and let T, S be two operators for $End_{\mathscr{A}}^*(\mathscr{H})$. If $R(S)$ is closed, then the following statements are equivalent:*

$$R(T) \subset R(S) \tag{4}$$

(i) $TT^* \leq \lambda^2 SS^*$ for some $\lambda \geq 0$

(i) There exists $Q \in End_{\mathscr{A}}^*(\mathscr{H})$ such that $T = SQ$

2. Controlled Continuous $*$-K-g-Frames for Hilbert C^*-Modules

Let X be a Banach space, (Ω, μ) be a measure space, and $f : \Omega \rightarrow X$ be a measurable function. Integral of the Banach-valued function f has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions. Since every C^*-algebra and Hilbert C^*-module are a Banach space, thus we can use this integral and its properties.

Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert C^*-modules, $\{\mathscr{K}_w : w \in \Omega\}$ is a family of subspaces of $\mathscr{K}$, and $End_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}_w)$ is the collection of all adjointable $\mathscr{A}$-linear maps from $\mathscr{H}$ into $\mathscr{K}_w$. We define

$$l^2(\Omega, \{\mathscr{K}_w\}_{w\in\Omega}) = \left\{x = \{x_w\}_{w\in\Omega} : x_w \in \mathscr{K}_w, \left\|\int_\Omega |x_w|^2 d\mu(w)\right\| < \infty\right\}. \tag{5}$$

For any $x = \{x_w : w \in \Omega\}$ and $y = \{y_w : w \in \Omega\}$, if the $\mathscr{A}$-valued inner product is defined by $\langle x,y\rangle = \int_\Omega \langle x_w, y_w\rangle_{\mathscr{A}} d\mu(w)$, the norm is defined by $\|x\| = \|\langle x,x\rangle_{\mathscr{A}}\|^{1/2}$. The $l^2(\Omega, \{\mathscr{K}_w\}_{\omega\in\Omega})$ is a Hilbert C^*-module.

Let $\mathscr{A}$ be a C*-algebra; $l^2(\mathscr{A})$ is defined by

$$l^2(\mathscr{A}) = \left\{\{a_\omega\}_{w\in\Omega} \subseteq \mathscr{A} : \left\|\int_\Omega a_\omega a_\omega^* d\mu(\omega)\right\| < \infty\right\}. \tag{6}$$

$l^2(\mathscr{A})$ is a Hilbert C^*-module (Hilbert $\mathscr{A}$-module) with pointwise operations and the inner product defined by

$$\left\langle \{a_\omega\}_{w\in\Omega}, \{b_\omega\}_{w\in\Omega}\right\rangle = \int_\Omega a_\omega b_\omega^* d\mu(\omega), \{a_\omega\}_{w\in\Omega}, \{b_\omega\}_{w\in\Omega} \in l^2(\mathscr{A}),$$
$$\|\{a_\omega\}_{w\in\Omega}\| = \left(\int_\Omega a_\omega a_\omega^* d\mu(\omega)\right)^{1/2}. \tag{7}$$

Let $GL^+(\mathscr{H})$ be the set of all positive bounded linear invertible operators on $\mathscr{H}$ with bounded inverse.

Definition 6 [16]. Let $\Lambda = \{\Lambda_w\}_{w\in\Omega}$ be a family in $End^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K}_w)$ for all $\omega \in \Omega$, and $C, C' \in GL^+(\mathscr{H})$. We say that the family Λ *is a* (C, C') -controlled continuous g-frame for Hilbert C^* -module $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$ if it is a continuous g -Bessel family and there is a pair of constants $0 < A, B$ such that, for any $f \in \mathscr{H}$,

$$A\langle f, f\rangle_{\mathscr{A}} \leq \int_{\Omega} \left\langle \Lambda_w C f, \Lambda_w C' f \right\rangle_{\mathscr{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathscr{A}}. \quad (8)$$

A and B are called the (C, C')-controlled continuous g -frame bounds.

If $C' = I$, then we call $\Lambda = \{\Lambda_w\}_{w\in\Omega}$ a C-controlled continuous g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$.

Definition 7. Let $C, C' \in GL^+(\mathscr{H})$. The family $\{\Lambda_w \in End^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K}_w)\colon w \in \Omega\}$ is called a (C, C') -controlled continuous $*$ -g-frame for Hilbert C^* -module $\mathscr{H}$ with respect to $\{V_w : w \in \Omega\}$ if Λ is continuous $*$ -g-Bessel sequence and if there exist two strictly nonzero elements A and B in $\mathscr{A}$ such that

$$A\langle x, x\rangle_{\mathscr{A}} A^* \leq \int_{\Omega} \left\langle \Lambda_w C x, \Lambda_w C' x \right\rangle_{\mathscr{A}} d\mu(w) \leq B\langle x, x\rangle_{\mathscr{A}} B^*, \quad \forall x \in U. \quad (9)$$

A and B are called the (C, C')-controlled continuous $*$-g-frame bounds.

If $C' = I$, then we call Λ a C-controlled continuous $*$-g-frames for $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$.

Definition 8. Let $\mathscr{H}$ be a Hilbert $\mathscr{A}$ -module over a unital C^* -algebra, and $C, C' \in GL^+(\mathscr{H})$. A family of adjointable operators $\{\Lambda_\omega\}_{w\in\Omega} \subset End^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K}_w)$ is said to be a (C, C') -controlled continuous $*$ - K - g -frame for Hilbert C^* -module $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$ if there exist two strictly nonzero elements A and B in $\mathscr{A}$ such that

$$A\langle K^* f, K^* f\rangle_{\mathscr{A}} A^* \leq \int_{\Omega} \left\langle \Lambda_\omega C f, \Lambda_\omega C' f \right\rangle_{\mathscr{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathscr{A}} B^*, \quad f \in \mathscr{H}. \quad (10)$$

The elements A and B are called continuous (C, C') -controlled $*$-K-g-frame bounds.

If

$$A\langle K^* f, K^* f\rangle_{\mathscr{A}} A^* = \int_{\Omega} \left\langle \Lambda_\omega C f, \Lambda_\omega C' f \right\rangle_{\mathscr{A}} d\mu(w), \quad f \in \mathscr{H}, \quad (11)$$

we call this continuous (C, C')-controlled $*$-K-g-frame a continuous (C, C')-controlled tight K-g-frame, and if $A = 1_{\mathscr{A}}$ it is called a continuous (C, C')-controlled Parseval $*$-K -g-frame. If only the right-hand inequality of (10) is satisfied, we call a continuous (C, C')-controlled Bessel $*$-g-frame with Bessel bound B.

Example 1. Let $\mathscr{H} = \{M = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \end{pmatrix} / x, y \in \mathbb{C}\}$, and $\mathscr{A} \ll \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} / a, b \in \mathbb{C}\}$.

It is known that $\mathscr{A}$ is a $\mathbb{C}^*$-algebra and $\mathscr{H}$ is a Hilbert $\mathscr{A}$ -module.

