The Product-Type Operators from Hardy Spaces into \( n \)th Weighted-Type Spaces

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1. Introduction

Let \( \mathbb{D} \) denote the open unit disc of the complex plane \( \mathbb{C} \) and \( H(\mathbb{D}) \) denotes the space of all holomorphic functions on \( \mathbb{D} \). The space of bounded holomorphic functions on \( \mathbb{D} \) is denoted by \( H^\infty \); it is a Banach space with the equipped norm

\[
\|g\|_{H^\infty} = \sup_{z \in \mathbb{D}} |g(z)|.
\] (1)

Let 0 < \( p \) < \( \infty \). A Hardy space \( H^p \) consists of all \( g \in H(\mathbb{D}) \) such that

\[
\|g\|_{H^p} = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty.
\] (2)

When 1 \( \leq p < \infty \), \( H^p \) is a Banach space with the norm \( \|\cdot\|_{H^p} \). If 0 < \( p < 1 \), \( H^p \) is a nonlocally convex topological vector space and it is a complete metric space (see [1]).

Let \( n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\} \) and \( \mu(z) \) be a weight, continuous, and positive function on \( \mathbb{D} \). The \( n \)th weighted-type space \( \mathcal{W}^n_\mu \), consists of all \( g \in H(\mathbb{D}) \) such that

\[
b_{\mathcal{W}^n_\mu}(g) = \sup_{z \in \mathbb{D}} \mu(z) |g^{(n)}(z)| < \infty.
\] (3)

It is a Banach space with the following norm

\[
\|g\|_{\mathcal{W}^n_\mu} = \sum_{j=0}^{n-1} |g^{(j)}(0)| + b_{\mathcal{W}^n_\mu}(g). \tag{4}
\]

The little \( n \)th weighted-type space \( \mathcal{W}^{(n)}_{\mu,0} \) is a closed subspace of \( \mathcal{W}^n_\mu \) such that for any \( g \in \mathcal{W}^{(n)}_{\mu,0} \)

\[
\lim_{|z| \to 1} \mu(z) |g^{(n)}(z)| = 0. \tag{5}
\]

For more information about \( n \)th weighted-type spaces, see [2–4]. Let \( \alpha > 0 \). Then, \( \mathcal{W}^{(\alpha)}_{1-|z|^2} = H^{(-\alpha)} \) (growth space), \( \mathcal{W}^{(1)}_{1-|z|^2} = \mathcal{B}^\alpha \) (Bloch-type space), and \( \mathcal{W}^{(2)}_{1-|z|^2} = \mathcal{L}^\alpha \) (Zygmund-type space). Also \( \mathcal{W}^{(0)}_\mu = H^\mu \) (weighted-type space), \( \mathcal{W}^{(1)}_\mu = \mathcal{B}^\mu \) (weighted Bloch space), \( \mathcal{W}^{(2)}_\mu = \mathcal{L}^\mu \) (weighted Zygmund space), and \( \mathcal{W}^{(1)}_{1-|z|^2 \log (2/(1-|z|^2))} \) coincide with the logarithmic Bloch space \( \mathcal{B}^\log \).

Let \( n, k \in \mathbb{N}_0 \) such that \( k \leq n \); the partial Bell polynomials are triangular

\[
B_{n,k}(x_1, \cdots, x_{n-k+1}) = \sum_{j_1, \cdots, j_{n-k+1}} \frac{n!}{j_1! \cdots j_{n-k+1}!} \frac{x_1^{j_1}}{1!} \cdots \frac{x_{n-k+1}^{j_{n-k+1}}}{(n-k+1)!}, \tag{6}
\]

where the sum is taken over all nonnegative integers \( j_1, \cdots, j_{n-k+1} \) such that \( j_1 + \cdots + j_{n-k+1} = n \).
where the sum is taken over all nonnegative integers \( j_1, \ldots, j_{n-k+1} \) such that

\[
j_1 + \cdots + (n-k+1)j_{n-k+1} = n, \quad j_1 + \cdots + j_{n-k+1} = k.
\]  

(7)

More information about Bell polynomials can be found in ([15], p 134).

Let \( m \in \mathbb{N}_0 \), \( u, v \in \mathcal{H}(D) \) and \( \varphi \in \mathcal{S}(D) \) be the set of all holomorphic self-map of \( D \). In [6], Stević, Sharma and Krishan defined a new product-type operator \( T^m_{u,v,\varphi} \) as follows:

\[
T^m_{u,v,\varphi}g(z) = u(z)g^{(m)}(\varphi(z)) + v(z)g^{(m+1)}(\varphi(z)), \quad g \in \mathcal{H}(D), z \in D.
\]  

(8)

When \( m = 0 \), we obtain the Stević - Sharma-type operator, and for \( v \equiv 0 \), we get the generalized weighted composition operators \( D^m_{u,\varphi} \). Product-type operators on some spaces of analytic functions on the unit disc have become a subject of increasing interest in the recent years. We refer the reader to [6–10] and the references therein.

