Hindawi
Abstract and Applied Analysis
Volume 2021, Article ID 5556275, 8 pages
https://doi.org/10.1155/2021/5556275

Research Article

The Product-Type Operators from Hardy Spaces into nth Weighted-Type Spaces

Ebrahim Abbasi

Department of Mathematics, Mahabad Branch, Islamic Azad University, Mahabad, Iran

Correspondence should be addressed to Ebrahim Abbasi; ebrahimabbasi81@gmail.com

Received 2 March 2021; Accepted 23 June 2021; Published 16 July 2021

Academic Editor: Alfred Peris

The main goal of this paper is to investigate the boundedness and essential norm of a class of product-type operators ($T^m_{u,v,\varphi}$, $m \in \mathbb{N}$) from Hardy spaces into nth weighted-type spaces. As a corollary, we obtain some equivalent conditions for compactness of such operators.

1. Introduction

Let $\mathbb{D}$ denote the open unit disc of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denotes the space of all holomorphic functions on $\mathbb{D}$. The space of bounded holomorphic functions on $\mathbb{D}$ is denoted by H^∞; it is a Banach space with the equipped norm

$$\|g\|_{H^\infty} = \sup_{z\in\mathbb{D}} |g(z)|. \tag{1}$$

Let $0 < p < \infty$. A Hardy space H^p consists of all $g \in H(\mathbb{D})$ such that

$$\|g\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| g\left(re^{i\theta}\right)\right|^p d\theta \right)^{1/p} < \infty. \tag{2}$$

When $1 \le p < \infty$, H^p is a Banach space with the norm $\|\cdot\|_{H^p}$. If $0 < p < 1$, H^p is a nonlocally convex topological vector space and it is a complete metric space (see [1]).

Let $n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and $\mu(z)$ be a weight, continuous, and positive function on $\mathbb{D}$. The nth weighted-type space $\mathscr{W}^{(n)}_\mu$, consists of all $g \in H(\mathbb{D})$ such that

$$b_{\mathscr{W}^{(n)}_\mu}(g) = \sup_{z\in\mathbb{D}} \mu(z) \left| g^{(n)}(z) \right| < \infty. \tag{3}$$

It is a Banach space with the following norm

$$\|g\|_{\mathscr{W}^{(n)}_\mu} = \sum_{i=0}^{n-1} \left| g^{(i)}(0) \right| + b_{\mathscr{W}^{(n)}_\mu}(g). \tag{4}$$

The little nth weighted-type space $\mathscr{W}^{(n)}_{\mu,0}$ is a closed subspace of $\mathscr{W}^{(n)}_\mu$ such that for any $g \in \mathscr{W}^{(n)}_{\mu,0}$

$$\lim_{|z|\longrightarrow 1} \mu(z) \left| g^{(n)}(z) \right| = 0. \tag{5}$$

For more information about nth weighted-type spaces, see [2–4]. Let $\alpha > 0$. Then, $\mathscr{W}^{(0)}_{(1-|z|^2)^\alpha} = H^{-\alpha}$(growth space), $\mathscr{W}^{(1)}_{(1-|z|^2)^\alpha} = \mathscr{B}^\alpha$(Bloch-type space), and $\mathscr{W}^{(2)}_{(1-|z|^2)^\alpha} = \mathscr{Z}^\alpha$(Zygmund-type space). Also $\mathscr{W}^{(0)}_\mu = H_\mu$(weighted-type space), $\mathscr{W}^{(1)}_\mu = \mathscr{B}\mu$(weighted Bloch space), $\mathscr{W}^{(2)}_\mu = \mathscr{Z}_\mu$(weighted Zygmund space), and $\mathscr{W}^{(1)}_{(1-|z|^2)\log(2/(1-|z|^2))}$ coincide with the logarithmic Bloch space $\mathscr{B}_{\log}$.

Let $n, k \in \mathbb{N}_0$ such that $k \le n$; the partial Bell polynomials are triangular

$$B_{n,k}(x_1, \cdots, x_{n-k+1}) = \sum \frac{n!}{j_1! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \tag{6}$$

where the sum is taken over all nonnegative integers $j_1, \cdots, j_{n-k+1}$ such that

$$\begin{aligned} j_1+\cdots+(n-k+1)j_{n-k+1} &= n, \\ j_1+\cdots+j_{n-k+1} &= k. \end{aligned} \tag{7}$$

More information about Bell polynomials can be found in ([5], p 134).

Let $m \in \mathbb{N}_0$, $u, v \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ be the set of all holomorphic self-map of $\mathbb{D}$. In [6], Stevic′, Sharma and Krishan defined a new product-type operator $T^m_{u,v,\varphi}$ as follows:

$$T^m_{u,v,\varphi} g(z) = u(z) g^{(m)}(\varphi(z)) + v(z) g^{(m+1)}(\varphi(z)), \quad g \in H(\mathbb{D}), z \in \mathbb{D}. \tag{8}$$

When $m = 0$, we obtain the Stevic′-Sharma-type operator, and for $v \equiv 0$, we get the generalized weighted composition operators $D^m_{u,\varphi}$. Product-type operators on some spaces of analytic functions on the unit disc have become a subject of increasing interest in the recent years. We refer the reader to [6–10] and the references therein.

Liu and Yu have considered boundedness and compactness of operator $T^0_{u,v,\varphi}$ from Hardy spaces and H^∞ into the logarithmic Bloch space in [11, 12]. Also, Zhang and Liu have found some characterizations for boundedness and compactness of operator $T^0_{u,v,\varphi}$ from Hardy spaces into the weighted Zygmund space in [10]. Recently, the boundedness, compactness, and norm of operator $T^0_{u,v,\varphi} : H^p \longrightarrow \mathcal{W}^n_\mu$ are considered in [13].

Motivated by previous works, the results found in them will be generalized for operator $T^m_{u,v,\varphi}$. For this purpose in the second section of this paper, we give some characterizations for boundedness of operator $T^m_{u,v,\varphi} : H^p \longrightarrow \mathcal{W}^{(n)}_\mu$ where $m, n \in \mathbb{N}$ and $0 < p \le \infty$. In the third section, some new estimates are obtained for the essential norm of such operators. As a corollary, some equivalent conditions are acquired for compactness of such operators.

Throughout this paper, if there exists a constant c such that $a \ge cb$, we use the notation $a \succeq b$. The symbol $a \approx b$ means that $a \succeq b \succeq a$.

2. Boundedness

In this section, some equivalent conditions are found for the boundedness of operator $T^m_{u,v,\varphi} (m \in \mathbb{N})$ from $H^p (0 < p \le \infty)$ into nth weighted-type spaces. Firstly, we state some lemmas.