Let C and C' be two operators, respectively, defined as follows:

$$\begin{aligned} C &: \mathscr{H} \to \mathscr{H}, \\ &M \to \alpha M, \\ C' &: \mathscr{H} \to \mathscr{H}, \\ &M \to \beta M, \end{aligned} \quad (12)$$

where α and β are two real numbers strictly positive.

It is clear that $C, C' \in Gl^+(\mathscr{H})$.

Indeed, for each $M \in \mathscr{H}$, one has

$$\begin{aligned} C^{-1}(M) &= \alpha^{-1} M, \\ \left(C'\right)^{-1}(M) &= \beta^{-1} M. \end{aligned} \quad (13)$$

Let $\Omega = [0, 1]$ endowed with the Lebesgue measure. It is clear that there is a measure space. Moreover, for $\omega \in \Omega$, we define the operator $\Lambda_w : \mathscr{H} \to \mathscr{H}$ by

$$\Lambda_w(M) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & wx & 0 \end{pmatrix}. \quad (14)$$

Λ_w is linear, bounded, and selfadjoint.

In addition, for $M \in \mathscr{H}$, we have

$$\begin{aligned} &\int_{\Omega} \left\langle \Lambda_w C M, \Lambda_w C' M \right\rangle_{\mathscr{A}} d\mu(\omega) \\ &= \int_{\Omega} \alpha\beta \begin{pmatrix} 0 & 0 & 0 \\ 0 & wy & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & w\bar{y} \\ 0 & 0 \end{pmatrix} d\mu(\omega) \\ &= \int_{\Omega} \alpha\beta w^2 \begin{pmatrix} 0 & 0 \\ 0 & |y|^2 \end{pmatrix} w^2 d\mu(\omega) = \frac{\alpha\beta}{3} \begin{pmatrix} 0 & 0 \\ 0 & |y|^2 \end{pmatrix} \\ &\leq \sqrt{\frac{\alpha\beta}{3}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix} \right\rangle_{\mathscr{A}} \sqrt{\frac{\alpha\beta}{3}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \quad (15)$$

which show that the family $(\Lambda_\omega)_{\omega\in\Omega}$ is a continuous (C, C') -controlled Bessel sequence for $\mathscr{H}$ with $\sqrt{\alpha\beta/3}$ as bounded.

Let $K : \mathscr{H} \to \mathscr{H}$ by

$$K(M) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix}. \quad (16)$$

$K = K^*$ because K is a projection. Furthermore, we have

$$\sqrt{\frac{\alpha\beta}{3}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\langle K^*M, K^*M\rangle_{\mathscr{A}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \le \int_\Omega \left\langle \Lambda_w CM, \Lambda_w C'M\right\rangle_{\mathscr{A}} d\mu(\omega) \le \sqrt{\frac{\alpha\beta}{3}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\langle M, M\rangle_{\mathscr{A}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{17}$$

for every $M \in \mathscr{H}$.

Then, $(\Lambda_\omega)_{w\in\Omega}$ is a continuous (C, C')-controlled K-g-frame for $\mathscr{H}$.

Remark 9. Every (C, C')-controlled continuous $*$-g-frame for $\mathscr{H}$ is a (C, C')-controlled continuous $*$-K-g-frame for $\mathscr{H}$.

Indeed, if $\{\Lambda_w \in End^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K}_w) \colon w \in \Omega\}$ is a (C, C')-controlled continuous $*$-g-frame for Hilbert C^*-module $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$, then there exist two strictly nonzero elements A and B in $\mathscr{A}$ such that

$$A\langle f, f\rangle_{\mathscr{A}}A^* \le \int_\Omega \left\langle \Lambda_\omega Cf, \Lambda_\omega C'f\right\rangle_{\mathscr{A}} d\mu(w) \le B\langle f, f\rangle_{\mathscr{A}}B^*, \quad f \in \mathscr{H}. \tag{18}$$

But

$$\langle K^*f, K^*f\rangle_{\mathscr{A}} \le \|K\|^2\langle f, f\rangle_{\mathscr{A}}, \quad f \in \mathscr{H}. \tag{19}$$

So,

$$A\|K\|^{-2}\langle K^*f, K^*f\rangle_{\mathscr{A}}A^* \le \int_\Omega \left\langle \Lambda_\omega Cf, \Lambda_\omega C'f\right\rangle_{\mathscr{A}} d\mu(w) \le B\langle f, f\rangle_{\mathscr{A}}B^*, \quad f \in \mathscr{H}. \tag{20}$$

Then,

$$A\|K\|^{-1}\langle K^*f, K^*f\rangle_{\mathscr{A}}\left(A\|K\|^{-1}\right)^* \le \int_\Omega \left\langle \Lambda_\omega Cf, \Lambda_\omega C'f\right\rangle_{\mathscr{A}} d\mu(w) \le B\langle f, f\rangle_{\mathscr{A}}B^*, \quad f \in \mathscr{H}. \tag{21}$$

Hence, $\{\Lambda_\omega\}_{w\in\Omega}$ is a controlled continuous $*$-K-g-frame for Hilbert C^*-module $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$ with bounds $A\|K\|^{-1}$ and B.

Remark 10. Let $K \in End^*_{\mathscr{A}}(\mathscr{H})$ be a surjective operator. Every (C, C')-controlled continuous $*$-K-g-frame for $\mathscr{H}$ is a (C, C')-controlled continuous $*$-g-frame for $\mathscr{H}$.

Proof. Let $K \in End^*_{\mathscr{A}}(\mathscr{H})$ be a surjective operator; then, there exists $m > 0$ such that

$$m\langle f, f\rangle_{\mathscr{A}} \le \langle K^*f, K^*f\rangle_{\mathscr{A}}. \tag{22}$$

Hence,

$$A\sqrt{m}\langle f, f\rangle_{\mathscr{A}}\left(A\sqrt{m}\right)^* \le A\langle K^*f, K^*f\rangle_{\mathscr{A}}A^*. \tag{23}$$

Since $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C, C')-controlled continuous $*$-K-g-frame, then there exist two strictly nonzero elements A and B in $\mathscr{A}$ such that

$$A\langle K^*f, K^*f\rangle_{\mathscr{A}}A^* \le \int_\Omega \left\langle \Lambda_\omega Cf, \Lambda_\omega C'f\right\rangle_{\mathscr{A}} d\mu(w) \le B\langle f, f\rangle_{\mathscr{A}}B^*, \quad f \in \mathscr{H}. \tag{24}$$

Therefore,

$$A\sqrt{m}\langle f, f\rangle_{\mathscr{A}}\left(A\sqrt{m}\right)^* \le \int_\Omega \left\langle \Lambda_\omega Cf, \Lambda_\omega C'f\right\rangle_{\mathscr{A}} d\mu(w) \le B\langle f, f\rangle_{\mathscr{A}}B^*, \quad f \in \mathscr{H}. \tag{25}$$

This gives that $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C, C')-controlled continuous $*$-g-frame with bounds $A\sqrt{m}$ and B, respectively.

