

## Research Article

# Fixed Point Results for an Almost Generalized $\alpha$ -Admissible $Z$ -Contraction in the Setting of Partially Ordered $b$ -Metric Spaces

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In this paper, we introduce an almost generalized  $\alpha$ -admissible  $Z$ -contraction with the help of a simulation function and study fixed point results in the setting of partially ordered  $b$ -metric spaces. The presented results generalize and unify several related fixed point results in the existing literature. Finally, we verify our results by using two examples. Moreover, one of our fixed point results is applied to guarantee the existence of a solution of an integral equation.

## 1. Introduction

Metric fixed point theory is a vivid topic, which furnishes useful methods and notions for dealing with various problems. In particular, we refer to the existence of solutions of mathematical problems reducible to equivalent fixed point problems. Thus, we recall that Banach contraction principle [1] is at the foundation of this theory. Due to its usefulness, Banach contraction principle has been extended and generalized in various spaces using different conditions either by modifying the basic contractive condition or by generalizing the ambient spaces or both. For some extensions of Banach contraction principle in metric spaces, see References [2–12].

The existence of fixed point in partially ordered sets has been considered by Turinici in ordered metrizable uniform spaces [11]. The applications of fixed point results in partially ordered metric spaces were studied by Ran and Reurings [13] to solve matrix equations and by Nieto and Rodríguez-López [14] to obtain solutions of certain partial differential equations with periodic boundary conditions. Many researchers have focused on different contractive conditions in complete metric spaces endowed with a partial order and obtained many fixed point results in such spaces. For further works in this direction, see References [3,

15–19]. The concept of metric spaces has been generalized in many directions. The notion of a  $b$ -metric space was introduced by Bakhtin in [20] and later extensively used by Czerwik in [21, 22]. Since then, several papers have been published on the fixed point theory of various classes of single-valued and multivalued operators in (ordered)  $b$ -metric spaces. For further works in this direction, see References [2, 4, 23–27].

Khojasteh et al. [28] introduced the notion of  $Z$ -contraction and studied existence and uniqueness of fixed points for  $Z$ -contraction type operators. This class of  $Z$ -contractions unifies large types of nonlinear contractions existing in the literature. Afterwards, Karapinar [25] originated the concept of  $\alpha$ -admissible  $Z$ -contraction. For more works in this line of research, see References [3, 4, 9, 12, 25]. Recently, Melliani et al. [9] introduced a new concept of  $\alpha$ -admissible almost type  $Z$ -contraction and proved the existence of fixed points for admissible almost type  $Z$ -contractions in a complete metric space.

Inspired and motivated by the works of [9, 25], the purpose of this paper is to introduce a new class of mappings, namely, an almost generalized  $\alpha$ -admissible  $Z$ -contraction, and prove the existence and uniqueness of fixed points for such mappings in the setting of partially ordered  $b$ -metric spaces.

## 2. Preliminaries

In this section, we present some notions, definitions, and theorems used in the sequel.

Throughout this paper, we shall use  $\mathbb{R}$  and  $\mathbb{R}^+$  to represent the set of real numbers and the set of nonnegative real numbers, respectively.

**Definition 1** (see [21]). Given a nonempty set  $X$ . A function  $d : X \times X \rightarrow \mathbb{R}^+$  is called b-metric if there is a real number  $s \geq 1$  such that for all  $x, y, z \in X$ , the following conditions hold:

- (i)  $d(x, y) = 0$  if and only if  $x = y$
- (ii)  $d(x, y) = d(y, x)$
- (iii)  $d(x, z) \leq s[d(x, y) + d(y, z)]$

The triplet  $(X, d, s)$  is called a b-metric space.

**Definition 2** (see [29]). Let  $(X, d)$  be a b-metric space. Then, a sequence  $\{x_n\}$  in  $X$  is said to be

- (a) b-convergent if there exists  $x \in X$  such that  $d(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ . In this case, we write  $\lim_{n \rightarrow \infty} x_n = x$
- (b) b-Cauchy sequence if  $d(x_n, x_m) \rightarrow 0$  as  $n, m \rightarrow \infty$
- (c) b-complete if every b-Cauchy sequence in  $X$  is b-convergent.

**Remark 3** (see [29]). In a b-metric space  $(X, d)$ , the following assertions hold:

- (1) (R1) A convergent sequence has a unique limit.
- (2) (R2) Each convergent sequence is a Cauchy sequence.
- (3) (R3) In general, a b-metric is not continuous.
- (4) (R4) In general, a b-metric does not induce a topology on  $X$

**Definition 4.** A partially ordered set (poset) is a system  $(X, \preceq)$ , where  $X$  is nonempty set and  $\preceq$  is a binary relation of  $X$  satisfying

- (i)  $x \preceq x$  (reflexivity).
- (ii) if  $x \preceq y$  and  $y \preceq x$ , then  $x = y$  (antisymmetry).
- (iii) if  $x \preceq y$  and  $y \preceq z$ , then  $x \preceq z$  (transitivity) for all  $x, y, z \in X$

**Definition 5.** Let  $X$  be a nonempty set. Then,  $(X, d, \preceq)$  is called partially ordered b-metric spaces if

- (i)  $(X, d)$  is a b-metric space and

- (ii)  $(X, \preceq)$  is a partially ordered set.

Now, we give an example to show that a b-metric is not necessarily metric.