Lemma 1 (see [14], Propositions 8). *Let $\alpha > 0$. Then, for any $g \in \mathcal{B}^\alpha$ and $n \in \mathbb{N}$,*

$$\sum_{i=0}^{n-1} \left| g^{(i)}(0) \right| + \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^{\alpha-1+n} \left| g^{(n)}(z) \right| \approx \|g\|_{\mathcal{B}^\alpha}. \tag{9}$$

Lemma 2 (see [15], Lemma 2.1). *Let $\alpha > 0$ and $\mathcal{B}^\alpha_0 = \mathcal{W}^{(1)}_{(1-|z|^2)^a,0}$. The sequence $\{j^{\alpha-1} z^j\}_1^\infty$ is bounded in $\mathcal{B}^\alpha_0$ and*

$$\lim_{j\longrightarrow\infty} j^{\alpha-1} \left\| z^j \right\|_{\mathcal{B}^\alpha} = \left(\frac{2\alpha}{e}\right)^a. \tag{10}$$

From Lemma 1, Proposition 5.1.2 [16] and [1], the next lemma is obtained.

Lemma 3. *Let $0 < p \le \infty$, $n \in \mathbb{N}_0$ and $g \in H^p$. Then,*

$$\left| g^{(n)}(z) \right| \preceq \frac{\|g\|_{H^p}}{\left(1 - |z|^2\right)^{(1/p)+n}}, \quad z \in \mathbb{D}. \tag{11}$$

Lemma 4 (see [4]). *Let $\varphi \in S(\mathbb{D})$ and $u, g \in H(\mathbb{D})$. Then, for any $m, n \in \mathbb{N}_0$*

$$\begin{aligned} \left(u(z) g^{(m)}(\varphi(z))\right)^{(n)} &= \sum_{i=0}^{n} g^{(m+i)}(\varphi(z)) \\ &\cdot \sum_{l=i}^{n} \binom{n}{l} u^{(n-l)}(z) B_{l,i}\left(\varphi'(z), \varphi''(z), \cdots, \varphi^{(l-i+1)}(z)\right). \end{aligned} \tag{12}$$

In this paper, we set

$$\begin{aligned} f_{i,a}(z) &= \frac{\left(1 - |a|^2\right)^i}{(1 - \bar{a}z)^{(1/p)+i}}, \quad 0 \ne a, z \in \mathbb{D}, i \in \mathbb{N}, \\ I^{n,u}_{i,\varphi}(z) &= \sum_{l=i}^{n} \binom{n}{l} u^{(n-l)}(z) B_{l,i}\left(\varphi'(z), \varphi''(z), \cdots, \varphi^{(l-i+1)}(z)\right), \\ I^{n,u}_{n+1,\varphi}(z) &= I^{n,v}_{-1,\varphi}(z) = 0, \quad p_j(z) = z^j. \end{aligned} \tag{13}$$

By using the functions $f_{i,a}$, we obtain the following lemma. Since the proof of it resembles the proof of Lemma 1 [2], therefore, it is omitted.

Lemma 5. *Let δ_{ik} be Kronecker delta. For any $0 \ne a \in \mathbb{D}$, $m \in \mathbb{N}_0$, and $i \in \{0, \cdots, n+1\}$, there exists a function $g_{i,a} \in H^p$ such that*

$$g^{(m+k)}_{i,a}(a) = \frac{\delta_{ik} \bar{a}^{m+k}}{\left(1 - |a|^2\right)^{m+k+(1/p)}}. \tag{14}$$

In this case, $g_{i,a}(z) = \sum_{j=1}^{n+2} c^i_j f_{j,a}(z)$, where c^i_j are independent of choice a.

Theorem 6. *Let $m, n \in \mathbb{N}$, $0 < p \le \infty$, $u, v \in H(\mathbb{D})$, μ be a weight and $\varphi \in S(\mathbb{D})$. Then, the following statements are equivalent*

(a) The operator $T^m_{u,v,\varphi} : H^p \longrightarrow \mathcal{W}^{(n)}_\mu$ is bounded

(b) *The operator* $T^m_{u,v,\varphi} : \mathscr{B}^{1+(1/p)} \longrightarrow \mathscr{W}^{(n)}_\mu$ *is bounded*

(c) *The operator* $T^m_{u,v,\varphi} : \mathscr{B}^{1+(1/p)}_0 \longrightarrow \mathscr{W}^{(n)}_\mu$ *is bounded*

(d) $\sup_{j\geq 1} j^{1/p}\|T^m_{u,v,\varphi}p_j\|_{\mathscr{W}^{(n)}_\mu} < \infty$

(e) *For each* $i \in \{0, 1, \cdots, n+1\}$, $\sup_{a\in\mathbb{D}}\|T^m_{u,v,\varphi}f_{i+1,a}\|_{\mathscr{W}^{(n)}_\mu} < \infty$ *and* $\sup_{z\in\mathbb{D}}\mu(z)|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(z)| < \infty$

(f) *For each* $i \in \{0, \cdots, n+1\}$,

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)\left|\left(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi}\right)(z)\right|}{\left(1-|\varphi(z)|^2\right)^{m+i+(1/p)}} < \infty \tag{15}$$

Proof. $(b) \Longrightarrow (c)$ Since $\mathscr{B}^{1+(1/p)}_0 \subset \mathscr{B}^{1+(1/p)}$, we get (c).

$(c) \Longrightarrow (d)$ It follows from Lemma 2.

$(d) \Longrightarrow (e)$ For each $i \in \{0, \cdots, n+1\}$ and $a \in \mathbb{D}$,

$$f_{i+1,a}(z) = \left(1-|a|^2\right)^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma(i+1+(1/p)+j)}{j!\Gamma(i+1+(1/p))} \bar{a}^j z^j. \tag{16}$$

So,

$$\begin{aligned}\left\|T^m_{u,v,\varphi}f_{i+1,a}\right\|_{\mathscr{W}^{(n)}_\mu} &\preceq \left(1-|a|^2\right)^{i+1} \sum_{j=0}^{\infty} j^i|\bar{a}|^j j^{1/p}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathscr{W}^{(n)}_\mu} \\ &\leq \sup_{j\geq 1} j^{1/p}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathscr{W}^{(n)}_\mu}\left(1-|a|^2\right)^{i+1}\sum_{j=0}^{\infty} j^i|\bar{a}|^j, \\ &\preceq \sup_{j\geq 1} j^{1/p}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathscr{W}^{(n)}_\mu}\left(1-|a|^2\right)^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma(i+1+j)}{j!\Gamma(i+1)}|\bar{a}|^j, \\ &\preceq 2^{i+1}\sup_{j\geq 1} j^{1/p}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathscr{W}^{(n)}_\mu}.\end{aligned} \tag{17}$$