Proposition 11. *Let $K \in End^*_{\mathscr{A}}(\mathscr{H})$ and $C, C' \in GL^+(\mathscr{H})$. Suppose that C and C' commute with each other and commute with the operators $\Lambda^*_\omega\Lambda_\omega$ for each $\omega \in \Omega$. A family $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C, C')-controlled continuous $*$-K-g-frame Bessel family for $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$ with bounds B if and only if the operator*

$$T_{(C,C')}\left(\{y_w\}_{w\in\Omega}\right) = \int_\Omega \left(CC'\right)^{1/2}\Lambda^*_\omega y_\omega d\mu(w), \{y_w\}_{w\in\Omega} \in l^2 \cdot \left(\Omega, \{\mathscr{K}_w\}_{w\in\Omega}\right) \tag{26}$$

is well defined and bounded with $\|T_{(C,C')}\| \le \|B\|$.

Proof. Assume that $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C, C')-controlled continuous Bessel $*$-K-g-frame family for $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$ with bounds B. As a result of (2.3),

$$\left\|\int_{\Omega}\left\langle \Lambda_{\omega}Cx, \Lambda_{\omega}C'x\right\rangle_{\mathscr{A}}\right\| \leq \|B\|^2\|\langle x, x\rangle_{\mathscr{A}}\|,$$

$$\begin{aligned}
\|T_{(C,C')}(\{y_\omega\}_{w\in\Omega})\|^2 &= \sup_{x\in\mathscr{H},\|x\|=1}\left\|\left\langle\int_{\Omega}\left(CC'\right)^{1/2}\Lambda_{\omega}^{*}y_{\omega}, x\right\rangle_{\mathscr{A}} d\mu(\omega)\right\|^2 \\
&= \sup_{x\in\mathscr{H},\|x\|=1}\left\|\int_{\Omega}\left\langle\left(CC'\right)^{1/2}\Lambda_{\omega}^{*}y_{\omega}, x\right\rangle_{\mathscr{A}} d\mu(\omega)\right\|^2 \\
&= \sup_{x\in\mathscr{H},\|x\|=1}\left\|\int_{\Omega}\left\langle y_{\omega}, \Lambda_{\omega}\left(CC'\right)^{1/2}x\right\rangle_{\mathscr{A}} d\mu(\omega)\right\|^2 \\
&\leq \sup_{x\in\mathscr{H},\|x\|=1}\left\|\int_{\Omega}\langle y_{\omega}, y_{\omega}\rangle_{\mathscr{A}}\right\|\left\|\int_{\Omega}\right. \\
&\quad \cdot \left.\left\langle\Lambda_{\omega}\left(CC'\right)^{1/2}x, \Lambda_{\omega}\left(CC'\right)^{1/2}x\right\rangle_{\mathscr{A}} d\mu(\omega)\right\| \\
&= \sup_{x\in\mathscr{H},\|x\|=1}\left\|\int_{\Omega}\langle y_{\omega}, y_{\omega}\rangle_{\mathscr{A}} d\mu(\omega)\right\|\left\|\int_{\Omega}\right. \\
&\quad \cdot \left.\left\langle\Lambda_{\omega}Cx, \Lambda_{\omega}C'x\right\rangle_{\mathscr{A}} d\mu(\omega)\right\| \\
&\leq \sup_{x\in\mathscr{H},\|x\|=1}\left\|\int_{\Omega}\langle y_{\omega}, y_{\omega}\rangle_{\mathscr{A}} d\mu(\omega)\right\|\|B\|^2\|x\|^2 \\
&= \|B\|^2\|\{y_\omega\}_{w\in\Omega}\|^2.
\end{aligned} \tag{27}$$

Then, the sum $\int_{\Omega}\langle(CC')^{1/2}\Lambda_{\omega}^{*}y_{\omega}$ is convergent and we have

$$\left\|T_{(C,C')}(\{y_\omega\}_{w\in\Omega})\right\|^2 \leq \|B\|^2\|\{y_w\}_{w\in\Omega}\|^2. \tag{28}$$

Hence,

$$\|T_{(C,C')}\|^2 \leq \|B\|^2. \tag{29}$$

Thus, the operator $T_{(C,C')}$ is well defined and bounded and

$$\|T_{(C,C')}\| \leq \|B\|. \tag{30}$$

For the converse, suppose that the operator $T_{(C,C')}$ is well defined and bounded and $\|T_{(C,C')}\|\leq\|B\|$. For all $x\in\mathscr{H}$, we have

$$\begin{aligned}
\int_{\Omega}\left\langle\Lambda_{\omega}Cx, \Lambda_{\omega}C'x\right\rangle_{\mathscr{A}} d\mu(\omega) &= \int_{\Omega}\left\langle C'\Lambda_{\omega}^{*}\Lambda_{\omega}Cx, x\right\rangle_{\mathscr{A}} d\mu(\omega) \\
&= \int_{\Omega}\left\langle\left(CC'\right)^{1/2}\Lambda_{\omega}^{*}\Lambda_{\omega}\left(CC'\right)^{1/2}x, x\right\rangle_{\mathscr{A}} d\mu(\omega) \\
&= \left\langle T_{(C,C')}(\{y_\omega\}_{w\in\Omega}), x\right\rangle_{\mathscr{A}} \\
&\leq \|T_{(C,C')}\|\|(\{y_\omega\}_{w\in\Omega})\|\|x\| \\
&\leq \|T_{(C,C')}\|\left(\int_{\Omega}\|\Lambda_{\omega}\left(CC'\right)^{1/2}x\|^2 d\mu(\omega)\right)^{1/2}\|x\| \\
&= \|T_{(C,C')}\|\left(\int_{\Omega}\left\langle\Lambda_{\omega}Cx, \Lambda_{\omega}C'x\right\rangle_{\mathscr{A}} d\mu(\omega)\right)^{1/2}\|x\|,
\end{aligned} \tag{31}$$

where $y_\omega = \Lambda_{\omega}(CC')^{1/2}x$.

Therefore,

$$\int_{\Omega}\left\langle\Lambda_{\omega}Cx, \Lambda_{\omega}C'x\right\rangle_{\mathscr{A}} d\mu(\omega) \leq \|T_{(C,C')}\|^2\|x\|^2. \tag{32}$$

Hence,

$$\int_{\Omega}\left\langle\Lambda_{\omega}Cx, \Lambda_{\omega}C'x\right\rangle_{\mathscr{A}} \leq \|B\|^2\|x\|^2, \quad \text{as}\|T_{(C,C')}\| \leq \|B\|. \tag{33}$$

This gives that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a (C, C')-controlled continuous $*$-K-g-frame Bessel family for $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w\in\Omega\}$.

Let $\Lambda = \{\Lambda_\omega\}_{w\in\Omega}$ be a (C, C')-controlled continuous Bessel $*$-K-g-frame for Hilbert C^*-module $\mathscr{H}$ over $\mathscr{A}$ with respect to $\{\mathscr{K}_w : w\in\Omega\}$ with bounds A and B.