Liu and Yu have considered boundedness and compactness of operator \( T^0_{u,v,\varphi} \) from Hardy spaces and \( H^{\infty} \) into the logarithmic Bloch space in [11, 12]. Also, Zhang and Liu have found some characterizations for boundedness and compactness of operator \( T^0_{u,v,\varphi} \) from Hardy spaces into the weighted Zygmund space in [10]. Recently, the boundedness, compactness, and norm of operator \( T^0_{u,v,\varphi} : H^p \rightarrow \mathbb{H}^n_{\mu} \) are considered in [13].

Motivated by previous works, the results found in them will be generalized for operator \( T^m_{u,v,\varphi} \). For this purpose in the second section of this paper, we give some characterizations for boundedness of operator \( T^m_{u,v,\varphi} : H^p \rightarrow \mathcal{W}^{m}_{\mu} \) where \( m, n \in \mathbb{N} \) and \( 0 < p \leq \infty \). In the third section, some new estimates are obtained for the essential norm of such operators. As a corollary, some equivalent conditions are acquired for compactness of such operators.

Throughout this paper, if there exists a constant \( c \) such that \( a \geq cb \), we use the notation \( a \succeq b \). The symbol \( a \asymp b \) means that \( a \succeq b \succeq a \).

2. Boundedness

In this section, some equivalent conditions are found for the boundedness of operator \( T^m_{u,v,\varphi}(m \in \mathbb{N}) \) from \( H^p(0 < p \leq \infty) \) into \( \mathcal{W}^{m}_{\mu} \) weighted-type spaces. Firstly, we state some lemmas.

**Lemma 2** (see [15], Lemma 2.1). Let \( a > 0 \) and \( \mathbb{B}^a = \mathcal{W}^{a(1)}_{(1-|\cdot|^2)^a} \). The sequence \( \{ j^{m-1}z \}^\infty_{j=1} \) is bounded in \( \mathbb{B}^a \) and

\[
\lim_{j \to \infty} j^{m-1}\|z\|^j_{\mathcal{B}^a} = \left( \frac{2a}{e} \right)^n.
\]  

(10)

From Lemma 1, Proposition 5.1.2 [16] and [1], the next lemma is obtained.

**Lemma 3.** Let \( 0 < p \leq \infty \), \( n \in \mathbb{N}_0 \) and \( g \in H^p \). Then,

\[
|g(n)(z)| \leq \frac{\|g\|_{H^p}}{(1-|z|^2)^{(1/p)n}}, \quad z \in D.
\]  

(11)

**Lemma 4** (see [4]). Let \( \varphi \in \mathcal{S}(D) \) and \( u, g \in \mathcal{H}(D) \). Then, for any \( m, n \in \mathbb{N}_0 \)

\[
\left( u(z)g^{(m)}(\varphi(z)) \right)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} u^{(n-i)}(z)B_{ij}(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l+i)}(z)).
\]  

(12)

In this paper, we set

\[
f_{ia}(z) = \frac{(1-|a|^2)^i}{(1-az)^{(1/p)i}}, \quad 0 \neq a, z \in D, i \in \mathbb{N},
\]

\[
P_{i,j}^{u,\varphi}(z) = \sum_{l=1}^{n} \binom{n}{l} u^{(n-l)}(z)B_{ij}(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l+i)}(z)),
\]

\[
P_{i,j}^{u,\varphi}(z) = T_{-i,j}^{n}(z) = 0, \quad p_j(z) = z^j.
\]  

(13)

By using the functions \( f_{ia} \), we obtain the following lemma. Since the proof of it resembles the proof of Lemma 1 [2], therefore, it is omitted.

**Lemma 5.** Let \( \delta_{ik} \) be Kronecker delta. For any \( 0 \neq a \in \mathbb{D}, m \in \mathbb{N}_0, n \in \mathbb{N}_0, \) and \( i \in \{0, \ldots, n+1\}, \) there exists a function \( g_{ia} \in H^p \) such that

\[
g_{ia}^{(m+k)}(a) = \frac{\delta_{ia} a^{m+k}}{(1-|a|^2)^{(m+k+1/p)}}.
\]  

(14)

In this case, \( g_{ia}(z) = \sum_{j=1}^{n+1} c_j f_{ja}(z) \), where \( c_j \) are independent of choice of \( a \).

**Theorem 6.** Let \( m, n \in \mathbb{N}, 0 < p \leq \infty, u, v \in \mathcal{H}(D), \mu \) be a weight and \( \varphi \in \mathcal{S}(D) \). Then, the following statements are equivalent.

(a) The operator \( T^m_{u,v,\varphi} : H^p \rightarrow \mathcal{W}^{m}_{\mu} \) is bounded
(b) The operator $T_{u,v}^m : B^{1+1/(2p)} \rightarrow \mathcal{W}^{(n)}_\mu$ is bounded.

(c) The operator $T_{u,v}^m : B^0_{1+1/(2p)} \rightarrow \mathcal{W}^{(n)}_\mu$ is bounded.

(d) $\sup_{j \in \mathbb{Z}} j^{1/p} \| T_{u,v}^m p_j \|_{\mathcal{W}^{(n)}_\mu} < \infty$.