Hence, $\sup_{a\in\mathbb{D}}\|T^m_{u,v,\varphi}f_{i+1,a}\|_{\mathscr{W}^{(n)}_\mu} < \infty$. It is remained to show that for each $i \in \{0, 1, \cdots, n+1\}$, $\sup_{z\in\mathbb{D}}\mu(z)|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(z)| < \infty$. Applying the operator $T^m_{u,v,\varphi}$ to $p_m(z)$, by using Lemma 4, we have

$$\sup_{z\in\mathbb{D}}\mu(z)\left|I^{n,u}_{0,\varphi}(z) + \underbrace{I^{n,v}_{-1,\varphi}(z)}_{0}\right| \leq \frac{\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathscr{W}^{(n)}_\mu}}{m!} < \infty. \tag{18}$$

Now, assume that we have the following inequalities for $0 \leq i \leq j-1$,

$$\sup_{z\in\mathbb{D}}\mu(z)\left|\left(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi}\right)(z)\right| < \infty, \tag{19}$$

where $j \leq n+1$. By applying the operator $T^m_{u,v,\varphi}$ for $p_{j+m}(z)$ and using Lemma 4, we get

$$\begin{aligned}&\sup_{z\in\mathbb{D}}\mu(z)\Big|\frac{(j+m)!}{j!}(\varphi(z))^j I^{n,u}_{0,\varphi}(z) + \sum_{k=1}^{j}\frac{(j+m)!}{(j-k)!}(\varphi(z))^{j-k} \\ &\quad\cdot\left(I^{n,u}_{k,\varphi} + I^{n,v}_{k-1,\varphi}\right)(z)\Big| \leq \left\|T^m_{u,v,\varphi}p_j\right\|_{\mathscr{W}^{(n)}_\mu} < \infty.\end{aligned} \tag{20}$$

Since $\|\varphi\| \leq 1$, so from the triangle inequality, we have

$$\sup_{z\in\mathbb{D}}\mu(z)\left|\left(I^{n,u}_{j,\varphi}(z) + I^{n,v}_{j-1,\varphi}\right)(z)\right| < \infty. \tag{21}$$

$(e) \Rightarrow (f)$ For any $\varphi(a) \neq 0$ and $i \in \{0, \cdots, n+1\}$, by using Lemmas 4 and 5, we obtain

$$\begin{aligned}\frac{\mu(a)|\varphi(a)|^{m+i}\left|\left(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi}\right)(a)\right|}{\left(1-|\varphi(a)|^2\right)^{m+i+(1/p)}} &\leq \sup_{a\in\mathbb{D}}\left\|T^m_{u,v,\varphi}g_{i+1,(a)}\right\|_{\mathscr{W}^{(n)}_\mu} \\ &\leq \sum_{j=1}^{n+2}\left|c^i_j\right|\sup_{a\in\mathbb{D}}\left\|T^m_{u,v,\varphi}f_{i+1,a}\right\|_{\mathscr{W}^{(n)}_\mu} < \infty.\end{aligned} \tag{22}$$

Therefore from the last inequality,

$$\sup_{|\varphi(a)|>(1/2)}\frac{\mu(a)\left|\left(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi}\right)(a)\right|}{\left(1-|\varphi(a)|^2\right)^{m+i+(1/p)}} \preceq \sum_{j=1}^{n+2}\left|c^i_j\right|\sup_{a\in\mathbb{D}}\left\|T^m_{u,v,\varphi}f_{i+1,a}\right\|_{\mathscr{W}^{(n)}_\mu} < \infty. \tag{23}$$

On the other hand, from (e), we have

$$\sup_{|\varphi(a)|\leq(1/2)}\frac{\mu(a)\left|\left(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi}\right)(a)\right|}{\left(1-|\varphi(a)|^2\right)^{m+i+(1/p)}} \preceq \sup_{|\varphi(a)|\leq(1/2)}\mu(a)\left|\left(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi}\right)(a)\right| < \infty. \tag{24}$$

$(f) \Longrightarrow (b)$ For any $f \in \mathscr{B}^{1+1/p}$, by using Lemmas 1 and 4, we obtain

$$\begin{aligned}\mu(z)\left|\left(T^m_{u,v,\varphi}f\right)^{(n)}(z)\right| &\leq \mu(z)\left|\sum_{i=0}^{n+1} f^{(m+i)}(\varphi(z))\left(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi}\right)(z)\right| \\ &\leq \|f\|_{\mathscr{B}^{1+(1/p)}}\sum_{i=0}^{n+1}\sup_{z\in\mathbb{D}}\frac{\mu(z)\left|\left(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi}\right)(z)\right|}{\left(1-|\varphi(z)|^2\right)^{m+i+(1/p)}}.\end{aligned} \tag{25}$$

Also for each $0 \leq k < n$, we have

$$\left|\left(T^m_{u,v,\varphi}f\right)^{(k)}(0)\right| \leq \|f\|_{\mathscr{B}^{1+(1/p)}}\sum_{i=0}^{k+1}\frac{\left|\left(I^{k,u}_{i,\varphi} + I^{k,v}_{i-1,\varphi}\right)(0)\right|}{\left(1-|\varphi(0)|^2\right)^{m+i+(1/p)}}. \tag{26}$$

So, the operator $T^m_{u,v,\varphi} : \mathscr{B}^{1+(1/p)} \longrightarrow \mathscr{W}^{(n)}_\mu$ is bounded.

$(b) \Longrightarrow (a)$ From Lemma 3, $H^p \subset \mathscr{B}^{1+(1/p)}$, so we obtain (a).

$(a) \Longrightarrow (e)$ It is clear that $f_{i,a} \in H^p$ and $\sup_{a\in\mathbb{D}}\|f_{i,a}\|_{H^p} < \infty$. Hence, for each $i \in \{0, \cdots, n+1\}$,

$$\sup_{a\in\mathbb{D}}\left\|T^m_{u,v,\varphi} f_{i+1,a}\right\|_{\mathscr{W}^{(n)}_\mu} \le \left\|T^m_{u,v,\varphi}\right\|_{H^p \longrightarrow \mathscr{W}^{(n)}_\mu} \sup_{a\in\mathbb{D}}\left\|f_{i+1,a}\right\|_{H^p} < \infty. \tag{27}$$

The proof of the second part of (e) is similar to the proof $(d) \Longrightarrow (e)$, so it is omitted. The proof is complete. □

3. Essential Norm

In this section, we find some approximations for the essential norm of operator $T^m_{u,v,\varphi}$ from Hardy spaces into nth weighted type-spaces. As a corollary, we give some equivalent conditions for compactness of such operators.

Let X and Y be Banach spaces and $T : X \longrightarrow Y$ be the continuous linear operator. The essential norm of T is the distance from T to the compact operators, that is,

$$\|T\|_{e,X\longrightarrow Y} = \inf\left\{\|T-K\| : K : X \longrightarrow \text{Y is compact}\right\}. \tag{28}$$

It is clear that T is compact if and only if $\|T\|_{e,X\longrightarrow Y} = 0$.