We define the operator $T_{(C,C')}$ by

$$T_{(C,C')} : l^2(\Omega, \{\mathscr{K}_w\}_{w\in\Omega}) \rightarrow \mathscr{H}, \tag{34}$$

such that

$$\begin{aligned}
T_{(C,C')}(\{y_w\}_{w\in\Omega}) &= \int_{\Omega}\left(CC'\right)^{1/2}\Lambda_{\omega}^{*}y_{\omega}d\mu(w), \{y_w\}_{w\in\Omega} \in l^2 \\
&\quad \cdot (\Omega, \{\mathscr{K}_w\}_{w\in\Omega}).
\end{aligned} \tag{35}$$

The bounded linear operator $T_{(C,C')}$ is called the synthesis operator of Λ.

The operator

$$T^{*}_{(C,C')} : \mathscr{H} \rightarrow l^2(\Omega, \{\mathscr{K}_w\}_{w\in\Omega}), \tag{36}$$

given by

$$T^{*}_{(C,C')}(x) = \left\{\Lambda_{\omega}\left(C'C\right)^{1/2}x\right\}_{\omega\in\Omega}, \quad x\in\mathscr{H}, \tag{37}$$

is called the analysis operator for Λ.

Indeed, we have for all $x\in\mathscr{H}$ and $\{y_w\}_{w\in\Omega} \in l^2(\Omega, \{\mathscr{K}_w\}_{w\in\Omega})$

$$\begin{aligned}
\left\langle T_{(C,C')}(\{y_w\}_{w\in\Omega}), x\right\rangle_{\mathscr{A}} &= \left\langle\int_{\Omega}\left(CC'\right)^{1/2}\Lambda_{\omega}^{*}y_{\omega}d\mu(w), x\right\rangle_{\mathscr{A}} \\
&= \int_{\Omega}\left\langle\left(CC'\right)^{1/2}\Lambda_{\omega}^{*}y_{\omega}, x\right\rangle_{\mathscr{A}} d\mu(w) \\
&= \int_{\Omega}\left\langle y_{\omega}, \Lambda_{\omega}\left(CC'\right)^{1/2}x\right\rangle_{\mathscr{A}} d\mu(w) \\
&= \left\langle\{y_w\}_{w\in\Omega}, \left\{\Lambda_{\omega}\left(C'C\right)^{1/2}x\right\}_{\omega\in\Omega}\right\rangle_{l^2(\Omega,\{\mathscr{K}_w\}_{w\in\Omega})} \\
&= \left\langle\{y_w\}_{w\in\Omega}, T^{*}_{(C,C')}(x)\right\rangle_{l^2(\Omega,\{\mathscr{K}_w\}_{w\in\Omega})},
\end{aligned} \tag{38}$$

which shows that $T^*_{(C,C')}$ is the adjoint of $T_{(C,C')}$. If C and C' commute between them and commute with the operators $\Lambda^*_\omega \Lambda_\omega$ for each $\omega \in \Omega$, we define the (C, C')-controlled continuous $*$-k-g-frame operator by

$$S_{(C,C')} : \mathscr{H} \rightarrow \mathscr{H},$$
$$x \rightarrow S_{(C,C')} x = T_{(C,C')} T^*_{(C,C')} x = \int_\Omega C' \Lambda^*_w \Lambda_w C x d\mu(w). \tag{39}$$

As consequence, one has the following proposition.

Proposition 12. *The operator $S_{(C,C')}$ is positive, selfadjoint, and bounded and*

$$\|A^{-1}\|^{-2} \leq \|S\| \leq \|B\|^2. \tag{40}$$

If K is surjective, then $S_{(C,C')}$ is invertible.

Proof. By definition, we have $\forall x, y \in U$

$$\begin{aligned}\left\langle S_{(C,C')} x, y\right\rangle &= \left\langle \int_\Omega C' \Lambda^*_w \Lambda_w C x d\mu(w), y\right\rangle \\ &= \int_\Omega \left\langle C' \Lambda^*_w \Lambda_w C x, y\right\rangle d\mu(w) \\ &= \int_\Omega \left\langle x, C \Lambda^*_w \Lambda_w C' y\right\rangle d\mu(w) \\ &= \left\langle x, \int_\Omega C \Lambda^*_w \Lambda_w C' y d\mu(w)\right\rangle = \left\langle x, S_{(C',C)} y\right\rangle.\end{aligned} \tag{41}$$

Then, S is a selfadjoint.
Clearly, S is positive.
By definition of a (C, C')-controlled continuous $*$-K-g-frame, we have

$$A\langle K^* x, K^* x\rangle A^* \leq \int_\Omega \left\langle \Lambda_w C x, \Lambda_w C' x\right\rangle d\mu(w) \leq B\langle x, x\rangle B^*. \tag{42}$$

So

$$A\langle K^* x, K^* x\rangle A^* \leq \langle S_{CC'} x, x\rangle \leq B\langle x, x\rangle B^*. \tag{43}$$

This gives

$$\|A^{-1}\|^{-2} \|\langle K K^* x, x\rangle\| \leq \|\langle S_{CC'} x, x\rangle\| \leq \|B\|^2 \|\langle x, x\rangle\|. \tag{44}$$

If we take supremum on all $x \in \mathscr{H}$, where $\|x\| \leq 1$, we have

$$\|A^{-1}\|^{-2} \|K\|^2 \leq \|S\| \leq \|B\|^2. \tag{45}$$

If K is surjective, then, using Lemma 4, we have $S_{(C,C')}$ which is invertible.

Lemma 13. *Let $\{\Lambda_\omega\}_{w\in\Omega} \subset End^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K}_w)$ be a (C, C') -controlled continuous Bessel $*$ - K - g -frame family for Hilbert C^* -module $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$. Then, for any $K \in End^*_{\mathscr{A}}(\mathscr{H})$ such that $KC = CK$ and $KC' = C'K$, the family $\{\Lambda_\omega K\}_{w\in\Omega}$ is a (C, C') -controlled continuous Bessel $*$ - K - g -frame for Hilbert C^* -module $\mathscr{H}$.*

Proof. Assume that $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C, C')-controlled continuous Bessel $*$-K-g-frame family for Hilbert C^*-module $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$ with bound B. Then,

$$\int_\Omega \left\langle \Lambda_\omega C f, \Lambda_\omega C' f\right\rangle_{\mathscr{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathscr{A}} B^*, \quad f \in \mathscr{H}. \tag{46}$$

So,

$$\int_\Omega \left\langle \Lambda_\omega C K f, \Lambda_\omega C' K f\right\rangle_{\mathscr{A}} d\mu(w) \leq B\langle K f, K f\rangle_{\mathscr{A}} B^*, \quad f \in \mathscr{H}. \tag{47}$$

Hence,

$$\begin{aligned}\int_\Omega \left\langle \Lambda_\omega K C f, \Lambda_\omega K C' f\right\rangle_{\mathscr{A}} d\mu(w) &\leq B\langle K f, K f\rangle_{\mathscr{A}} B^* \\ &\leq B\|K\| \langle f, f\rangle_{\mathscr{A}} (B\|K\|)^*, \quad f \in \mathscr{H}.\end{aligned} \tag{48}$$

The results hold.