(e) For each $i \in \{0, 1, \ldots, n+1\}$, $\sup_{x \in D} \| T_{u,v}^m f_{i+1,a} \|_{\mathcal{W}^{(n)}_\mu} < \infty$ and $\sup_{i \in \mathbb{Z}} \mu(z) | \{ f_{i+1,a} + f_{i-1,a} \}(z) | < \infty$.

(f) For each $i \in \{0, \ldots, n+1\}$,

$$\sup_{z \in \mathbb{C}} \mu(z) \left( | f_{i+1,a} + f_{i-1,a} \right)(z) \right) \left| \frac{1}{m+1} \right| < \infty \quad (15)$$

Proof. (b) $\implies$ (c) Since $B^0_{1+1/(2p)} \subset B^{1+1/(2p)}$, we get (c).

(c) $\implies$ (d) It follows from Lemma 2.

(d) $\implies$ (e) For each $i \in \{0, \ldots, n+1\}$ and $a \in \mathbb{D}$,

$$f_{i+1,a}(z) = (1-|a|^2)^{i+1} \sum_{j=0}^{\infty} \frac{f^j(a) | a |^{j^2}}{j! \Gamma(i+1+(1/p))} | a |^j. \quad (16)$$

So,

$$\| T_{u,v}^m f_{i+1,a} \|_{\mathcal{W}^{(n)}_\mu} \leq (1-|a|^2)^i \sum_{j=0}^{\infty} \frac{f^j(a) | a |^{j^2}}{j! \Gamma(i+1+(1/p))} | a |^j \leq \sup_{j \in \mathbb{I}} j^{1/p} \| T_{u,v}^m p_j \|_{\mathcal{W}^{(n)}_\mu} \leq \sup_{j \in \mathbb{I}} j^{1/p} \left| T_{u,v}^m p_j \right|_{\mathcal{W}^{(n)}_\mu} \leq 2^{i+1} \sup_{j \in \mathbb{I}} j^{1/p} \left| T_{u,v}^m p_j \right|_{\mathcal{W}^{(n)}_\mu}. \quad (17)$$

Hence, $\sup_{x \in D} \| T_{u,v}^m f_{i+1,a} \|_{\mathcal{W}^{(n)}_\mu} < \infty$. It is remained to show that for each $i \in \{0, 1, \ldots, n+1\}$, $\sup_{x \in D} \mu(z) | \{ f_{i+1,a} + f_{i-1,a} \}(z) | < \infty$. Applying the operator $T_{u,v}^m$ to $p_m(z)$, by using Lemma 4, we have

$$\sup_{z \in \mathbb{D}} \mu(z) \left( | f_{i+1,a} + f_{i-1,a} \right)(z) \right) \left| \frac{1}{m+1} \right| < \infty. \quad (18)$$

Now, assume that we have the following inequalities for

$$0 \leq i \leq j-1,$$

$$\sup_{x \in D} \mu(z) \left( | f_{i+1,a} + f_{i-1,a} \right)(z) \right) < \infty, \quad (19)$$

where $j \leq n+1$. By applying the operator $T_{u,v}^m$ for $p_{j+m}(z)$ and using Lemma 4, we get

$$\sup_{z \in \mathbb{D}} \mu(z) \left( | f_{i+1,a} + f_{i-1,a} \right)(z) \right) \left| \frac{1}{m+1} \right| < \infty.$$
(b) \implies (a) From Lemma 3, \( H^p \subset B^{1+(1/p)} \), so we obtain (a).

(a) \implies (c) It is clear that \( f_{r,a} \in H^p \) and sup\(_{\omega \in \Omega} \| f_{r,a} \|_{H^p} \) is finite. Hence, for each \( i \in \{0, \ldots, n + 1\} \),

\[
\sup_{\omega \in \Omega} \left\| \left. T^m_{u,\nu,\rho} f_{i+1,a} \right|_{\mathcal{H}_\mu^n} \right\|_{\mathcal{Y}_\mu^n} \leq \left\| T^m_{u,\nu,\rho} \right\|_{H^p \to \mathcal{H}_\mu^n} \sup_{\omega \in \Omega} \| f_{i+1,a} \|_{H^p} < \infty.
\]

(27)

The proof of the second part of (e) is similar to the proof (d) \implies (e), so it is omitted. The proof is complete.

3. Essential Norm

In this section, we find some approximations for the essential norm of operator \( T^m_{u,\nu,\rho} \) from Hardy spaces into \( n \)th weighted type-spaces. As a corollary, we give some equivalent conditions for compactness of such operators.

Let \( X \) and \( Y \) be Banach spaces and \( T : X \to Y \) be the continuous linear operator. The essential norm of \( T \) is the distance from \( T \) to the compact operators, that is,

\[
\| T \|_{e, X \to Y} = \inf \{ \| T - K \| : K : X \to Y \text{ is compact} \}.
\]  

(28)

It is clear that \( T \) is compact if and only if \( \| T \|_{e, X \to Y} = 0 \).