**Example 1** (see [23]). Let  $(X, d)$  be a metric space and  $\rho(x, y) = (d(x, y))^p$ , where  $p > 1$  is a real number. Then,  $\rho$  is a b-metric with  $s = 2^{p-1}$ . However, if  $(X, d)$  is a metric space, then  $\rho(x, y)$  is not necessarily a metric space. For example, if  $X = \mathbb{R}$  and  $d(x, y) = |x - y|$ , then  $\rho(x, y) = ((d(x, y))^s)$  is a b-metric on  $\mathbb{R}$  with  $s = 2$  but it is not a metric on  $\mathbb{R}$ .

**Definition 6** (see [18]). Let  $(X, \preceq)$  be a partially ordered set and  $T : X \rightarrow X$  is a self-mapping; we say  $T$  is monotone nondecreasing with respect to  $\preceq$  if for  $x, y \in X$ ,

$$x \preceq y \implies Tx \preceq Ty. \tag{1}$$

**Definition 7.** Let  $(X, \preceq)$  be a partially ordered set and  $x, y \in X$ ; then,  $x$  and  $y$  are said to be comparable elements of  $X$  if

$$x \preceq y \text{ or } y \preceq x. \tag{2}$$

**Theorem 8** (see [30]). Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  be a map satisfying

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y), \tag{3}$$

for all  $x, y \in X$  where  $\alpha, \beta \geq 0$  with  $\alpha + \beta < 1$ . Then,  $T$  has a unique fixed point.

**Theorem 9** (see [18]). Let  $(X, d, \preceq)$  be a complete partially ordered metric space. Let  $T : X \rightarrow X$  be a continuous and nondecreasing map satisfying

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y), \tag{4}$$

for all  $x, y \in X$  with  $x \preceq y$  and  $\alpha, \beta \geq 0$  with  $\alpha + \beta < 1$ . If there exists  $x_0 \in X$  such that  $x_0 \preceq Tx_0$ , then  $T$  has a fixed point.

**Definition 10** (see [5]). Let  $(X, d)$  be a metric space. A map  $T : X \rightarrow X$  is called an almost contraction or  $(\delta, L)$  contraction if there exist constants  $\delta \in (0, 1)$  and  $L \geq 0$  such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx), \tag{5}$$

for all  $x, y \in X$ .

**Definition 11** (see [8]). A function  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is called an altering distance function if

- (i)  $\phi$  is nondecreasing and continuous;
- (ii)  $\phi(t) = 0$  if and only if  $t = 0$

**Definition 12** (see [28]). A simulation function is a map  $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$  that satisfies the following conditions:

- ( $\zeta_1$ )  $\zeta(0, 0) = 0$
- ( $\zeta_2$ )  $\zeta(t, s) < s - t$  for each  $t, s > 0$
- ( $\zeta_3$ ) If  $\{s_n\}, \{t_n\}$  are sequences in  $(0, \infty)$  such that

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} t_n = l \in \mathbb{R}^+, \tag{6}$$

then  $\lim_{n \rightarrow \infty} \sup \zeta(t_n, s_n) < 0$ .

The collection of all simulation functions is denoted by  $Z$ .

**Example 2** (see [28]). Let  $\zeta_i : [0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}$  where  $i = 1, 2, 3$  is defined as follows:

- (i)  $\zeta_1(t, s) = \lambda s - t$  for all  $s, t \in [0, +\infty)$  where  $\lambda \in [0, 1)$
- (ii)  $\zeta_2(t, s) = \varphi(s) - t$  for all  $s, t \in [0, +\infty)$  where  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is an upper semicontinuous function such that  $\varphi(t) = 0$  if and only if  $t = 0$  and  $\varphi(t) < t$  for all  $t > 0$
- (iii)  $\zeta_3(t, s) = \varphi(s) - \theta(t)$  for all  $s, t \in [0, +\infty)$  where  $\varphi, \theta : [0, +\infty) \rightarrow [0, +\infty)$  are continuous functions such that  $\varphi(t) = \theta(t) = 0$  if and only if  $t = 0$  and  $\varphi(t) < t \leq \theta(t)$  for all  $t > 0$

**Definition 13** (see [22]). Let  $(X, d)$  be a metric space,  $T : X \rightarrow X$  be a map, and  $\zeta \in Z$ . Then,  $T$  is called  $Z$ -contraction with respect to  $\zeta$  if the following condition is satisfied:

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0, \tag{7}$$

for all  $x, y \in X$ .

**Definition 14** (see [10]). Let  $X$  be a nonempty set. Let  $T : X \rightarrow X$  and  $\alpha : X \times X \rightarrow \mathbb{R}^+$  be the maps. Then,  $T$  is called  $\alpha$ -admissible if  $\alpha(x, y) \geq 1$  implies  $\alpha(Tx, Ty) \geq 1$ , for each  $x, y \in X$ .

**Definition 15** (see [31]). Let  $X$  be a nonempty set. Let  $T : X \rightarrow X$  and  $\alpha : X \times X \rightarrow \mathbb{R}^+$  be the maps. Then,  $T$  is said to be  $\alpha$ -orbital admissible mapping if  $\alpha(x, Tx) \geq 1$  implies  $\alpha(Tx, T^2x) \geq 1$ , for each  $x \in X$ .