Theorem 7. *Let $m, n \in \mathbb{N}$, $0 < p \le \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, and μ be a weight such that $T^m_{u,v,\varphi} : \mathscr{B}^{1+(1/p)} \longrightarrow \mathscr{W}^{(n)}_\mu$ be bounded. Then*

$$\left\|T^m_{u,v,\varphi}\right\|_{e,\mathscr{B}^{1+(1/p)}\longrightarrow \mathscr{W}^{(n)}_\mu} \approx \max\{A_i\}_{i=0}^{n+1} \approx \max\{B_i\}_{i=0}^{n+1}, \tag{29}$$

where

$$A_i = \limsup_{|a|\longrightarrow 1}\left\|T^m_{u,v,\varphi} f_{i+1,a}\right\|_{\mathscr{W}^{(n)}_\mu},$$
$$B_i = \limsup_{|\varphi(z)|\longrightarrow 1}\frac{\mu(z)\left|\left(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi}\right)(z)\right|}{\left(1-|\varphi(z)|^2\right)^{m+i+(1/p)}}. \tag{30}$$

Proof. For each $i \in \{0, \cdots, n+1\}$, $\sup_{a\in\mathbb{D}}\|f_{i+1,a}\|_{\mathscr{B}^{1+(1/p)}} < \infty$ and $f_{i+1,a} \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \longrightarrow 1$. Applying Lemma 2.10 from [17], for any compact operator K from $\mathscr{B}^{1+(1/p)}$ into $\mathscr{W}^{(n)}_\mu$, we have

$$\lim_{|a|\longrightarrow 1}\left\|K f_{i+1,a}\right\|_{\mathscr{W}^{(n)}_\mu} = 0. \tag{31}$$

Hence, for any $i \in \{0, \cdots, n+1\}$,

$$\begin{aligned} A_i &= \limsup_{|a|\longrightarrow 1}\left\|T^m_{u,v,\varphi} f_{i+1,a}\right\|_{\mathscr{W}^{(n)}_\mu} - \lim_{|a|\longrightarrow 1}\left\|K f_{i+1,a}\right\|_{\mathscr{W}^{(n)}_\mu} \\ &\le \limsup_{|a|\longrightarrow 1}\left\|\left(T^m_{u,v,\varphi} - K\right) f_{i+1,a}\right\|_{\mathscr{W}^{(n)}_\mu} \\ &\le \left\|T^m_{u,v,\varphi} - K\right\|_{\mathscr{B}^{1+(1/p)}\longrightarrow \mathscr{W}^{(n)}_\mu}. \end{aligned} \tag{32}$$

So,

$$\max\{A_i\}_{i=0}^{n+1} \preceq \inf_K \left\|T^m_{u,v,\varphi} - K\right\|_{\mathscr{B}^{1+(1/p)}\longrightarrow \mathscr{W}^{(n)}_\mu} = \left\|T^m_{u,v,\varphi}\right\|_{e,\mathscr{B}^{1+(1/p)}\longrightarrow \mathscr{W}^{(n)}_\mu}. \tag{33}$$

Now, we prove that

$$\max\{B_i\}_{i=0}^{n+1} \preceq \left\|T^m_{u,v,\varphi}\right\|_{e,\mathscr{B}^{1+(1/p)}\longrightarrow \mathscr{W}^{(n)}_\mu}. \tag{34}$$

Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim_{j\longrightarrow\infty}|\varphi(z_j)| \longrightarrow 1$. Since $T^m_{u,v,\varphi} : \mathscr{B}^{1+(1/p)} \longrightarrow \mathscr{W}^{(n)}_\mu$ is bounded, by using Lemmas 4 and 5 for any compact operator $K : \mathscr{B}^{1+(1/p)} \longrightarrow \mathscr{W}^{(n)}_\mu$ and $i \in \{0, \cdots, n+1\}$, we obtain

$$\begin{aligned} \left\|T^m_{u,v,\varphi} - K\right\|_{\mathscr{B}^{1+(1/p)}\longrightarrow \mathscr{W}^{(n)}_\mu} &\succeq \limsup_{j\longrightarrow\infty}\left\|T^m_{u,v,\varphi}\left(g_{i,\varphi(z_j)}\right)\right\|_{\mathscr{W}^{(n)}_\mu} \\ &\quad - \lim_{j\longrightarrow\infty}\left\|K\left(g_{i,\varphi(z_j)}\right)\right\|_{\mathscr{W}^{(n)}_\mu} \\ &\succeq \limsup_{j\longrightarrow\infty}\frac{\mu(z_j)\left|\varphi(z_j)\right|^{m+i}\left|\left(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi}\right)(z_j)\right|}{\left(1-\left|\varphi(z_j)\right|^2\right)^{m+i+(1/p)}}. \end{aligned} \tag{35}$$

So, from the definition of the essential norm, we get (34).

For $r \in [0,1)$, we define $K_r f(z) = f_r(z) = f(rz)$. It is apparent that K_r is a compact operator on $\mathscr{B}^{1+(1/p)}$. Let $\{r_j\} \subset (0,1)$ be a sequence such that $r_j \longrightarrow 1$ as $j \longrightarrow \infty$. Since $f_r \longrightarrow f$ uniformly on compact subsets of $\mathbb{D}$ as $r \longrightarrow 1$, then, for any positive integer j, the operator $T^m_{u,v,\varphi}K_{r_j} : \mathscr{B}^{1+(1/p)} \longrightarrow \mathscr{W}^{(n)}_\mu$ is compact. Based on the definition of the essential norm, we obtain

$$\left\|T^m_{u,v,\varphi}\right\|_{e,\mathscr{B}^{1+(1/p)}\longrightarrow \mathscr{W}^{(n)}_\mu} \le \limsup_{j\longrightarrow\infty}\left\|T^m_{u,v,\varphi} - T^m_{u,v,\varphi}K_{r_j}\right\|. \tag{36}$$

So, it is sufficient to show that

$$\limsup_{j\longrightarrow\infty}\left\|T^m_{u,v,\varphi} - T^m_{u,v,\varphi}K_{r_j}\right\| \preceq \min\left\{\max\{A_i\}_{i=0}^{n+1}, \max\max\{B_i\}_{i=0}^{n+1}\right\}. \tag{37}$$