Theorem 7. Let \( m, n \in \mathbb{N}, 0 < p \leq \infty, \nu, \nu \in H(D), \nu \in S(D), \) and \( \mu \) be a weight such that \( T^m_{u,\nu,\rho} : B^{1+(1/p)} \to \mathcal{H}_\mu^n \) is bounded. Then

\[
\| T^m_{u,\nu,\rho} \|_{e, B^{1+(1/p)} \to \mathcal{H}_\mu^n} = \max \{ A_i \}_{i=0}^{n+1} = \max \{ B_i \}_{i=0}^{n+1},
\]  

(29)

where

\[
A_i = \limsup_{|a| \to 1} \| T^m_{u,\nu,\rho} f_{i+1,a} \|_{\mathcal{H}_\mu^n}, \\
B_i = \limsup_{|\phi(z)| \to 1} \frac{\mu(z)}{|1 - |\phi(z)||^2} \left[ \left( I_{B^p_{\mu}} + I_{B^p_{\mu}} \right)(z) \right]
\]  

(30)

Proof. For each \( i \in \{0, \ldots, n + 1\} \), sup\(_{|a| \to 1} \| f_{i+1,a} \|_{\mathcal{H}_\mu^n} \) is finite and \( f_{i+1,a} \to 0 \) uniformly on compact subsets of \( D \) as \( |a| \to 1 \). Applying Lemma 2.10 from [17], for any compact operator \( K \) from \( B^{1+(1/p)} \) into \( \mathcal{H}_\mu^n \), we have

\[
\lim_{|a| \to 1} \| K f_{i+1,a} \|_{\mathcal{H}_\mu^n} = 0.
\]  

(31)

Hence, for any \( i \in \{0, \ldots, n + 1\} \),

\[
A_i = \limsup_{|a| \to 1} \| T^m_{u,\nu,\rho} f_{i+1,a} \|_{\mathcal{H}_\mu^n} - \lim_{|a| \to 1} \| K f_{i+1,a} \|_{\mathcal{H}_\mu^n} \\
\leq \limsup_{|a| \to 1} \| T^m_{u,\nu,\rho} - K \|_{\mathcal{H}_\mu^n} \\
\leq \| T^m_{u,\nu,\rho} - K \|_{B^{1+(1/p)} \to \mathcal{H}_\mu^n}.
\]  

(32)

So,

\[
\max \{ A_i \}_{i=0}^{n+1} \leq \| T^m_{u,\nu,\rho} - K \|_{B^{1+(1/p)} \to \mathcal{H}_\mu^n} = \| T^m_{u,\nu,\rho} \|_{B^{1+(1/p)} \to \mathcal{H}_\mu^n}.
\]  

(33)

Now, we prove that

\[
\max \{ B_i \}_{i=0}^{n+1} \leq \| T^m_{u,\nu,\rho} \|_{\mathcal{H}_\mu^n \to \mathcal{H}_\mu^n}.
\]  

(34)

Let \( \{ z_j \}_{j \in \mathbb{N}} \) be a sequence in \( D \) such that \( \lim_{j \to \infty} |\phi(z_j)| \to 1 \). Since \( T^m_{u,\nu,\rho} : B^{1+(1/p)} \to \mathcal{H}_\mu^n \) is bounded, by using Lemmas 4 and 5 for any compact operator \( K : B^{1+(1/p)} \to \mathcal{H}_\mu^n \) and \( i \in \{0, \ldots, n + 1 \} \), we obtain

\[
\left\| T^m_{u,\nu,\rho} - K \right\|_{\mathcal{H}_\mu^n \to \mathcal{H}_\mu^n} \geq \limsup_{j \to \infty} \left\| T^m_{u,\nu,\rho} \left( g_{\phi(z_j)} \right) \right\|_{\mathcal{H}_\mu^n} \\
- \lim_{j \to \infty} \left\| K \left( g_{\phi(z_j)} \right) \right\|_{\mathcal{H}_\mu^n} \\
\geq \limsup_{j \to \infty} \mu(z_j) \left| \phi(z_j) \right|^{m+1} \left( \left( I_{B^p_{\mu}} + I_{B^p_{\mu}} \right)(z_j) \right). \]

(35)

So, from the definition of the essential norm, we get (34). For \( r \in \{0,1\} \), we define \( K_r(f) = f_r(f) = f(rz) \). It is apparent that \( K_r \) is a compact operator on \( B^{1+(1/p)} \). Let \( \{ r_j \} \subset (0,1) \) be a sequence such that \( r_j \to 1 \) as \( j \to \infty \). Since \( f_r \to f \) uniformly on compact subsets of \( D \) as \( r \to 1 \), then, for any positive integer \( j \), the operator \( T^m_{u,\nu,\rho} K_{r_j} : B^{1+(1/p)} \to \mathcal{H}_\mu^n \) is compact. Based on the definition of the essential norm, we obtain

\[
\left\| T^m_{u,\nu,\rho} \right\|_{\mathcal{H}_\mu^n \to \mathcal{H}_\mu^n} \leq \limsup_{j \to \infty} \left\| T^m_{u,\nu,\rho} - T^m_{u,\nu,\rho} K_{r_j} \right\|_{\mathcal{H}_\mu^n \to \mathcal{H}_\mu^n}.
\]  

(36)

So, it is sufficient to show that

\[
\limsup_{j \to \infty} \left\| T^m_{u,\nu,\rho} - T^m_{u,\nu,\rho} K_{r_j} \right\| = \max \{ A_i \}_{i=0}^{n+1}, \max \{ B_i \}_{i=0}^{n+1}.
\]  