Lemma 14. *Let $K \in End^*_{\mathscr{A}}(\mathscr{H})$ and $C, C' \in GL^+(\mathscr{H})$. Let $\{\Lambda_\omega\}_{w\in\Omega}$ be a (C, C') -controlled continuous Bessel $*$ - K - g -frame for Hilbert C^* -module $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$. $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C, C') -controlled continuous $*$ - K - g -frame if and only if there exists $A \in \mathscr{A}$ such that*

$$AKK^* \leq S_{(C,C').} \tag{49}$$

Proof. The family $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C, C')-controlled continuous $*$-K-g-frame if and only if

$$\begin{aligned}A\langle K^* f, K^* f\rangle_{\mathscr{A}} A^* &\leq \int_\Omega \left\langle \Lambda_\omega C f, \Lambda_\omega C' f\right\rangle_{\mathscr{A}} d\mu(w) \\ &\leq B\langle f, f\rangle_{\mathscr{A}} B^*, \quad f \in \mathscr{H},\end{aligned} \tag{50}$$

if and only if

$$\langle AK^* f, AK^* f\rangle_{\mathscr{A}} \leq \left\langle S_{(C,C')} f, f\right\rangle_{\mathscr{A}} \leq \langle B f, B f\rangle_{\mathscr{A}}, \quad f \in \mathscr{H}, \tag{51}$$

if

$$A\langle K^* f, K^* f\rangle_{\mathscr{A}} \leq \langle S f, f\rangle_{\mathscr{A}}, \tag{52}$$

and the family $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous (C, C')-controlled Bessel K-g-frame sequence; then,

$$\langle Sf,f\rangle_{\mathscr{A}} \le B\langle f,f\rangle_{\mathscr{A}}, \quad f\in\mathscr{H}, \tag{53}$$

which completes the proof.

Theorem 15. *Let $K\in End^*_{\mathscr{A}}(\mathscr{H})$ and $C,C'\in GL^+(\mathscr{H})$. Suppose that K^* commute with C and C'. If $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C,C') -controlled continuous $*$ - g -frame for Hilbert C^* -module $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w\in\Omega\}$, then $\{\Lambda_\omega K^*\}_{w\in\Omega}$ is a (C,C') -controlled continuous $*$ - K - g -frame for Hilbert C^* -module $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w\in\Omega\}$ with frame operator $KS_{(C,C')}K^*$.*

Proof. Let $\{\Lambda_\omega\}_{w\in\Omega}$ be a (C,C')-controlled continuous $*$-g-frame for Hilbert C^*-module $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w\in\Omega\}$; then,

$$A\langle f,f\rangle_{\mathscr{A}}A^* \le \int_\Omega \left\langle \Lambda_\omega Cf, \Lambda_\omega C'f\right\rangle_{\mathscr{A}} d\mu(w) \le B\langle f,f\rangle_{\mathscr{A}}B^*, \quad f\in\mathscr{H}. \tag{54}$$

Hence,

$$\begin{aligned} A\langle K^*f,K^*f\rangle_{\mathscr{A}}A^* &\le \int_\Omega \left\langle \Lambda_\omega CK^*f, \Lambda_\omega C'K^*f\right\rangle_{\mathscr{A}} d\mu(w) \\ &\le B\langle K^*f,K^*f\rangle_{\mathscr{A}}B^*, \quad f\in\mathscr{H}. \end{aligned} \tag{55}$$

Therefore,

$$\begin{aligned} A\langle K^*f,K^*f\rangle_{\mathscr{A}}A^* &\le \int_\Omega \left\langle \Lambda_\omega K^*Cf, \Lambda_\omega K^*C'f\right\rangle_{\mathscr{A}} d\mu(w) \\ &\le B\|K^*\|^2\langle f,f\rangle_{\mathscr{A}}B^*, \quad f\in\mathscr{H}. \end{aligned} \tag{56}$$

Thus,

$$\begin{aligned} A\langle K^*f,K^*f\rangle_{\mathscr{A}}A^* &\le \int_\Omega \left\langle \Lambda_\omega K^*Cf, \Lambda_\omega K^*C'f\right\rangle_{\mathscr{A}} d\mu(w) \\ &\le B\|K^*\|\langle f,f\rangle_{\mathscr{A}}(B\|K^*\|)^*, \quad f\in\mathscr{H}. \end{aligned} \tag{57}$$

This concludes that $\{\Lambda_\omega K^*\}_{w\in\Omega}$ is a (C,C')-controlled continuous $*$-K-g-frame for Hilbert C^*-module $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w\in\Omega\}$.

$$\begin{aligned} KS_{(C,C')}K^*f &= K\int_\Omega C'\Lambda_w^*\Lambda_w CK^*f d\mu(w) \\ &= \int_\Omega KC'\Lambda_w^*\Lambda_w CK^*f d\mu(w) \\ &= \int_\Omega C'K\Lambda_w^*\Lambda_w K^*Cf d\mu(w) \\ &= \int_\Omega C'(\Lambda_w K^*)^*(\Lambda_w K^*)Cf d\mu(w). \end{aligned} \tag{58}$$

Proposition 16. *Let $K\in End^*_{\mathscr{A}}(\mathscr{H})$ and $C,C'\in GL^+(\mathscr{H})$. Suppose that C and C' commute with each other and commute with S. Then, $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C,C') -controlled continuous $*$ - K - g -frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w\in\Omega\}$ if and only if $\{\Lambda_\omega\}_{w\in\Omega}$ is a $(C'C,I_{\mathscr{H}})$ -controlled continuous $*$ - K - g -frame for Hilbert C^* -module $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w\in\Omega\}$.*

Proof. For all $f\in\mathscr{H}$, we have

$$\begin{aligned} \left\langle \left(C'\right)^{-1} S_{(C,C')C^{-1}f,f}\right\rangle \mathscr{A} &= \int_\Omega \left\langle C'\Lambda_\omega^*\Lambda_\omega CC^{-1}f, \left(C'\right)^{-1}f\right\rangle_{\mathscr{A}} d\mu(w) \\ &= \int_\Omega \langle \Lambda_\omega^*\Lambda_\omega f,f\rangle_{\mathscr{A}} d\mu(w) = \langle Sf,f\rangle_{\mathscr{A}}, \end{aligned} \tag{59}$$

where

$$Sf = \int_\Omega \Lambda_\omega^*\Lambda_\omega f d\mu(w). \tag{60}$$

Hence,

$$S = \left(C'\right)^{-1} S_{(C,C')}C^{-1}. \tag{61}$$

For any $f\in\mathscr{H}$, we have

$$\begin{aligned} \int_\Omega \left\langle \Lambda_\omega Cf, \Lambda_\omega C'f\right\rangle_{\mathscr{A}} d\mu(w) &= \int_\Omega \left\langle C'\Lambda_\omega^*\Lambda_\omega Cf, f\right\rangle_{\mathscr{A}} d\mu(w) \\ &= \left\langle S_{(C,C')}f,f\right\rangle_{\mathscr{A}} = \left\langle C'SCf,f\right\rangle_{\mathscr{A}} \\ &= \left\langle CSC'f,f\right\rangle_{\mathscr{A}} = \left\langle SC'Cf,f\right\rangle_{\mathscr{A}} \\ &= \int_\Omega \left\langle \Lambda_\omega^*\Lambda_\omega C'Cf, f\right\rangle_{\mathscr{A}} d\mu(w) \\ &= \int_\Omega \left\langle \Lambda_\omega C'Cf, \Lambda_\omega f\right\rangle_{\mathscr{A}} d\mu(w). \end{aligned} \tag{62}$$