Let $f \in \mathscr{B}^{1+(1/p)}$ such that $\|f\|_{\mathscr{B}^{1+(1/p)}} \le 1$ and for all $j \ge N$, $r_j \ge (3/4)$, therefore,

$$\left\|\left(T^m_{u,v,\varphi}-T^m_{u,v,\varphi}K_{r_j}\right)f\right\|_{\mathcal{W}^{(n)}_\mu}\le\sum_{t=0}^{n-1}\underbrace{\left|\left(T^m_{u,v,\varphi}\left(f-f_{r_j}\right)\right)^{(t)}(0)\right|}_{S_t}$$
$$+\underbrace{\sup\mu(z)\left|\sum_{k=0}^{n+1}\left(f-f_{r_j}\right)^{(k+m)}(\varphi(z))\left(I^{n,u}_{k,\varphi}+I^{n,v}_{k-1,\varphi}\right)(z)\right|}_{H_1}$$
$$+\underbrace{\sup\mu(z)\left|\sum_{k=0}^{n+1}\left(f-f_{r_j}\right)^{(k+m)}(\varphi(z))\left(I^{n,u}_{k,\varphi}+I^{n,v}_{k-1,\varphi}\right)(z)\right|}_{H_2}. \tag{38}$$

For any $s\in\mathbb{N}_0$ and compact subset of $\mathbb{D}$, $(f-f_{r_j})^{(s)}\longrightarrow 0$ uniformly, hence, from Theorem 6, we obtain

$$\limsup_{j\longrightarrow\infty}H_1=0,$$
$$\limsup_{j\longrightarrow\infty}S_t=0(t=0,\cdots,n-1). \tag{39}$$

On the other hand

$$H_2\le\sum_{k=0}^{n+1}\underbrace{\sup\mu(z)\left|f^{(k+m)}(\varphi(z))\right|\left|\left(I^{n,u}_{k,\varphi}+I^{n,v}_{k-1,\varphi}\right)(z)\right|}_{M_k}$$
$$+\sum_{k=0}^{n+1}\underbrace{\sup\mu(z)\left|r_j^kf^{(k+m)}(r_j\varphi(z))\right|\left|\left(I^{n,u}_{k,\varphi}+I^{n,v}_{k-1,\varphi}\right)(z)\right|}_{N_k}. \tag{40}$$

Now, estimate for M_k is obtained. Employing Lemmas 1 and 5,

$$M_k=\sup_{|\varphi(z)|>r_N}\mu(z)\frac{\left(1-|\varphi(z)|^2\right)^{m+k+(1/p)}\left|f^{(m+k)}(\varphi(z))\right|}{|\varphi(z)|^{m+k}}$$
$$\times\frac{|\varphi(z)|^{m+k}\left|\left(I^{n,u}_{k,\varphi}+I^{n,v}_{k-1,\varphi}\right)(z)\right|}{\left(1-|\varphi(z)|^2\right)^{m+k+(1/p)}}$$
$$\preceq\|f\|_{\mathcal{B}^{1+(1/p)}}\sup_{|\varphi(z)|>r_N}\left\|T^m_{u,v,\varphi}g_{k,\varphi(z)}\right\|_{\mathcal{W}^{(n)}_\mu}$$
$$\preceq\sum_{i=0}^{n+1}\left|c^k_{i+1}\right|\sup_{|a|>r_N}\left\|T^m_{u,v,\varphi}f_{i+1,a}\right\|_{\mathcal{W}^{(n)}_\mu}. \tag{41}$$

Taking the limit when $N\longrightarrow\infty$, we get

$$\limsup_{j\longrightarrow\infty}M_k\preceq\sum_{i=0}^{n+1}\underbrace{\limsup\left\|T^m_{u,v,\varphi}f_{i+1,a}\right\|_{\mathcal{W}^{(n)}_\mu}}_{A_i}\preceq\max\{A_i\}_{i=0}^{n+1},$$
$$\limsup_{j\longrightarrow\infty}M_k\preceq B_k. \tag{42}$$

Likewise, we have

$$\limsup_{j\longrightarrow\infty}N_k\preceq\sum_{i=0}^{n+1}\underbrace{\limsup\left\|T^m_{u,v,\varphi}f_{i+1,a}\right\|_{\mathcal{W}^{(n)}_\mu}}_{A_i}\preceq\max\{A_i\}_{i=0}^{n+1},$$
$$\limsup_{j\longrightarrow\infty}N_k\preceq B_k. \tag{43}$$

Thus, by using (38), (39), (40), (42) and (43), we obtain

$$\limsup_{j\longrightarrow\infty}\left\|T^m_{u,v,\varphi}-T^m_{u,v,\varphi}K_{r_j}\right\|_{\mathcal{B}^{1+(1/p)}\longrightarrow\mathcal{W}^{(n)}_\mu}$$
$$=\limsup_{j\longrightarrow\infty}\sup_{\|f\|_{\mathcal{B}^{1+(1/p)}}\le1}\left\|\left(T^m_{u,v,\varphi}-T^m_{u,v,\varphi}K_{r_j}\right)f\right\|_{\mathcal{W}^{(n)}_\mu}\preceq\max\{A_i\}_{i=0}^{n+1},$$
$$\limsup_{j\longrightarrow\infty}\left\|T^m_{u,v,\varphi}-T^m_{u,v,\varphi}K_{r_j}\right\|_{\mathcal{B}^{1+(1/p)}\longrightarrow\mathcal{W}^{(n)}_\mu}\preceq\max\{B_i\}_{i=0}^{n+1}. \tag{44}$$

Hence, from (36),

$$\left\|T^m_{u,v,\varphi}\right\|_{\mathcal{B}^{1+(1/p)}\longrightarrow\mathcal{W}^{(n)}_\mu}\preceq\min\left\{\max\{A_i\}_{i=0}^{n+1},\max\{B_i\}_{i=0}^{n+1}\right\}. \tag{45}$$

Consequently,

$$\left\|T^m_{u,v,\varphi}\right\|_{\mathcal{B}^{1+(1/p)}\longrightarrow\mathcal{W}^{(n)}_\mu}\approx\max\{A_i\}_{i=0}^{n+1}\approx\max\{B_i\}_{i=0}^{n+1}. \tag{46}$$

The proof is complete. □

Theorem 8. *Let $m,n\in\mathbb{N}$, $0<p\le\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, μ be a weight. If $T^m_{u,v,\varphi}:H^p\longrightarrow\mathcal{W}^{(n)}_\mu$ be bounded then*

$$\left\|T^m_{u,v,\varphi}\right\|_{e,H^p\longrightarrow\mathcal{W}^{(n)}_\mu}\approx\left\|T^m_{u,v,\varphi}\right\|_{e,\mathcal{B}^{1+(1/p)}\longrightarrow\mathcal{W}^{(n)}_\mu}. \tag{47}$$

Proof. It is evident that

$$\left\|T^m_{u,v,\varphi}\right\|_{e,H^p\longrightarrow\mathcal{W}^{(n)}_\mu}\preceq\left\|T^m_{u,v,\varphi}\right\|_{e,\mathcal{B}^{1+(1/p)}\longrightarrow\mathcal{W}^{(n)}_\mu}. \tag{48}$$