**Definition 16** (see [31]). Let  $X$  be a nonempty set. Let  $T : X \rightarrow X$  and  $\alpha : X \times X \rightarrow \mathbb{R}^+$  be the maps. Then,  $T$  is said to be a triangular  $\alpha$ -orbital admissible if

- (i)  $T$  is  $\alpha$ -orbital admissible;
- (ii)  $\alpha(x, y) \geq 1$  and  $\alpha(y, Ty) \geq 1$  implies  $\alpha(x, Ty) \geq 1$

for each  $x, y \in X$ .

**Definition 17** (see [25]). Let  $X$  be a nonempty set and  $T : X \rightarrow X$  be a self-map defined on a metric space  $(X, d)$ . If there exist  $\zeta \in Z$  and  $\alpha : X \times X \rightarrow \mathbb{R}^+$  such that

$$\zeta(\alpha(x, y)d(Tx, Ty), d(x, y)) \geq 0, \tag{8}$$

for all  $x, y \in X$ , then  $T$  is called an  $\alpha$ -admissible  $Z$ -contraction with respect to  $\zeta$ .

**Theorem 18** (see [25]). Let  $(X, d)$  be a complete metric space and  $T$  be an  $\alpha$ -admissible  $Z$ -contraction with respect to  $\zeta$ . Suppose that

- (i)  $T$  is triangular  $\alpha$ -orbital admissible
- (ii) There exists  $x_0 \in X$  such that  $\alpha(x_0, Tx_0) \geq 1$
- (iii)  $T$  is continuous

Then, there exists  $u \in X$  such that  $u = Tu$ .

### 3. Main Result

In this section, we present our main findings.

**Definition 19.** Let  $(X, d, \preceq)$  be a partially ordered  $b$ -metric space with parameter  $s \geq 1$ . Let  $T : X \rightarrow X$  and  $\alpha : X \times X \rightarrow \mathbb{R}^+$  be the maps. Assume that there exist a simulation function  $\zeta$  and a constant  $L \geq 0$  such that

$$\zeta(\alpha(x, y)d(Tx, Ty), M(x, y) + L \bullet m(x, y)) \geq 0, \tag{9}$$

for all  $x, y \in X$  with  $x \preceq y$  where

$$M(x, y) = \max \left\{ \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, \frac{d(y, Tx)[1 + d(x, Ty)]}{s^2[1 + d(x, y)]}, \frac{d(x, y)}{s} \right\}, \tag{10}$$

$$m(x, y) = \min \left\{ \frac{d(x, Ty)d(y, Tx)}{1 + d(x, y)}, \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} \right\}. \tag{11}$$

Then,  $T$  is called an almost generalized  $\alpha$ -admissible  $Z$ -contraction with respect to  $\zeta$ .

**Theorem 20.** Let  $(X, d, \preceq)$  be a complete partially ordered  $b$ -metric space and  $T$  be an almost generalized  $\alpha$ -admissible  $Z$ -contraction with respect to  $\zeta$ . Suppose that

- (i)  $T$  is nondecreasing and continuous
- (ii) There exists  $x_0 \in X$  such that  $x_0 \preceq Tx_0$  and  $\alpha(x_0, Tx_0) \geq 1$
- (iii)  $T$  is triangular  $\alpha$ -orbital admissible

Then,  $T$  has a fixed point.

*Proof.* Due to condition (ii), there exists  $x_0 \in X$  such that  $x_0 \preceq Tx_0$  and  $\alpha(x_0, Tx_0) \geq 1$ .

Define an iterative sequence  $\{x_n\}$  in  $X$  as follows:

$$x_{n+1} = Tx_n, \tag{12}$$

for every  $n \geq 0$ . □

If there exists some nonnegative integer  $n_0$  such that  $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ , then  $x_{n_0}$  is a fixed point of  $T$ .

Now, we shall assume that  $x_n \neq x_{n+1}$  for all  $n \geq 0$ . So, we have  $d(x_{n+1}, x_n) \neq 0$  for all integers  $n \geq 0$ . Due to (ii) and  $T$  is  $\alpha$ -admissible, we have  $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$  which in turn implies that  $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$ .

Inductively, we obtain that

$$\alpha(x_n, x_{n+1}) \geq 1. \tag{13}$$

Since  $T$  is nondecreasing map and  $x_0 \leq Tx_0$ , we have the following:

$$x_0 \leq Tx_0 = x_1 \leq Tx_1 = x_2 \leq Tx_2 = x_3 \leq \dots \leq Tx_{n-1} = x_n \leq Tx_n = x_{n+1} \leq \dots \tag{14}$$

Hence,  $x_n \leq x_{n+1}$  for  $n \geq 0$ .

Using Equation (9), putting  $x = x_{n-1}$  and  $y = x_n$  for  $n \geq 1$ , we get the following:

$$\zeta(\alpha(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n), M(x_{n-1}, x_n) + L \bullet m(x_{n-1}, x_n)) \geq 0, \tag{15}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max \left\{ \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)}, \frac{d(x_n, x_n)[1 + d(x_{n-1}, x_{n+1})]}{s^2[1 + d(x_{n-1}, x_n)]}, \frac{d(x_{n-1}, x_n)}{s} \right\} \\ &= \max \left\{ d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n)}{s} \right\}, \\ m(x_{n-1}, x_n) &= \min \left\{ \frac{d(x_{n-1}, x_{n+1})d(x_n, x_n)}{1 + d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)} \right\} = \min \left\{ 0, \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)} \right\} = 0. \end{aligned} \tag{16}$$

If  $M(x_{n-1}, x_n) = \max \{d(x_n, x_{n+1}), d(x_{n-1}, x_n)/s\} = d(x_n, x_{n+1})$ , then Equation (15) becomes the following:

$$\zeta(\alpha(x_{n-1}, x_n)d(x_n, x_{n+1}), d(x_n, x_{n+1})) \geq 0. \tag{17}$$

It follows that  $\alpha(x_{n-1}, x_n)d(x_n, x_{n+1}) < d(x_n, x_{n+1})$ .