Hence, $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(CC',I_{\mathscr{H}})$-controlled $*$-K-g-frame for $\mathscr{H}$ with bounds A and B with respect to $\{\mathscr{K}_w : w\in\Omega\}$ if and only if

$$\begin{aligned} A\langle K^*f,K^*f\rangle_{\mathscr{A}}A^* &\le \int_\Omega \left\langle \Lambda_\omega C'Cf, \Lambda_\omega f\right\rangle_{\mathscr{A}} d\mu(w) \\ &\le B\langle f,f\rangle_{\mathscr{A}}B^*, \quad f\in\mathscr{H}. \end{aligned} \tag{63}$$

The results hold.

Proposition 17. *Let $K\in End^*_{\mathscr{A}}(\mathscr{H})$ and $C,C'\in GL^+(\mathscr{H})$. Then, $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C,C') -controlled continuous $*$ - K - g -frame for Hilbert C^* -module $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w\in\Omega\}$ if and only if $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $((C'C)^{1/2},(C'C)^{1/2})$ -controlled $*$ - K - g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w\in\Omega\}$.*

Proof. The proof is similar as the proof of Lemma 2.12.

Proposition 18. *Let $K \in End^*_{\mathscr{A}}(\mathscr{H})$ and $C, C' \in GL^+(\mathscr{H})$. Suppose that $CK = KC$, $C'K = KC'$, and $CS = SC$. Then, $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C, C') -controlled continuous $*$ - K - g -frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$ if and only if $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $*$ - K - g -frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$.*

Proof. Assume that $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $*$-K-g-frame for $\mathscr{H}$ with bounds A and B with respect to $\{\mathscr{K}_w : w \in \Omega\}$. Then,

$$A\langle KK^*f, f\rangle_{\mathscr{A}}A^* \le \langle Sf, f\rangle_{\mathscr{A}} \le B\langle f, f\rangle_{\mathscr{A}}B^*, \quad f \in \mathscr{H}. \tag{64}$$

Since C, C' are bounded positive operators, there exist constants $m, m', M, M' (0 < m, m', M, M' < \infty)$ such that

$$\begin{aligned} mI_{\mathscr{H}} &\le C \le MI_{\mathscr{H}}, \\ m'I_{\mathscr{H}} &\le C' \le M'I_{\mathscr{H}}. \end{aligned} \tag{65}$$

From

$$\langle CSf, f\rangle_{\mathscr{A}} = \langle f, SCf\rangle_{\mathscr{A}} = \langle f, CSf\rangle_{\mathscr{A}}, \tag{66}$$

we have

$$mAKK^* \le SC = CS \le MBI_{\mathscr{H}}. \tag{67}$$

Then,

$$mm'AKK^* \le C'SC \le MM'BI_{\mathscr{H}}. \tag{68}$$

Therefore,

$$\begin{aligned} mm'A\langle K^*f, K^*f\rangle_{\mathscr{A}}A^* &\le \int_\Omega \left\langle \Lambda_\omega Cf, \Lambda_\omega C'f\right\rangle_{\mathscr{A}} d\mu(w) \\ &\le MM'B\langle f, f\rangle_{\mathscr{A}}B^*, \quad f \in \mathscr{H}. \end{aligned} \tag{69}$$

So

$$\begin{aligned} \sqrt{mm'}A\langle K^*f, K^*f\rangle_{\mathscr{A}}\left(\sqrt{mm'}A\right)^* &\le \int_\Omega \left\langle \Lambda_\omega Cf, \Lambda_\omega C'f\right\rangle_{\mathscr{A}} d\mu(w) \\ &\le \sqrt{MM'}B\langle f, f\rangle_{\mathscr{A}}\left(\sqrt{MM'}B\right)^*, \quad f \in \mathscr{H}. \end{aligned} \tag{70}$$

Hence, $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous (C, C')-controlled $*$-K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$.

Conversely, assume that $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C, C')-controlled continuous $*$-K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$ with bounds A and B. On the one hand, we have for any $f \in \mathscr{H}$,

$$\begin{aligned} A\langle K^*f, K^*f\rangle_{\mathscr{A}}A^* &= A\left\langle \left(CC'\right)^{1/2}\left(CC'\right)^{-1/2}K^*f, \left(CC'\right)^{1/2} \cdot \left(CC'\right)^{-1/2}K^*f\right\rangle_{\mathscr{A}} A^* \\ &= A\left\langle \left(CC'\right)^{1/2}K^*\left(CC'\right)^{-1/2}f, \left(CC'\right)^{1/2}K^* \cdot \left(CC'\right)^{-1/2}f\right\rangle_{\mathscr{A}} A^* \\ &\le \|\left(CC'\right)^{1/2}\|^2\int_\Omega \left\langle \Lambda_\omega C\left(CC'\right)^{-1/2}f, \Lambda_\omega C' \cdot \left(CC'\right)^{-1/2}f\right\rangle_{\mathscr{A}} d\mu(w) \\ &= \|\left(CC'\right)^{1/2}\|^2\left\langle SC\left(CC'\right)^{-1/2}f, C'\left(CC'\right)^{-1/2}f\right\rangle_{\mathscr{A}} \\ &= \|\left(CC'\right)^{1/2}\|^2\left\langle SC^{1/2}\left(C'\right)^{-1/2}f, \left(C'\right)^{1/2}(C)^{-1/2}f\right\rangle_{\mathscr{A}} \\ &= \|\left(CC'\right)^{1/2}\|^2\left\langle (C)^{-1/2}\left(C'\right)^{1/2}SC^{1/2}\left(C'\right)^{-1/2}f, f\right\rangle_{\mathscr{A}} \\ &= \|\left(CC'\right)^{1/2}\|^2\langle Sf, f\rangle_{\mathscr{A}}. \end{aligned} \tag{71}$$

So,

$$A\|\left(CC'\right)^{1/2}\|^{-2}\langle K^*f, K^*f\rangle_{\mathscr{A}}A^* \le \int_\Omega \langle \Lambda_\omega f, \Lambda_\omega f\rangle_{\mathscr{A}} d\mu(w) = \langle Sf, f\rangle_{\mathscr{A}}. \tag{72}$$