On the other hand, since $f_{i,a}(z)=(1-|a|^2)^i/(1-\bar{a}z)^{i+(1/p)}\in H^p$, for any compact operator $K:H^p\longrightarrow\mathcal{W}^{(n)}_\mu$, from Lemma 2.10 in [17], for any $i\in\{0,\cdots,n+1\}$, we get

$$\left\|T^m_{u,v,\varphi}-K\right\|_{H^p\longrightarrow\mathcal{W}^{(n)}_\mu}\geq\limsup_{|a|\longrightarrow1}\left\|\left(T^m_{u,v,\varphi}-K\right)f_{i+1,a}\right\|_{\mathcal{W}^{(n)}_\mu}$$
$$\geq\limsup_{|a|\longrightarrow1}\left\|T^m_{u,v,\varphi}f_{i+1,a}\right\|_{\mathcal{W}^{(n)}_\mu}-\lim_{|a|\longrightarrow1}\left\|Kf_{i+1,a}\right\|_{\mathcal{W}^{(n)}_\mu}=A_i. \tag{49}$$

So, from the last inequality and Theorem 7

$$\left\|T^m_{u,v,\varphi}\right\|_{e,H^p\longrightarrow\mathcal{W}^{(n)}_\mu}\geq\max\{A_i\}^{n+1}_{i=0}\approx\left\|T^m_{u,v,\varphi}\right\|_{e,\mathcal{B}^{1+(1/p)}\longrightarrow\mathcal{W}^{(n)}_\mu}. \tag{50}$$

The proof is complete. □

Theorem 9. *Let $m,n\in\mathbb{N}$, $0<p\leq\infty$, $u,v\in H(\mathbb{D})$, μ be a weight and $\varphi\in S(\mathbb{D})$ such that $T^m_{u,v,\varphi}:\mathcal{B}^{1+(1/p)}(\mathcal{B}^{1+(1/p)}_0)\longrightarrow\mathcal{W}^{(n)}_\mu$ be bounded. Then,*

$$\limsup_{j\longrightarrow\infty}j^{1/p}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathcal{W}^{(n)}_\mu}\approx\left\|T^m_{u,v,\varphi}\right\|_{e,\mathcal{B}^{1+(1/p)}_0\longrightarrow\mathcal{W}^{(n)}_\mu}\approx\left\|T^m_{u,v,\varphi}\right\|_{e,\mathcal{B}^{1+(1/p)}\longrightarrow\mathcal{W}^{(n)}_\mu}. \tag{51}$$

Proof. Let j be any positive integer and $h_j(z)=j^{(1/p)}p_j(z)$. It is clear that $\|h_j\|_{\mathcal{B}^{1+(1/p)}}\approx1$, $h_j\in\mathcal{B}^{1+(1/p)}_0$, and $\{h_j\}_{j\in\mathbb{N}}$ converge to 0 uniformly on compact subsets of $\mathbb{D}$. By using Lemma 2.10 in [17], for any compact operator K from $\mathcal{B}^{1+(1/p)}_0$ into $\mathcal{W}^n_\mu$, we get

$$\lim_{j\longrightarrow\infty}\left\|Kh_j\right\|_{\mathcal{W}^{(n)}_\mu}=0. \tag{52}$$

Hence,

$$\left\|T^m_{u,v,\varphi}-K\right\|_{\mathcal{B}^{1+(1/p)}_0\longrightarrow\mathcal{W}^{(n)}_\mu}\geq\limsup_{j\longrightarrow\infty}\left\|\left(T^m_{u,v,\varphi}-K\right)h_j\right\|_{\mathcal{W}^{(n)}_\mu}$$
$$\geq\limsup_{j\longrightarrow\infty}\left\|T^m_{u,v,\varphi}h_j\right\|_{\mathcal{W}^{(n)}_\mu}\|-\limsup_{j\longrightarrow\infty}\left\|Kh_j\right\|_{\mathcal{W}^{(n)}_\mu}$$
$$=\limsup_{j\longrightarrow\infty}j^{(1/p)}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathcal{W}^{(n)}_\mu}. \tag{53}$$

So, $\|T^m_{u,v,\varphi}\|_{e,\mathcal{B}^{1+(1/p)}_0\longrightarrow\mathcal{W}^{(n)}_\mu}\geq\limsup_{j\longrightarrow\infty}j^{(1/p)}\|T^m_{u,v,\varphi}p_j\|_{\mathcal{W}^{(n)}_\mu}$. Now, we prove that

$$\limsup_{j\longrightarrow\infty}j^{(1/p)}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathcal{W}^{(n)}_\mu}\succeq\left\|T^m_{u,v,\varphi}\right\|_{e,\mathcal{B}^{1+(1/p)}\longrightarrow\mathcal{W}^{(n)}_\mu}. \tag{54}$$

From Theorem 6, for any fixed positive integer $k\geq m$ and $0\leq i\leq n+1$, we have

$$\left\|T^m_{u,v,\varphi}f_{i+1,a}\right\|_{\mathcal{W}^{(n)}_\mu}\leq C_{i+1}\left(1-|a|^2\right)^{i+1}\sum_{j=0}^{\infty}j^i|a|^jj^{(1/p)}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathcal{W}^{(n)}_\mu}$$
$$=C_{i+1}\left(1-|a|^2\right)^{i+1}\left(\sum_{j=0}^{k-1}j^i|a|^jj^{(1/p)}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathcal{W}^{(n)}_\mu}+\sum_{j=k}^{\infty}j^i|a|^jj^{(1/p)}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathcal{W}^{(n)}_\mu}\right)$$
$$\leq2QC_{i+1}(k-1)^{i+1}\left(1-|a|^k\right)\left(1-|a|^2\right)^i+2^{i+1}C_{i+1}C'_{i+1}\sup_{j\geq k}j^{(1/p)}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathcal{W}^{(n)}_\mu}, \tag{55}$$

where $Q=\sup_{j\geq m}j^{(1/p)}\|T^m_{u,v,\varphi}p_j\|_{\mathcal{W}^{(n)}_\mu}$. Letting $|a|\longrightarrow1$, we obtain

$$A_{i+1}=\limsup_{|a|\longrightarrow1}\left\|T^m_{u,v,\varphi}f_{i+1,a}\right\|_{\mathcal{W}^{(n)}_\mu}\preceq\sup_{j\geq k}j^{(1/p)}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathcal{W}^{(n)}_\mu}. \tag{56}$$

Applying Theorem 7, we get

$$\left\|T^m_{u,v,\varphi}\right\|_{e,\mathcal{B}^{1+(1/p)}\longrightarrow\mathcal{W}^{(n)}_\mu}\approx\max\{A_i\}^{n+1}_{i=0}\preceq\limsup_{j\longrightarrow\infty}j^{(1/p)}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathcal{W}^{(n)}_\mu}. \tag{57}$$