Using Equation (13), we have the following:

$$d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n)d(x_n, x_{n+1}) < d(x_n, x_{n+1}), \tag{18}$$

which is a contradiction.

Hence,  $\max \{d(x_n, x_{n+1}), d(x_{n-1}, x_n)/s\} = d(x_{n-1}, x_n)/s$

Now Equation (15) becomes  $\zeta(\alpha(x_{n-1}, x_n)d(x_n, x_{n+1}), d(x_{n-1}, x_n)/s) \geq 0$ .

It follows that

$$\alpha(x_{n-1}, x_n)d(x_n, x_{n+1}) < \frac{d(x_{n-1}, x_n)}{s}. \tag{19}$$

Furthermore, using Equation (15) and since  $s \geq 1$ , we have the following:

$$d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n)d(x_n, x_{n+1}) < \frac{d(x_{n-1}, x_n)}{s} < d(x_{n-1}, x_n). \tag{20}$$

It follows that  $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ .

Hence, the sequence  $\{d(x_{n-1}, x_n)\}$  is nonincreasing and bounded below. Accordingly, there exists  $r \geq 0$  such that

$$\lim_{n \rightarrow \infty} d(x_{n-1}, x_n) = r. \tag{21}$$

Assuming  $r > 0$  and using Equation (20), we get the following:

$$\lim_{n \rightarrow \infty} \alpha(x_{n-1}, x_n)d(x_n, x_{n+1}) = r > 0. \tag{22}$$

Letting  $t_n = \{\alpha(x_{n-1}, x_n)d(x_n, x_{n+1})\}$ ,  $s_n = \{d(x_{n-1}, x_n)\}$ , and using  $\zeta_3$ , we obtain the following:

$$0 \leq \lim_{n \rightarrow \infty} \sup \zeta(t_n, s_n) < 0, \tag{23}$$

which is a contradiction.

Thus, we have the following:

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \tag{24}$$

Now, we need to show that  $\{x_n\}$  is a Cauchy sequence in  $X$ .

Suppose  $\{x_n\}$  is not a Cauchy sequence in  $X$ , then there exists an  $\varepsilon > 0$  such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon, \tag{25}$$

where  $\{m_k\}$  and  $\{n_k\}$  are two sequences of positive integers with  $n_k > m_k > k$  for all positive integers  $k$ . Moreover,  $m_k$  is chosen as the smallest integer satisfying Equation (25).

Thus, we have the following:

$$d(x_{m_k}, x_{n_{k-1}}) < \varepsilon. \tag{26}$$

By applying the triangle inequality and using Equations (25) and (26), we get the following:

$$\begin{aligned} \varepsilon &\leq d(x_{m_k}, x_{n_k}) \leq sd(x_{m_k}, x_{n_{k-1}}) + sd(x_{n_{k-1}}, x_{n_k}) \\ &< s\varepsilon + sd(x_{n_{k-1}}, x_{n_k}). \end{aligned} \tag{27}$$

Now, taking the upper limit as  $k \rightarrow \infty$  in Equation (27) and using Equation (24), we get the following:

$$\varepsilon \leq \limsup d(x_{m_k}, x_{n_k}) < s\varepsilon. \tag{28}$$

Similarly, applying the triangle inequality twice, we get the following:

$$\begin{aligned} \varepsilon &\leq d(x_{m_k}, x_{n_k}) \leq sd(x_{m_k}, x_{n_{k-1}}) + sd(x_{n_{k-1}}, x_{n_k}) \\ &\leq s^2 d(x_{m_k}, x_{m_{k-1}}) + s^2 d(x_{m_{k-1}}, x_{n_{k-1}}) + sd(x_{n_{k-1}}, x_{n_k}). \end{aligned} \tag{29}$$

Furthermore,

$$\begin{aligned} d(x_{m_{k-1}}, x_{n_{k-1}}) &\leq sd(x_{m_{k-1}}, x_{m_k}) + sd(x_{m_k}, x_{n_{k-1}}) \\ &\leq sd(x_{m_{k-1}}, x_{m_k}) + s\varepsilon. \end{aligned} \tag{30}$$

Taking the upper limit as  $k \rightarrow \infty$  in Equations (29) and (30) and combining, we get the following:

$$\frac{\varepsilon}{s^2} \leq \limsup d(x_{m_{k-1}}, x_{n_{k-1}}) < s\varepsilon. \tag{31}$$

Again,

$$\varepsilon \leq d(x_{m_k}, x_{n_k}) \leq sd(x_{m_k}, x_{n_{k-1}}) + sd(x_{n_{k-1}}, x_{n_k}). \tag{32}$$

Furthermore,

$$d(x_{m_k}, x_{n_{k-1}}) \leq sd(x_{m_k}, x_{m_{k-1}}) + sd(x_{m_{k-1}}, x_{n_{k-1}}). \tag{33}$$

Taking the upper limit as  $k \rightarrow \infty$  in Equations (32) and (33), using Equation (31), and combining, we get the following:

$$\frac{\varepsilon}{s} \leq \limsup d(x_{m_k}, x_{n_{k-1}}) < s^2\varepsilon. \tag{34}$$