On the other hand, we have

$$\begin{aligned} \int_\Omega \langle \Lambda_\omega f, \Lambda_\omega f\rangle_{\mathscr{A}} d\mu(w) = \langle Sf, f\rangle_{\mathscr{A}} &= \left\langle \left(CC'\right)^{-1/2}\left(CC'\right)^{1/2}Sf, f\right\rangle_{\mathscr{A}} \\ &= \left\langle \left(CC'\right)^{1/2}Sf, \left(CC'\right)^{-1/2}f\right\rangle_{\mathscr{A}} \\ &= \left\langle \left(CC'\right)\left(CC'\right)^{-1/2}Sf, \left(CC'\right)^{-1/2}f\right\rangle_{\mathscr{A}} \\ &= \left\langle C'SC\left(CC'\right)^{-1/2}f, \left(CC'\right)^{-1/2}f\right\rangle_{\mathscr{A}} \\ &\le B\left\langle \left(CC'\right)^{-1/2}f, \left(CC'\right)^{-1/2}f\right\rangle_{\mathscr{A}} B^* \\ &\le B\|\left(CC'\right)^{-1/2}\|^2\langle f, f\rangle_{\mathscr{A}}B^* \\ &\le B\|\left(CC'\right)^{-1/2}\|\langle f, f\rangle_{\mathscr{A}}\left(B\|\left(CC'\right)^{-1/2}\|\right)^*. \end{aligned} \tag{73}$$

Therefore, $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $*$-K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_w : w \in \Omega\}$ with bounds $A\|(CC')^{1/2}\|^{-2}$ and $B\|(CC')^{-1/2}\|^2$.

3. Operators Preserving Controlled $*$-K-g-Frames

In this section, for the (C, C')-controlled $*$-K-g-frame $\{\Lambda_\omega\}_{w\in\Omega}$, we consider some proper relations between the operators $T, K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $C, C' \in GL^+(\mathcal{H})$ and investigate the cases that $\{\Lambda_\omega T\}_{w\in\Omega}$, $\{\Lambda_\omega T^*\}_{w\in\Omega}$ can also be (C, C')-controlled $*$-K-g-frame. Next, by putting connections between the operators $S(C, C')$, K, C, and C', we reach to necessary and sufficient conditions that $\{\Lambda_\omega\}_{w\in\Omega}$ can be a Parseval (C, C')-controlled $*$-K-g-frame.

Proposition 19. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $C, C' \in GL^+(\mathcal{H})$. Let $\{\Lambda_\omega\}_{w\in\Omega}$ be a (C, C') -controlled continuous $*$ - K - g -frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$. Suppose that $R(K)$ is closed. If $T \in End^*_{\mathcal{A}}(\mathcal{H})$ with $R(T) \subset R(K)$, then $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C, C') -controlled continuous $*$ - T - g -frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.*

Proof. Let $\{\Lambda_\omega\}_{w\in\Omega}$ be a (C, C')-controlled continuous $*$-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$. Then, there exist $A, B > 0$ such that

$$A\langle K^*f, K^*f\rangle_{\mathcal{A}} A^* \le \int_\Omega \left\langle \Lambda_\omega Cf, \Lambda_\omega C'f \right\rangle_{\mathcal{A}} d\mu(w) \le B\langle f, f\rangle_{\mathcal{A}} A^*. \tag{74}$$

From Lemma 1.5 and $R(T) \subset R(K)$, there exists some $m > 0$ such that

$$TT^* \le mKK^*. \tag{75}$$

Hence,

$$\begin{aligned} \frac{A}{m}\langle T^*f, T^*f\rangle_{\mathcal{A}} A^* &\le A\langle K^*f, K^*f\rangle_{\mathcal{A}} A^* \\ &\le \int_\Omega \left\langle \Lambda_\omega Cf, \Lambda_\omega C'f \right\rangle_{\mathcal{A}} d\mu(w) \\ &\le B\langle f, f\rangle_{\mathcal{A}} B^*. \end{aligned} \tag{76}$$

So, $\{\Lambda_\omega\}_{w\in\Omega}$ is a (C, C')-controlled continuous $*$-T-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.

Theorem 20. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ be a surjective operator and $\{\Lambda_w \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_w): w \in \Omega\}$ be a (C, C')-controlled continuous $*$-K-g-frame for $\mathcal{H}$, with lower and upper bounds A and B, respectively, and with the (C, C')-controlled continuous $*$-K-g-frame operator $S_{(C,C')}$. Let $T \in End^*_{\mathcal{A}}(\mathcal{H})$ be an invertible such that T commute with C and C', then $\{\Lambda_w T : w \in \Omega\}$ is a continuous $*$-K-g-frame for $\mathcal{H}$ with (C, C')-controlled continuous $*$-K-g-frame operator $T^* S_{(C,C')} T$.*

Proof. Let $\{\Lambda_w\}_{w\in\Omega}$ be a (C, C')-controlled continuous $*$-K-g-frame for $\mathcal{H}$, with lower and upper bounds A and B, then

$$\begin{aligned} A\langle K^*Tx, K^*Tx\rangle A^* &\le \int_\Omega \left\langle \Lambda_w CTx, \Lambda_w C'Tx \right\rangle d\mu(w) \\ &\le B\langle Tx, Tx\rangle B^*, \quad \forall x \in \mathcal{H}. \end{aligned} \tag{77}$$

Using Lemma 4, we have $\|(T^*T)^{-1}\|^{-1}\langle x, x\rangle \le \langle Tx, Tx\rangle$, $\forall x \in \mathcal{H}$.

K is surjective; then, there exist m such that

$$m\langle Tx, Tx\rangle \le \langle K^*Tx, K^*Tx\rangle, \tag{78}$$

then

$$m\|(T^*T)^{-1}\|^{-1}\langle x, x\rangle \le \langle K^*Tx, K^*Tx\rangle, \tag{79}$$

so

$$m\|(T^*T)^{-1}\|^{-1} A\langle x, x\rangle A^* \le A\langle K^*Tx, K^*Tx\rangle A^*, \tag{80}$$

or $\|T^{-1}\|^{-2} \le \|(T^*T)^{-1}\|^{-1}$; this implies

$$\begin{aligned} &\left(\|T^{-1}\|^{-1}\sqrt{m}A\right)\langle x, x\rangle \left(\|T^{-1}\|^{-1}\sqrt{m}A\right)^* \\ &\quad \le A\langle K^*Tx, K^*Tx\rangle A^*, \quad \forall x \in \mathcal{H}. \end{aligned} \tag{81}$$

And we know that $\langle Tx, Tx\rangle \le \|T\|^2\langle x, x\rangle$, $\forall x \in \mathcal{H}$. This implies that

$$B\langle Tx, Tx\rangle B^* \le (\|T\|B)\langle x, x\rangle(\|T\|B)^*, \quad \forall x \in \mathcal{H}. \tag{82}$$

Using (77), (81), and (82), we have

$$\begin{aligned} &\left(\|T^{-1}\|^{-1}\sqrt{m}A\right)\langle x, x\rangle \left(\|T^{-1}\|^{-1}\sqrt{m}A\right)^* \\ &\quad \le \int_\Omega \left\langle \Lambda_w CTx, \Lambda_w C'Tx \right\rangle d\mu(w) \le (\|T\|B)\langle x, x\rangle(\|T\|B)^*. \end{aligned} \tag{83}$$

So $\{\Lambda_w T : w \in \Omega\}$ is a (C, C')-controlled continuous $*$-K-g-frame for $\mathcal{H}$.