It is clear that $\|T^m_{u,v,\varphi}\|_{e,\mathcal{B}^{1+(1/p)}_0\longrightarrow\mathcal{W}^{(n)}_\mu}\leq\|T^m_{u,v,\varphi}\|_{e,\mathcal{B}^{1+(1/p)}\longrightarrow\mathcal{W}^{(n)}_\mu}$; so, from the last inequalities, we have

$$\limsup_{j\longrightarrow\infty}j^{(1/p)}\left\|T^m_{u,v,\varphi}p_j\right\|_{\mathcal{W}^{(n)}_\mu}\approx\left\|T^m_{u,v,\varphi}\right\|_{e,\mathcal{B}^{1+(1/p)}_0\longrightarrow\mathcal{W}^{(n)}_\mu}\approx\left\|T^m_{u,v,\varphi}\right\|_{e,\mathcal{B}^{1+(1/p)}\longrightarrow\mathcal{W}^{(n)}_\mu}. \tag{58}$$

The proof is complete. □

4. Some Applications

For $0<p<\infty$, by using Lemma 3, we have $H^p\subset\mathcal{B}^{1+(1/p)}_0$. Also, for $p=\infty$, $H^\infty\not\subset\mathcal{B}_0$ and $H^\infty\cap\mathcal{B}_0$ are a Banach space with the norm $\|\cdot\|_{H^\infty}$. In this case, we get the following corollary.

Corollary 10. *Let $m,n\in\mathbb{N}$, $u,v\in H(\mathbb{D})$, and μ be a weight and $\varphi\in S(\mathbb{D})$. The operator $T^m_{u,v,\varphi}:H^\infty\longrightarrow\mathcal{W}^{(n)}_\mu$ is bounded if and only if the operator $T^m_{u,v,\varphi}:H^\infty\cap\mathcal{B}_0\longrightarrow\mathcal{W}^{(n)}_\mu$ be bounded.*

Corollary 11. *Let $m,n\in\mathbb{N}$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, and μ be a weight. If $T^m_{u,v,\varphi}:H^\infty\longrightarrow\mathcal{W}^{(n)}_\mu$ be bounded, then,*

$$\left\|T^m_{u,v,\varphi}\right\|_{e,H^\infty\longrightarrow\mathcal{W}^{(n)}_\mu}\approx\left\|T^m_{u,v,\varphi}\right\|_{e,H^\infty\cap\mathcal{B}_0\longrightarrow\mathcal{W}^{(n)}_\mu}. \tag{59}$$

Proof. It is clear that $\|T^m_{u,v,\varphi}\|_{e,H^\infty\cap\mathscr{B}_0\longrightarrow\mathscr{W}^{(n)}_\mu} \preceq \|T^m_{u,v,\varphi}\|_{e,H^\infty\longrightarrow\mathscr{W}^{(n)}_\mu}$ and $f_{i,a}(z)=((1-|a|^2)^i/(1-\bar{a}z)^i)\in H^\infty\cap\mathscr{B}_0$. So, for any compact operator $K:H^\infty\cap\mathscr{B}_0\longrightarrow\mathscr{W}^{(n)}_\mu$, from Lemma 2.10 in [17], for any $i\in\{0,\cdots,n+1\}$, we obtain

$$\begin{aligned}\left\|T^m_{u,v,\varphi}\right\|_{e,H^\infty\cap\mathscr{B}_0\longrightarrow\mathscr{W}^{(n)}_\mu} &\ge \limsup_{|a|\longrightarrow1}\left\|\left(T^m_{u,v,\varphi}-K\right)f_{i+1,a}\right\|_{\mathscr{W}^{(n)}_\mu}\\ &\ge \limsup_{|a|\longrightarrow1}\left\|T^m_{u,v,\varphi}f_{i+1,a}\right\|_{\mathscr{W}^{(n)}_\mu}-\lim_{|a|\longrightarrow1}\|Kf_{i+1,a}\|_{\mathscr{W}^{(n)}_\mu}=A_i.\end{aligned} \tag{60}$$

Hence, from the last inequality and Theorem 7,

$$\left\|T^m_{u,v,\varphi}\right\|_{e,H^\infty\cap\mathscr{B}_0\longrightarrow\mathscr{W}^{(n)}_\mu}\ge\max\{A_i\}_{i=0}^{n+1}\approx\left\|T^m_{u,v,\varphi}\right\|_{e,\mathscr{B}_0^{1+(1/p)}\longrightarrow\mathscr{W}^{(n)}_\mu}. \tag{61}$$

The proof is complete. □

From Theorems 7, 8 and 9 and Corollary 11, the next corollaries are obtained.

Corollary 12. *Let $m,n\in\mathbb{N}$, $0<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, and μ be a weight such that $T^m_{u,v,\varphi}:H^p\longrightarrow\mathscr{W}^{(n)}_\mu$ be bounded. Then, the following statements are equivalent.*

(a) The operator $T^m_{u,v,\varphi}:H^p\longrightarrow\mathscr{W}^{(n)}_\mu$ is compact

(b) The operator $T^m_{u,v,\varphi}:\mathscr{B}^{1+(1/p)}\longrightarrow\mathscr{W}^{(n)}_\mu$ is compact

(c) The operator $T^m_{u,v,\varphi}:\mathscr{B}_0^{1+(1/p)}\longrightarrow\mathscr{W}^{(n)}_\mu$ is compact

(d) $\lim_{j\longrightarrow\infty}j^{(1/p)}\|T^m_{u,v,\varphi}p_j\|_{\mathscr{W}^{(n)}_\mu}=0$

(e) For each $i\in\{0,\cdots,n+1\}$, $\limsup_{|a|\longrightarrow1}\|T^m_{u,v,\varphi}f_{i+1,a}\|_{\mathscr{W}^{(n)}_\mu}=0$

(f) For each $i\in\{0,\cdots,n+1\}$,

$$\limsup_{|\varphi(z)|\longrightarrow1}\frac{\mu(z)\left|\left(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi}\right)(z)\right|}{\left(1-|\varphi(z)|^2\right)^{m+i+(1/p)}}=0 \tag{62}$$

Corollary 13. *Let $m,n\in\mathbb{N}$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, and μ be a weight such that $T^m_{u,v,\varphi}:H^\infty\longrightarrow\mathscr{W}^{(n)}_\mu$ be bounded. Then, the following statements are equivalent.*

(a) The operator $T^m_{u,v,\varphi}:H^\infty\longrightarrow\mathscr{W}^{(n)}_\mu$ is compact

(b) The operator $T^m_{u,v,\varphi}:H^\infty\cap\mathscr{B}_0\longrightarrow\mathscr{W}^{(n)}_\mu$ is compact

(c) The operator $T^m_{u,v,\varphi}:\mathscr{B}\longrightarrow\mathscr{W}^{(n)}_\mu$ is compact

(d) The operator $T^m_{u,v,\varphi}:\mathscr{B}_0\longrightarrow\mathscr{W}^{(n)}_\mu$ is compact

(e) $\lim_{j\longrightarrow\infty}\|T^m_{u,v,\varphi}p_j\|_{\mathscr{W}^{(n)}_\mu}=0$

(f) For each $i\in\{0,\cdots,n+1\}$, $\limsup_{|a|\longrightarrow1}\|T^m_{u,v,\varphi}f_{i+1,a}\|_{\mathscr{W}^{(n)}_\mu}=0$

(g) For each $i\in\{0,\cdots,n+1\}$

$$\limsup_{|\varphi(z)|\longrightarrow1}\frac{\mu(z)\left|\left(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi}\right)(z)\right|}{\left(1-|\varphi(z)|^2\right)^{m+i}}=0 \tag{63}$$

Remark 14. By putting $v\equiv0$ in Theorems 6, 7, 8, and 9 and Corollaries 12 and 13, some characterizations are acquired for boundedness, essential norm, and compactness of the generalized weighted composition operator from Hardy spaces ($0<p\le\infty$) into nth weighted-type spaces.