Similarly, we can show that

$$\frac{\varepsilon}{s} \leq \limsup d(x_{m_{k-1}}, x_{n_k}) < s^2\varepsilon. \tag{35}$$

Now, for simplicity, we denote the following expressions as follows:

$$\begin{aligned} a_k &= \frac{d(x_{n_{k-1}}, x_{n_k}) [1 + d(x_{m_{k-1}}, x_{m_k})]}{1 + d(x_{m_{k-1}}, x_{n_{k-1}})}, \\ b_k &= \frac{d(x_{n_{k-1}}, x_{m_k}) [1 + d(x_{m_{k-1}}, x_{n_k})]}{s^2 [1 + d(x_{m_{k-1}}, x_{n_{k-1}})]}, \\ c_k &= \frac{d(x_{m_{k-1}}, x_{n_{k-1}})}{s}, \\ d_k &= \frac{d(x_{m_{k-1}}, x_{n_k}) d(x_{n_{k-1}}, x_{m_k})}{1 + d(x_{m_{k-1}}, x_{n_{k-1}})}, \\ e_k &= \frac{d(x_{m_{k-1}}, x_{m_k}) d(x_{n_{k-1}}, x_{n_k})}{1 + d(x_{m_{k-1}}, x_{n_{k-1}})}, \end{aligned} \tag{36}$$

which give  $M(x_{m_{k-1}}, x_{n_{k-1}}) = \max \{a_k, b_k, c_k\}$  and  $m(x_{m_{k-1}}, x_{n_{k-1}}) = \min \{d_k, e_k\}$ .

Using Equations (24), (34), and (35), and the property of  $\limsup$ , we can see that

$$\limsup m(x_{m_{k-1}}, x_{n_{k-1}}) = \min \{\limsup d_k, \limsup e_k\} = \min \{\limsup d_k, 0\} = 0. \tag{37}$$

Since  $T$  is triangular  $\alpha$ -orbital admissible, we have the following:

$$\alpha(x_{m_{k-1}}, x_{n_{k-1}}) \geq 1. \tag{38}$$

Using Equation (38), we have the following:

$$d(x_{m_k}, x_{n_k}) \leq \alpha(x_{m_{k-1}}, x_{n_{k-1}}) d(x_{m_k}, x_{n_k}). \tag{39}$$

Using Equation (9) with  $x = x_{m_{k-1}}$  and  $y = x_{n_{k-1}}$ , we obtain the following:

$$\zeta(\alpha(x_{m_{k-1}}, x_{n_{k-1}}) d(x_{m_k}, x_{n_k}), M(x_{m_{k-1}}, x_{n_{k-1}}) + L \bullet m(x_{m_{k-1}}, x_{n_{k-1}})) \geq 0. \tag{40}$$

It follows that

$$\alpha(x_{m_{k-1}}, x_{n_{k-1}}) d(x_{m_k}, x_{n_k}) < M(x_{m_{k-1}}, x_{n_{k-1}}) + L \bullet m(x_{m_{k-1}}, x_{n_{k-1}}). \tag{41}$$

Combining Equations (39) and (41), we get the following:

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq \alpha(x_{m_{k-1}}, x_{n_{k-1}}) d(x_{m_k}, x_{n_k}) \\ &< M(x_{m_{k-1}}, x_{n_{k-1}}) + L \bullet m(x_{m_{k-1}}, x_{n_{k-1}}). \end{aligned} \tag{42}$$

If  $M(x_{m_{k-1}}, x_{n_{k-1}}) = b_k$ , then after rearranging, collecting like terms, and applying the triangle inequality, Equation (42) becomes the following:

$$\begin{aligned}
 & s^2 [1 + d(x_{m_{k-1}}, x_{n_{k-1}})] d(x_{m_k}, x_{n_k}) \\
 & < d(x_{n_{k-1}}, x_{m_k}) [1 + d(x_{m_{k-1}}, x_{n_k})] + s^2 L \cdot \min \\
 & \quad \cdot \{d(x_{m_{k-1}}, x_{n_k}) d(x_{n_{k-1}}, x_{m_k}), d(x_{m_{k-1}}, x_{m_k}) d(x_{n_{k-1}}, x_{n_k})\} \\
 & \leq [sd(x_{n_{k-1}}, x_{m_{k-1}}) + sd(x_{m_{k-1}}, x_{m_k})] \\
 & \quad \cdot [1 + sd(x_{m_{k-1}}, x_{m_k}) + sd(x_{m_k}, x_{n_k})] + s^2 L \cdot d \\
 & \quad \cdot (x_{m_{k-1}}, x_{m_k}) d(x_{n_{k-1}}, x_{n_k}) = sd(x_{n_{k-1}}, x_{m_{k-1}}) + s^2 d \\
 & \quad \cdot (x_{n_{k-1}}, x_{m_{k-1}}) d(x_{m_{k-1}}, x_{m_k}) s^2 d(x_{n_{k-1}}, x_{m_{k-1}}) d \\
 & \quad \cdot (x_{m_k}, x_{n_k}) + sd(x_{m_{k-1}}, x_{m_k}) [1 + sd(x_{m_{k-1}}, x_{m_k}) + sd(x_{m_k}, x_{n_k})] \\
 & \quad + s^2 L d(x_{m_{k-1}}, x_{m_k}) d(x_{n_{k-1}}, x_{n_k}).
 \end{aligned} \tag{43}$$

Taking the upper limit as  $k \rightarrow \infty$  in Equation (43) and using Equation (31), we get the following:

$$\bar{\lim} d(x_{m_k}, x_{n_k}) \leq \bar{\lim} d(x_{m_{k-1}}, x_{n_{k-1}}) < \varepsilon, \tag{44}$$

which is a contradiction due to Equation (9).