Moreover, for every $x \in \mathcal{H}$, we have

$$\begin{aligned} T^* S_{(C,C')} Tx &= T^* \int_\Omega C'\Lambda_w^*\Lambda_w CTx d\mu(w) \\ &= \int_\Omega T^*C'\Lambda_w^*\Lambda_w CTx d\mu(w) \\ &= \int_\Omega C'T^*\Lambda_w^*\Lambda_w TCx d\mu(w) \\ &= \int_\Omega C'(\Lambda_w T)^*(\Lambda_w T)Cx d\mu(w). \end{aligned} \tag{84}$$

This completes the proof.

Corollary 21. *Let $\{\Lambda_w \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_w): w \in \Omega\}$ be a (C, C')-controlled continuous $*$-K-g-frame for $\mathcal{H}$ and $K \in End^*_{\mathcal{A}}(\mathcal{H})$ be surjective, with continuous $*$-K-g-frame operator S. Then, $\{\Lambda_w S^{-1} : w \in \Omega\}$ is a (C, C')-controlled continuous $*$-K-g-frame for $\mathcal{H}$.*

Proof. Result from the last theorem by taking $T = S^{-1}_{(C,C')}$.

Theorem 22. *Let $K_1, K_2 \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $R(K_1) \perp R(K_2)$. If $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous (C, C') -controlled $*$ - K_1 - g -frame for $\mathcal{H}$ as well as $*$ - K_2 - g -frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$ and α and β are scalers. Then, $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous (C, C') -controlled $*$ - $(\alpha K_1 + \beta K_2)$ - g -frame and a continuous (C, C') -controlled $*$ - $(K_1 K_2)$ - g -frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.*

Proof. Let $\{\Lambda_\omega\}_{w\in\Omega} \subset End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_w)$ be a controlled continuous $*$-K_1-g-frame for $\mathcal{H}$ as well a $*$-K_2-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.

Then, there exist positive constants A_1, A_2, B_1, B_2 such that

$$\begin{aligned} A_1\langle K_1^* f, K_1^* f\rangle_{\mathcal{A}} A_1^* &\le \int_\Omega \left\langle \Lambda_\omega C f, \Lambda_\omega C' f \right\rangle_{\mathcal{A}} d\mu(w) \le B_1\langle f, f\rangle_{\mathcal{A}} B_1^*, \\ A_2\langle K_2^* f, K_2^* f\rangle_{\mathcal{A}} A_2^* &\le \int_\Omega \left\langle \Lambda_\omega C f, \Lambda_\omega C' f \right\rangle_{\mathcal{A}} d\mu(w) \le B_2\langle f, f\rangle_{\mathcal{A}} B_2^*. \end{aligned} \tag{85}$$

For any $f \in \mathcal{H}$, we have
$\langle (\alpha K_1 + \beta K_2)^* f, (\alpha K_1 + \beta K_2)^* f\rangle_{\mathcal{A}}$
$=\langle \bar{\alpha} K_1^* f + \bar{\beta} K_2^* f, \bar{\alpha} K_1^* f + \bar{\beta} K_2^* f\rangle_{\mathcal{A}} = |\alpha|^2 \langle K_1^* f, K_1^* f\rangle_{\mathcal{A}} + \bar{\alpha}\beta$
$\langle K_1^* f, K_2^* f\rangle + \alpha\bar{\beta}\langle K_2^* f, K_1^* f\rangle + |\beta|^2\langle K_2^* f, K_2^* f\rangle$.

Since $R(K_1) \perp R(K_2)$, then

$$\begin{aligned} &\langle (\alpha K_1 + \beta K_2)^* f, (\alpha K_1 + \beta K_2)^* f\rangle_{\mathcal{A}} \\ &\quad = |\alpha|^2\langle K_1^* f, K_1^* f\rangle + |\beta|^2\langle K_2^* f, K_2^* f\rangle_{\mathcal{A}}. \end{aligned} \tag{86}$$

Let $B = (|\alpha|^2 A_2 A_2^* + |\beta|^2 A_1 A_1^*)$.

Therefore, for each $f \in \mathcal{H}$, we have

$$\begin{aligned} &A_1 A_2 B^{-1}\langle (\alpha K_1 + \beta K_2)^* f, (\alpha K_1 + \beta K_2)^* f\rangle_{\mathcal{A}} (A_1 A_2)^* \\ &\quad = A_1 A_2 B^{-1/2}\langle (\alpha K_1 + \beta K_2)^* f, (\alpha K_1 + \beta K_2)^* f\rangle_{\mathcal{A}} \left(A_1 A_2 B^{-1/2}\right)^* \\ &\quad \le \int_\Omega \left\langle \Lambda_\omega C f, \Lambda_\omega C' f \right\rangle_{\mathcal{A}} d\mu(w) \le (B_1 + B_2)\langle f, f\rangle_{\mathcal{A}} (B_1 + B_2)^*. \end{aligned} \tag{87}$$

Therefore, $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous (C, C')-controlled $*$-$(\alpha K_1 + \beta K_2)$-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.

Also, for every $f \in \mathcal{H}$, we have

$$\begin{aligned} \langle (K_1 K_2)^* f, (K_1 K_2)^* f\rangle_{\mathcal{A}} &= \langle K_2^* K_1^* f, K_2^* K_1^* f\rangle_{\mathcal{A}} \\ &\le \|K_2^*\|^2\langle K_1^* f, K_1^* f\rangle_{\mathcal{A}}. \end{aligned} \tag{88}$$

Since $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous (C, C')-controlled $*$-K_1-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$, we have for every $f \in \mathcal{H}$

$$\begin{aligned} &A_1\|K_2^*\|^{-2}\langle (K_1 K_2)^* f, (K_1 K_2)^* f\rangle_{\mathcal{A}} A_1^* \\ &\quad \le \int_\Omega \left\langle \Lambda_\omega C f, \Lambda_\omega C' f \right\rangle_{\mathcal{A}} d\mu(w) \le B_1\langle f, f\rangle_{\mathcal{A}} B_1^*. \end{aligned} \tag{89}$$

Thus,

$$\begin{aligned} &A_1\|K_2^*\|^{-1}\langle (K_1 K_2)^* f, (K_1 K_2)^* f\rangle_{\mathcal{A}} \left(A_1\|K_2^*\|^{-1}\right)^* \\ &\quad \le \int_\Omega \left\langle \Lambda_\omega C f, \Lambda_\omega C' f \right\rangle_{\mathcal{A}} d\mu(w) \le B_1\langle f, f\rangle_{\mathcal{A}} B_1^*. \end{aligned} \tag{90}$$

So, $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous (C, C')-controlled $*$-$(K_1 K_2)$-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.

Corollary 23. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$. If $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous (C, C') -controlled $*$ - K - g -frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$; then, for any operator $\ominus$ in the subalgebra generated by K, the family $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous (C, C') -controlled $*$ - $\ominus$ - g -frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.*

Data Availability

No data were used to support this study.

Conflicts of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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