Since

$$\begin{aligned}\left(I^{1,u}_{0,\varphi}+I^{1,v}_{-1,\varphi}\right)(z)&=u'(z),\\ \left(I^{1,u}_{1,\varphi}+I^{1,v}_{0,\varphi}\right)(z)&=u(z)\varphi'(z)+v'(z),\\ \left(I^{1,u}_{2,\varphi}+I^{1,v}_{1,\varphi}\right)(z)&=v(z)\varphi'(z),\end{aligned} \tag{64}$$

we obtain the next remark.

Remark 15. Let $\alpha>0$. Setting $n=1(\mu(z)=(1-|z|^2)^a,(1-|z|^2)\log(2/(1-|z|)))$ in Theorems 6, 7, 8, and 9 and Corollaries12 and 13 and using (64) we get similar results for operator $T^m_{u,v,\varphi}:H^p\longrightarrow\mathscr{B}_\mu(T^m_{u,v,\varphi}:H^p\longrightarrow\mathscr{B}^\alpha, T^m_{u,v,\varphi}:H^p\longrightarrow\mathscr{B}_{\log})$(see [11, 12]).

$$\begin{aligned}\left(I^{2,u}_{0,\varphi}+I^{2,v}_{-1,\varphi}\right)(z)&=u''(z),\\ \left(I^{2,u}_{1,\varphi}+I^{2,v}_{0,\varphi}\right)(z)&=2u'(z)\varphi'(z)+u(z)\varphi''(z)+v''(z),\\ \left(I^{2,u}_{2,\varphi}+I^{2,v}_{1,\varphi}\right)(z)&=u(z)\varphi'^2(z)+2v'(z)\varphi'(z)+v(z)\varphi''(z),\\ \left(I^{2,u}_{3,\varphi}+I^{2,v}_{2,\varphi}\right)(z)&=v(z)\varphi'^2(z).\end{aligned} \tag{65}$$

Remark 16. Putting $n=2(\mu(z)=(1-|z|^2)^a)$ in Theorems 6, 7, 8, and 9 and Corollaries 12 and 13 and applying (65), similar results are achieved for operator $T^m_{u,v,\varphi}:H^p\longrightarrow\mathscr{Z}_\mu(T^m_{u,v,\varphi}:H^p\longrightarrow\mathscr{Z}^\alpha)$ (generalizing Theorems 7 and 9 [10]).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] P. L. Duren, *Theory of* $\mathrm{H^p}$ *Spaces*, Academic Press, New York and London, 1970.

[2] E. Abbasi and H. Vaezi, "Estimates of essential norm of generalized weighted composition operators from Bloch type spaces to nth weighted type spaces," *Mathematica Slovaca*, vol. 70, no. 1, pp. 71–80, 2020.

[3] E. Abbasi, H. Vaezi, and S. Li, "Essential norm of weighted composition operators from H^∞ to nth weighted type spaces," *Mediterranean Journal of Mathematics*, vol. 16, no. 5, p. 133, 2019.

[4] S. Stević, "Weighted differentiation composition operators from H^∞ and Bloch spaces to nth weighted-type spaces on the unit disk," *Applied Mathematics and Computation*, vol. 216, no. 12, pp. 3634–3641, 2010.

[5] L. Comtet, *Advanced Combinatorics*, D. Reidel, Dordrecht, 1974.

[6] S. Stević, A. K. Sharma, and R. Krishan, "Boundedness and compactness of a new product-type operator from a general space to Bloch-type spaces," *Journal of Inequalities and Applications*, vol. 2016, no. 1, 2016.

[7] Z. Jiang, "Product-type operators from Zygmund spaces to Bloch-Orlicz spaces," *Complex Variables and Elliptic Equations*, vol. 62, no. 11, pp. 1645–1664, 2017.

[8] Y. Liu and Y. Yu, "Products of composition, multiplication and radial derivative operators from logarithmic Bloch spaces to weighted-type spaces on the unit ball," *Journal of Mathematical Analysis and Applications*, vol. 423, no. 1, pp. 76–93, 2015.

[9] Y. Liu and Y. Yu, "On Stević-Sharma type operator from the Besov spaces into the weighted-type space H_μ^∞," *Mathematical Inequalities & Applications*, vol. 22, no. 3, pp. 1037–1053, 1998.

[10] F. Zhang and Y. Liu, "On a Stević–Sharma Operator from Hardy Spaces to Zygmund-Type Spaces on the Unit Disk," *Complex Analysis and Operator Theory*, vol. 12, no. 1, pp. 81–100, 2018.

[11] Y. Liu and Y. Yu, "On a Stević-Sharma operator from Hardy spaces to the logarithmic Bloch spaces," *Journal of Inequalities and Applications*, vol. 2015, no. 1, 2015.

[12] Y. Yu and Y. Liu, "On Stević Type Operator from H^∞ Space to the Logarithmic Bloch Spaces," *Complex Analysis and Operator Theory*, vol. 9, no. 8, pp. 1759–1780, 2015.

[13] Z. Guo and Y. Shu, "On Stević-Sharma operators from Hardy spaces to Stevic weighted spaces," *Mathematical Inequalities & Applications*, vol. 23, no. 1, pp. 217–229, 1998.

[14] K. Zhu, "Bloch type spaces of analytic functions," *The Rocky Mountain Journal of Mathematics*, vol. 23, no. 3, pp. 1143–1177, 1993.

[15] O. Hyvärinen and M. Lindström, "Estimates of essential norms of weighted composition operators between Bloch- type spaces," *Journal of Mathematical Analysis and Applications*, vol. 393, no. 1, pp. 38–44, 2012.

[16] K. Zhu, *Operator Theory in Function Spaces, Pure and Applied Mathematics*, Marcel Dekker, Inc., New York and Basel, 1990.

[17] M. Tjani, *Compact Composition Operators on Some Möbius Invariant Banach Space*, PhD dissertation, Michigan State university, 1996.