(37)

Let \( f \in B^{1+(1/p)} \) such that \( \| f \|_{B^{1+(1/p)}} \leq 1 \) and for all \( j \geq N \), \( r_j \geq (3/4) \), therefore,
\[
\left\| \left( T^m_{u,v} - T^m_{u,v} K_r \right) f \right\|_{\gamma^p_\mu} \leq \sum_{i=0}^{n+1} \left( T^m_{u,v} \left( f - f_r \right) \right) ^{(i)}_{s_i} \\
+ \sup \mu(z) \sum_{k=0}^{n+1} \left( f - f_r \right) ^{(k+m)} (\varphi(z)) \left( I_{k,v}^u + I_{k-1,v}^w \right) (z) \\
+ \sup \mu(z) \sum_{k=0}^{n+1} \left( f - f_r \right) ^{(k+m)} (\varphi(z)) \left( I_{k,v}^u + I_{k-1,v}^w \right) (z) \\
\leq \sup \mu(z) \left( 1 - \left\| \varphi(z) \right\|^2 \right) ^{m+k+1/(p)} f^{(m+k)} (\varphi(z)) \\
\times \frac{\left| \varphi(z) \right|^{m+k} \left( I_{k,v}^u + I_{k-1,v}^w \right) (z)}{1 - \left| \varphi(z) \right|^2} \quad \left( m+k+1/(p) \right) \\
\leq \| f \|_{\gamma^{1+1/(p)}_\mu} \sup \left( \left| \varphi(z) \right|^{m+k} \left( I_{k,v}^u + I_{k-1,v}^w \right) \right) \| \varphi(z) \|_{\gamma^p_\mu} \\\n\leq \sum_{i=0}^{n+1} c^i \sup \left( \left| f \right|_{\gamma^{1+1/(p)}_\mu} T^m_{u,v} f + f_{r+1} \right) \| \varphi(z) \|_{\gamma^p_\mu}. 
\]

Taking the limit when \( N \to \infty \), we get

\[
\limsup_{m \to \infty} M_k \leq \sum_{i=0}^{n+1} M_i \sup \left| f \right|_{\gamma^{1+1/(p)}_\mu} T^m_{u,v} f + f_{r+1} \| \varphi(z) \|_{\gamma^p_\mu} \leq \max \left\{ A_i \right\}_{i=0}^{n+1},
\]

\[
\limsup_{m \to \infty} M_k \leq B_k.
\]

Likewise, we have

\[
\limsup_{m \to \infty} N_k \leq \sum_{i=0}^{n+1} N_i \sup \left| f \right|_{\gamma^{1+1/(p)}_\mu} T^m_{u,v} f + f_{r+1} \| \varphi(z) \|_{\gamma^p_\mu} \leq \max \left\{ A_i \right\}_{i=0}^{n+1},
\]

\[
\limsup_{m \to \infty} N_k \leq B_k.
\]

Thus, by using (38), (39), (40), (42) and (43), we obtain

\[
\limsup_{m \to \infty} \left\| T^m_{u,v} - T^m_{u,v} K_r \right\|_{\gamma^{1+1/(p)}_\mu} \leq \min \left\{ \max \left\{ A_i \right\}_{i=0}^{n+1}, \max \left\{ B_i \right\}_{i=0}^{n+1} \right\}.
\]

Hence, from (36),

\[
\left\| T^m_{u,v} \right\|_{\gamma^{1+1/(p)}_\mu} \leq \min \left\{ \max \left\{ A_i \right\}_{i=0}^{n+1}, \max \left\{ B_i \right\}_{i=0}^{n+1} \right\}.
\]

Consequently,

\[
\left\| T^m_{u,v} \right\|_{\gamma^{1+1/(p)}_\mu} \leq \max \left\{ A_i \right\}_{i=0}^{n+1} = \max \left\{ B_i \right\}_{i=0}^{n+1}.
\]

The proof is complete.

\[
\textbf{Theorem 8.} \text{ Let } m, n \in \mathbb{N}, 0 < p \leq \infty, u, v \in H(D), \varphi \in S(D), \mu \text{ be a weight. If } T^m_{u,v} : H^p \to H^{(n)}_\mu \text{ be bounded then}
\]

\[
\left\| T^m_{u,v} \right\|_{L^p(H^p) \to H^{(n)}_\mu} \leq \left\| T^m_{u,v} \right\|_{L^p(H^p) \to H^{(n)}_\mu}.
\]

\[
\textbf{Proof.} \text{ It is evident that}
\]

\[
\left\| T^m_{u,v} \right\|_{L^p(H^p) \to H^{(n)}_\mu} \leq \left\| T^m_{u,v} \right\|_{L^p(H^p) \to H^{(n)}_\mu}.
\]