Hence,

$$M(x_{m_{k-1}}, x_{n_{k-1}}) = \max \{a_k, b_k, c_k\} = c_k. \tag{45}$$

Using Equations (32) and (45), we get the following:

$$\bar{\lim} c_k = \frac{1}{s} \bar{\lim} d(x_{m_{k-1}}, x_{n_{k-1}}) < \frac{s\varepsilon}{s} = \varepsilon. \tag{46}$$

Using Equations (45) and (46) together with Equations (24), (34), and (35) and applying the property of  $\lim \sup$ , it follows that

$$\bar{\lim} M(x_{m_{k-1}}, x_{n_{k-1}}) = \max \{\bar{\lim} a_k, \bar{\lim} b_k, \bar{\lim} c_k\} < \varepsilon. \tag{47}$$

Taking the lower limit in Equation (39) as  $k \rightarrow \infty$  and using Equation (25), we get the following:

$$\varepsilon \leq \underline{\lim} d(x_{m_k}, x_{n_k}) \leq \underline{\lim} (\alpha(x_{m_{k-1}}, x_{n_{k-1}}) d(x_{m_k}, x_{n_k})). \tag{48}$$

Taking the upper limit in Equation (40) and using Equations (37), (47), and (48), we get the following:

$$\begin{aligned}
 & 0 \leq \bar{\lim} \zeta(\alpha(x_{m_{k-1}}, x_{n_{k-1}}) d(x_{m_k}, x_{n_k}), M(x_{m_{k-1}}, x_{n_{k-1}}) + L \cdot m(x_{m_{k-1}}, x_{n_{k-1}})) \\
 & < \bar{\lim} (M(x_{m_{k-1}}, x_{n_{k-1}}) + L \cdot m(x_{m_{k-1}}, x_{n_{k-1}}) - \alpha(x_{m_{k-1}}, x_{n_{k-1}}) d(x_{m_k}, x_{n_k})) \\
 & \leq \bar{\lim} M(x_{m_{k-1}}, x_{n_{k-1}}) + L \cdot \bar{\lim} m(x_{m_{k-1}}, x_{n_{k-1}}) \\
 & \quad - \underline{\lim} (\alpha(x_{m_{k-1}}, x_{n_{k-1}}) d(x_{m_k}, x_{n_k})) < \varepsilon - \varepsilon = 0,
 \end{aligned} \tag{49}$$

which is a contradiction.

Hence,  $\{x_n\}$  is a Cauchy sequence in a complete b-metric space  $X$ . So, there exists  $u \in X$  such that

$$\lim_{n \rightarrow \infty} x_n = u. \tag{50}$$

Since  $T$  is continuous, we obtain the following:

$$u = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} Tx_n = T\left(\lim_{n \rightarrow \infty} x_n\right) = Tu. \tag{51}$$

Thus,  $u$  is a fixed point of  $T$ .

**Theorem 21.** Let  $(X, d, \leq)$  be a complete partially ordered b-metric space and  $T$  is an almost generalized  $\alpha$ -admissible  $Z$ -contraction with respect to  $\zeta$ . Suppose that

- (i)  $T$  is nondecreasing
- (ii) There exists  $x_0 \in X$  such that  $x_0 \leq Tx_0$  and  $\alpha(x_0, Tx_0) \geq 1$
- (iii)  $T$  is triangular  $\alpha$ -orbital admissible
- (iv) There exists a nondecreasing sequence  $\{x_n\}$  in  $X$  such that  $x_n \rightarrow x$  with  $x_n \leq x$  and  $\alpha(x_n, x) \geq 1$

Then,  $T$  has a fixed point.

*Proof.* Following the proof of Theorem 20, we know that the sequence  $\{x_n\}$  defined by  $x_{n+1} = Tx_n$  for all  $n \geq 0$  converges to some  $u \in X$ .

Hence, by (iii), we have the following:

$$\alpha(x_n, u) \geq 1, \tag{52}$$

for all  $n$ . □

By (i), there exists  $x_0 \in X$  with  $x_0 \leq Tx_0$  and since  $T$  is a nondecreasing map, we have the following:

$$x_0 \leq Tx_0 = x_1 \leq Tx_1 = x_2 \leq Tx_2 = x_3 \leq \dots \leq Tx_{n-1} = x_n \leq Tx_n = x_{n+1} \leq \dots \tag{53}$$

Thus,  $\{x_n\}$  is a nondecreasing sequence that converges to  $u$ .

It follows that  $x_n \leq u$ , for all  $n \in N$ .

Now, we show that  $u = Tu$ .

Applying Equation (9) with  $x = x_n$  and  $y = u$ , we get the following:

$$0 \leq \zeta(\alpha(x_n, u) d(x_{n+1}, Tu), M(x_n, u) + L \cdot m(x_n, u)), \tag{54}$$

where

$$M(x, y) = M(x_n, u) = \max \left\{ \frac{d(u, Tu)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, u)}, \frac{d(u, x_{n+1})[1 + d(x_n, Tu)]}{s^2 [1 + d(x_n, u)]}, \frac{d(x_n, u)}{s} \right\},$$

$$m(x, y) = m(x_n, u) = \min$$

$$\left\{ \frac{d(x_n, x_{n+1}) d(u, Tu)}{1 + d(x_n, u)}, \frac{d(x_n, Tu) d(u, x_{n+1})}{1 + d(x_n, u)} \right\}. \tag{55}$$

From Equation (54), it follows that

$$\alpha(x_n, u)d(x_{n+1}, Tu) < M(x_n, u) + L \bullet m(x_n, u). \quad (56)$$

Using Equations (52) and (56), we have the following:

$$d(x_{n+1}, Tu) \leq \alpha(x_n, u)d(x_{n+1}, Tu) < M(x_n, u) + L \bullet m(x_n, u). \quad (57)$$

Since  $\lim_{n \rightarrow \infty} d(x_{n+1}, Tu) = \lim_{n \rightarrow \infty} (M(x_n, u) + L \bullet m(x_n, u)) = d(u, Tu)$ , we have the following:

$$\lim_{n \rightarrow \infty} (\alpha(x_n, u)d(x_{n+1}, Tu)) = d(u, Tu). \quad (58)$$

If  $d(u, Tu) = 0$ , then  $u = Tu$ , then we are done. Assume  $d(u, Tu) \neq 0$ .

Letting  $q_n = \{\alpha(x_n, u)d(x_{n+1}, Tu)\}$  and  $p_n = \{M(x_n, u)\}$  and using  $\zeta_3$ , we obtain the following:

$$0 \leq \lim_{n \rightarrow \infty} \sup \zeta(q_n, p_n) < 0, \quad (59)$$

which is a contradiction. Thus, we have  $d(u, Tu) = 0$ ; that is,  $u = Tu$ .

**Theorem 22.** *In addition to the hypotheses of Theorem 20 or Theorem 21, suppose that for every  $x, y \in X$ , there exists  $u \in X$  such that  $u \preceq x$  and  $u \preceq y$  and  $\alpha(u, Tu) \geq 1$ . Then,  $T$  has a unique fixed point.*

*Proof.* Referring to Theorem 20 or Theorem 21, the sets of fixed points of  $T$  are nonempty. Now, we shall show the uniqueness of fixed point. To prove the uniqueness of the fixed point, assume that there exist  $z_1, z_2 \in X$  such that  $z_1 = Tz_1$  and  $z_2 = Tz_2$  with  $z_1 \neq z_2$ .

Assume that there exists  $u_0 \in X$  such that  $u_0 \preceq z_1$  and  $u_0 \preceq z_2$ , then as in the proof of Theorem 20, we define the sequence such that

$$u_{n+1} = Tu_n = T^{n+1}u_0, \quad (60)$$

for all  $n \geq 0$ . □

Using the hypotheses of Theorem 22, Equation (60), and proceeding inductively, we get the following:

$$\alpha(u_n, u_{n+1}) \geq 1. \quad (61)$$

Due to the monotone property of  $T$ , we have the following:

$$\begin{aligned} T^n u_0 &= u_n \preceq z_1 = T^n z_1, \\ T^n u_0 &= u_n \preceq z_2 = T^n z_2. \end{aligned} \quad (62)$$

Since  $z_1, z_2, u_n \in X$  for all  $n \geq 0$ , then  $z_1 = u_m$  for some positive integer  $m$ , and hence,  $z_1 = Tz_1 = Tu_m = u_{m+1}$  for all  $n \geq m$ . It follows that  $u_n \rightarrow z_1$  as  $n \rightarrow \infty$ .

Using Equation (61) and the fact that  $u_n \rightarrow z_1$  as  $n \rightarrow \infty$ , we have  $\alpha(u_n, z_1) \geq 1$  for all  $n \geq 0$ .

Now, suppose that  $z_1 \neq u_n$  for all  $n \geq 0$ , so  $u_n \preceq z_1$  for all  $n \geq 0$ , then  $d(u_n, z_1) \neq 0$  for all  $n \geq 0$ .

Applying Equation (9) with  $x = u_n$  and  $y = z_1$ , we get the following:

$$\begin{aligned} 0 &\leq \zeta(\alpha(u_n, z_1)d(Tu_n, Tz_1), M(u_n, z_1) + L \bullet m(u_n, z_1)) \\ &< M(u_n, z_1) + L \bullet m(u_n, z_1) - \alpha(u_n, z_1)d(Tu_n, Tz_1). \end{aligned} \quad (63)$$

It follows that

$$\alpha(u_n, z_1)d(Tu_n, Tz_1) < M(u_n, z_1) + L \bullet m(u_n, z_1). \quad (64)$$

Here,

$$\begin{aligned} M(u_n, z_1) &= \max \left\{ \frac{d(z_1, Tz_1)[1 + d(u_n, Tu_n)]}{1 + d(u_n, z_1)}, \right. \\ &\quad \left. \frac{d(z_1, Tu_n)[1 + d(u_n, Tz_1)]}{s^2[1 + d(u_n, z_1)]}, \frac{d(u_n, z_1)}{s} \right\} \\ &= \max \left\{ \frac{d(z_1, z_1)[1 + d(u_n, u_{n+1})]}{1 + d(u_n, z_1)}, \frac{d(z_1, u_{n+1})}{s^2}, \frac{d(u_n, z_1)}{s} \right\} \\ &= \max \left\{ \frac{d(z_1, u_{n+1})}{s^2}, \frac{d(u_n, z_1)}{s} \right\}. \end{aligned} \quad (65)$$