On the other hand, since \( f_{r+1}(z) = (1 - |a|^2)^i / (1 - \bar{a}z)^{i+1/(p)} \) \( \in H^p \), for any compact operator \( K : H^p \to H^{(n)}_\mu \), from Lemma 2.10 in [17], for any \( i \in \{0, \cdots, n+1\} \), we get
\[ \left\| T_{u,v}^{m} - K \right\|_{H^{p} \rightarrow \mathcal{H}_{w}^{(n)}} \geq \limsup_{[a] \to 1} \left\| \left( T_{u,v}^{m} - K \right) f_{n+1,a} \right\|_{\mathcal{H}_{w}^{(n)}} \]
\[ \geq \limsup_{[a] \to 1} \left\| T_{u,v}^{m} f_{n+1,a} \right\|_{\mathcal{H}_{w}^{(n)}} - \lim_{[a] \to 1} \left\| Kg_{n+1,a} \right\|_{\mathcal{H}_{w}^{(n)}} = A_{i}. \]

So, from the last inequality and Theorem 7
\[ \left\| T_{u,v}^{m} \right\|_{e,H^{p} \rightarrow \mathcal{H}_{w}^{(n)}} \geq \max \{ A_{i} \}_{i=0}^{n+1} = \left\| T_{u,v}^{m} \right\|_{e,B^{1+((p)/2)} \rightarrow \mathcal{H}_{w}^{(n)}}. \]

The proof is complete. \[ \square \]

**Theorem 9.** Let \( m, n \in \mathbb{N}, 0 < p \leq \infty, u, v \in H(\mathbb{D}), \mu \) be a weight and \( \phi \in \mathcal{S}(\mathbb{D}) \) such that \( T_{u,v}^{m} \) : \( B^{1+((p)/2)}(\mathcal{B}_{0}^{1+((p)/2)}) \rightarrow \mathcal{H}_{w}^{(n)} \) be bounded.

Then,
\[ \limsup_{j \to \infty} \left\| T_{u,v}^{m} f_{j} \right\|_{\mathcal{H}_{w}^{(n)}} = \left\| T_{u,v}^{m} \right\|_{e,B^{1+((p)/2)} \rightarrow \mathcal{H}_{w}^{(n)}} = \left\| T_{u,v}^{m} \right\|_{e,B^{1+((p)/2)} \rightarrow \mathcal{H}_{w}^{(n)}}. \]

**Proof.** Let \( j \) be any positive integer and \( h_{j}(z) = j^{1/p} p_{j}(z) \). It is clear that \( \left\| h_{j} \right\|_{\mathcal{B}^{1+((p)/2)}} \approx 1, h_{j} \to \mathcal{B}^{1+((p)/2)} \), and \( h_{j} \) converge to 0 uniformly on compact subsets of \( \mathbb{D} \). By using Lemma 2.10 in [17], for any compact operator \( K \) from \( \mathcal{B}^{1+((p)/2)} \) into \( \mathcal{H}_{w}^{n} \), we get
\[ \lim_{j \to \infty} \left\| Kh_{j} \right\|_{\mathcal{H}_{w}^{(n)}} = 0. \]

Hence,
\[ \left\| T_{u,v}^{m} - K \right\|_{e,B^{1+((p)/2)} \rightarrow \mathcal{H}_{w}^{(n)}} \geq \limsup_{j \to \infty} \left\| \left( T_{u,v}^{m} - K \right) h_{j} \right\|_{\mathcal{H}_{w}^{(n)}} \]
\[ \geq \limsup_{j \to \infty} \left\| T_{u,v}^{m} h_{j} \right\|_{\mathcal{H}_{w}^{(n)}} - \limsup_{j \to \infty} \left\| Kh_{j} \right\|_{\mathcal{H}_{w}^{(n)}} = \limsup_{j \to \infty} j^{1/p} \left\| T_{u,v}^{m} f_{j} \right\|_{\mathcal{H}_{w}^{(n)}}. \]

So, \( \left\| T_{u,v}^{m} \right\|_{e,B^{1+((p)/2)} \rightarrow \mathcal{H}_{w}^{(n)}} \geq \limsup_{j \to \infty} j^{1/p} \left\| T_{u,v}^{m} f_{j} \right\|_{\mathcal{H}_{w}^{(n)}} \). Now, we prove that
\[ \limsup_{j \to \infty} j^{1/p} \left\| T_{u,v}^{m} f_{j} \right\|_{\mathcal{H}_{w}^{(n)}} \geq \left\| T_{u,v}^{m} \right\|_{e,B^{1+((p)/2)} \rightarrow \mathcal{H}_{w}^{(n)}}. \]

From Theorem 6, for any fixed positive integer \( k \geq m \) and \( 0 \leq i \leq n + 1 \), we have
\[ \left\| T_{u,v}^{m} f_{i+1,a} \right\|_{\mathcal{H}_{w}^{(n)}} \leq C_{i+1}(1 - |a|^{2})^{1+\frac{i+1}{[a]}} \sum_{j=0}^{\infty} f_{ij} j^{1/p} \left\| T_{u,v}^{m} f_{j} \right\|_{\mathcal{H}_{w}^{(n)}} \]
\[ = C_{i+1}(1 - |a|^{2})^{1+\frac{i}{[a]}} \sum_{j=0}^{\infty} f_{ij} j^{1/p} \left\| T_{u,v}^{m} f_{j} \right\|_{\mathcal{H}_{w}^{(n)}} + \sum_{j=0}^{\infty} f_{ij} j^{1/p} \left\| T_{u,v}^{m} f_{j} \right\|_{\mathcal{H}_{w}^{(n)}} \]
\[ \leq 2Q_{i+1}(1 - |a|^{2})^{1+\frac{i}{[a]}} (1 - |a|^{2})^{1+\frac{i}{[a]}} + 2^{i+1} C_{i+1} \sum_{j=0}^{\infty} \sup_{j} \left\| T_{u,v}^{m} f_{j} \right\|_{\mathcal{H}_{w}^{(n)}}. \]

where \( Q = \sup_{j \geq m} j^{1/p} \left\| T_{u,v}^{m} f_{j} \right\|_{\mathcal{H}_{w}^{(n)}} \). Letting \( |a| \to 1 \), we obtain
\[ A_{i+1} = \limsup_{[a] \to 1} \left\| T_{u,v}^{m} f_{i+1,a} \right\|_{\mathcal{H}_{w}^{(n)}} \leq \sup_{j \geq m} j^{1/p} \left\| T_{u,v}^{m} f_{j} \right\|_{\mathcal{H}_{w}^{(n)}}. \]

Applying Theorem 7, we get
\[ \left\| T_{u,v}^{m} \right\|_{e,B^{1+((p)/2)} \rightarrow \mathcal{H}_{w}^{(n)}} \geq \limsup_{j \to \infty} j^{1/p} \left\| T_{u,v}^{m} f_{j} \right\|_{\mathcal{H}_{w}^{(n)}}. \]

It is clear that \( \left\| T_{u,v}^{m} \right\|_{e,B^{1+((p)/2)} \rightarrow \mathcal{H}_{w}^{(n)}} \leq \left\| T_{u,v}^{m} \right\|_{e,B^{1+((p)/2)} \rightarrow \mathcal{H}_{w}^{(n)}} \to \mathcal{H}_{w}^{(n)} \); so, from the last inequalities, we have
\[ \limsup_{j \to \infty} j^{1/p} \left\| T_{u,v}^{m} f_{j} \right\|_{\mathcal{H}_{w}^{(n)}} \approx \left\| T_{u,v}^{m} \right\|_{e,B^{1+((p)/2)} \rightarrow \mathcal{H}_{w}^{(n)}} \approx \left\| T_{u,v}^{m} \right\|_{e,B^{1+((p)/2)} \rightarrow \mathcal{H}_{w}^{(n)}}. \]

The proof is complete. \[ \square \]

4. **Some Applications**

For \( 0 < p < \infty \), by using Lemma 3, we have \( H^{p} \subset B^{1+((p)/2)} \). Also, for \( p = \infty, H^{\infty} \notin B_{0} \) and \( H^{\infty} \cap B_{0} \) are a Banach space with the norm \( \left\| \cdot \right\|_{H^{\infty}} \). In this case, we get the following corollary.

**Corollary 10.** Let \( m, n \in \mathbb{N}, u, v \in H(\mathbb{D}), \mu \) be a weight and \( \phi \in \mathcal{S}(\mathbb{D}) \). The operator \( T_{u,v}^{m} : H^{\infty} \rightarrow \mathcal{H}_{w}^{(n)} \) is bounded if and only if the operator \( T_{u,v}^{m} : H^{\infty} \cap B_{0} \rightarrow \mathcal{H}_{w}^{(n)} \) be bounded.

**Corollary 11.** Let \( m, n \in \mathbb{N}, u, v \in H(\mathbb{D}), \phi \in \mathcal{S}(\mathbb{D}), \mu \) be a weight. If \( T_{u,v}^{m} : H^{\infty} \rightarrow \mathcal{H}_{w}^{(n)} \) be bounded, then,
\[ \left\| T_{u,v}^{m} \right\|_{e,H^{\infty} \rightarrow \mathcal{H}_{w}^{(n)}} \approx \left\| T_{u,v}^{m} \right\|_{e,H^{\infty} \cap B_{0} \rightarrow \mathcal{H}_{w}^{(n)}}. \]

Proof. It is clear that \( \| T_{u,v,\varphi}^{m} \|_{c,H^0 \cap B_0 \rightarrow \mathcal{W}^{(n)}_{\mu}} \leq \| T_{u,v,\varphi}^{m} \|_{c,H^0 \cap B_0 \rightarrow \mathcal{W}^{(n)}_{\mu}} \) and \( f_{i,a}(z) = ((1 - |a|)^2 / (1 - \bar{a}z)^2) \in H^0 \) and \( f_{i,a}(z) = ((1 - |a|)^2 / (1 - \bar{a}z)^2) \in H^0 \) and \( \mathcal{B}_0 \), so for any compact operator \( K : H^0 \cap \mathcal{B}_0 \rightarrow \mathcal{W}^{(n)}_{\mu} \), from Lemma 2.10 in [17], for any \( i \in \{0, \ldots, n+1\} \), we obtain

\[
\| T_{u,v,\varphi}^{m} \|_{c,H^0 \cap B_0 \rightarrow \mathcal{W}^{(n)}_{\mu}} \geq \limsup_{|a| \rightarrow 1} \left( \| T_{u,v,\varphi}^{m} - K \|_{\mathcal{W}^{(n)}_{\mu}} \right)_{|a| \rightarrow 1} 
\geq \limsup_{|a| \rightarrow 1} \| T_{u,v,\varphi}^{m} f_{i+1,a} \|_{\mathcal{W}^{(n)}_{\mu}} - \lim_{|a| \rightarrow 1} \| K f_{i+1,a} \|_{\mathcal{W}^{(n)}_{\mu}} = A_i.
\]

(60)

Hence, from the last inequality and Theorem 7,

\[
\| T_{u,v,\varphi}^{m} \|_{c,H^0 \cap B_0 \rightarrow \mathcal{W}^{(n)}_{\mu}} \geq \max \{ A_i \}_{j=0}^{n+1} = \| T_{u,v,\varphi}^{m} \|_{c,B_0 \rightarrow \mathcal{W}^{(n)}_{\mu}}
\]

(61)

The proof is complete. \( \square \)

From Theorems 7, 8 and 9 and Corollary 11, the next corollaries are obtained.

**Corollary 12.** Let \( m, n \in \mathbb{N} \), \( 0 < p < \infty \), \( u, v \in H(D) \), \( \varphi \in S(D) \), and \( \mu \) be a weight such that \( T_{u,v,\varphi}^{m} : H^p \rightarrow \mathcal{W}^{(n)}_{\mu} \) is bounded. Then, the following statements are equivalent.

(a) The operator \( T_{u,v,\varphi}^{m} : H^p \rightarrow \mathcal{W}^{(n)}_{\mu} \) is compact

(b) The operator \( T_{u,v,\varphi}^{m} : \mathcal{B}^{1+1/p} \rightarrow \mathcal{W}^{(n)}_{\mu} \) is compact

(c) The operator \( T_{u,v,\varphi}^{m} : \mathcal{B}_0^{1+1/p} \rightarrow \mathcal{W}^{(n)}_{\mu} \) is compact

(d) \( \lim_{\lambda \rightarrow \infty} \| T_{u,v,\varphi}^{m} \|_{c,H^p \rightarrow \mathcal{W}^{(n)}_{\mu}} = 0 \)

(e) For each \( i \in \{0, \ldots, n+1\} \), \( \limsup_{|a| \rightarrow 1} \| T_{u,v,\varphi}^{m} f_{i+1,a} \|_{\mathcal{W}^{(n)}_{\mu}} = 0 \)

(f) For each \( i \in \{0, \ldots, n+1\} \), \( \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z)}{(1 - |\varphi(z)|^2)^{m_i+1/p}} = 0 \)

(62)

**Corollary 13.** Let \( m, n \in \mathbb{N} \), \( u, v \in H(D) \), \( \varphi \in S(D) \), and \( \mu \) be a weight such that \( T_{u,v,\varphi}^{m} : H^\infty \rightarrow \mathcal{W}^{(n)}_{\mu} \) is bounded. Then, the following statements are equivalent.

(a) The operator \( T_{u,v,\varphi}^{m} : H^\infty \rightarrow \mathcal{W}^{(n)}_{\mu} \) is compact

(b) The operator \( T_{u,v,\varphi}^{m} : H^\infty \cap \mathcal{B}_0 \rightarrow \mathcal{W}^{(n)}_{\mu} \) is compact

(c) The operator \( T_{u,v,\varphi}^{m} : \mathcal{B} \rightarrow \mathcal{W}^{(n)}_{\mu} \) is compact

(d) The operator \( T_{u,v,\varphi}^{m} : \mathcal{B}_0 \rightarrow \mathcal{W}^{(n)}_{\mu} \) is compact

(63)

Remark 14. By putting \( \nu \equiv 0 \) in Theorems 6, 7, 8, and 9 and Corollaries 12 and 13, some characterizations are acquired for boundedness, essential norm, and compactness of the generalized weighted composition operator from Hardy spaces \( 0 < p \leq \infty \) into \( n \)th weighted-type spaces.

Since

\[
\begin{align*}
&\left( I_{1,\varphi}^{1,u} + I_{1,\varphi}^{1,v} \right) (z) = u'(z), \\
&\left( I_{1,\varphi}^{1,u} + I_{1,\varphi}^{1,v} \right) (z) = u(z)\varphi'(z) + v'(z), \\
&\left( I_{1,\varphi}^{1,u} + I_{1,\varphi}^{1,v} \right) (z) = v(z)\varphi'(z),
\end{align*}
\]

we obtain the next remark.

**Remark 15.** Let \( \alpha > 0 \). Setting \( n = 1(\mu(z) = (1 - |z|^2)^{\alpha}, (1 - |z|^2) \log (2/(1 - |z|))) \) in Theorems 6, 7, 8, and 9 and Corollaries 12 and 13 and using (64) we get similar results for operator \( T_{u,v,\varphi}^{m} : H^p \rightarrow \mathcal{B}_0 \mu \left( T_{u,v,\varphi}^{m} : H^p \rightarrow \mathcal{B}_0 \right) \) (see [11, 12]).

(65)

Data Availability

No data were used to support this study.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