Moreover,

$$\begin{aligned} m(u_n, z_1) &= \min \left\{ \frac{d(z_1, Tz_1)d(u_n, Tu_n)}{1 + d(u_n, z_2)}, \frac{d(z_1, Tu_n)d(u_n, Tz_1)}{1 + d(u_n, z_2)} \right\} \\ &= \min \left\{ \frac{d(z_1, z_1)d(u_n, u_{n+1})}{1 + d(u_n, z_2)}, \frac{d(z_1, u_{n+1})d(u_n, z_1)}{1 + d(u_n, z_2)} \right\} \\ &= \max \left\{ 0, \frac{d(z_1, u_{n+1})d(u_n, z_1)}{1 + d(u_n, z_2)} \right\} = 0. \end{aligned} \quad (66)$$

If  $\max \{d(z_1, u_{n+1})/s^2, d(u_n, z_1)/s\} = d(z_1, u_{n+1})/s^2$ , then substituting the corresponding values of  $M(u_n, z_1)$  and  $m(u_n, z_1)$  in Equation (64) and using the fact that  $\alpha(u_n, z_1) \geq 1$ , it becomes the following:

$$\alpha(u_n, z_1)d(u_{n+1}, z_1) < \frac{d(z_1, u_{n+1})}{s^2} + L \bullet 0. \quad (67)$$

It follows that

$$d(u_{n+1}, z_1) \leq \alpha(u_n, z_1)d(u_{n+1}, z_1) < \frac{d(z_1, u_{n+1})}{s^2}, \quad (68)$$

which is a contradiction. Hence,  $\max \{d(z_1, u_{n+1})/s^2, d(u_n, z_1)/s\} = d(u_n, z_1)/s$ .

Again, substituting the corresponding values of  $M(u_n, z_1)$  and  $m(u_n, z_1)$  in Equation (64) and using the fact that  $\alpha(u_n, z_1) \geq 1$ , it becomes the following:

$$d(u_{n+1}, z_1) \leq \alpha(u_n, z_1)d(u_{n+1}, z_1) < \frac{d(u_n, z_1)}{s} + L \bullet 0 \leq d(u_n, z_1). \quad (69)$$

Hence, the sequence  $\{d(u_n, z_1)\}$  is nonincreasing and bounded below. Accordingly, there exists  $r \geq 0$  such that  $\lim_{n \rightarrow \infty} d(u_n, z_1) = r$ . Assume  $r > 0$ .

Using Equation (69), we get the following:

$$\lim_{n \rightarrow \infty} \alpha(u_n, z_1)d(u_{n+1}, z_1) = r. \tag{70}$$

Letting  $b_n = \{\alpha(u_n, z_1)d(u_{n+1}, z_1)\}$  and  $a_n = \{d(u_n, z_1)\}$ , and using  $\zeta_3$ , we obtain the following:

$$0 \leq \lim_{n \rightarrow \infty} \sup \zeta(b_n, a_n) < 0, \tag{71}$$

which is a contradiction.

Thus, we have  $\lim_{n \rightarrow \infty} d(u_n, z_1) = 0$ ; that is,  $u_n \rightarrow z_1$  as  $n \rightarrow \infty$ .

Similarly  $u_n \rightarrow z_2$  as  $n \rightarrow \infty$ .

Due to the uniqueness of the limit, it implies that  $z_1 = z_2$ . Thus,  $T$  has a unique fixed point.

Now, we give corollaries of our main theorem, Theorem 20.

If we take  $\alpha(x, y) = 1$  for all  $x, y \in X$  in Theorem 20, then we have the following result.

**Corollary 23.** *Let  $(X, d, \preceq)$  be a complete partially ordered b-metric space with parameter  $s \geq 1$ . Let  $T : X \rightarrow X$  be a map. Suppose there exists a simulation function  $\zeta$  such that*

$$\zeta(d(Tx, Ty), M(x, y) + L \bullet m(x, y)) \geq 0, \tag{72}$$

for all  $x, y \in X$  with  $x \preceq y$  where  $M(x, y)$  and  $m(x, y)$  are the same as in Theorem 20.

Also, assume that the following conditions hold;

- (i)  $T$  is nondecreasing and continuous
- (ii) There exists  $x_0 \in X$  such that  $x_0 \preceq Tx_0$

Then,  $T$  has a fixed point.

Similarly, we can deduce the following results.

**Corollary 24.** *Let  $(X, d, \preceq)$  be a complete partially ordered b-metric space with parameter  $s \geq 1$ . Let  $T : X \rightarrow X$  and  $\alpha : X \times X \rightarrow [0, \infty)$  be the maps. Suppose there exists a simulation function  $\zeta$  such that*

$$\zeta(\alpha(x, y)d(Tx, Ty), M(x, y)) \geq 0, \tag{73}$$

for all  $x, y \in X$  with  $x \preceq y$  where  $M(x, y)$  is defined in Theorem 20.

Also, assume that the following conditions hold;

- (i)  $T$  is nondecreasing and continuous
- (ii) There exists  $x_0 \in X$  such that  $x_0 \preceq Tx_0$  and  $\alpha(x_0, Tx_0) \geq 1$
- (iii)  $T$  is triangular  $\alpha$ -orbital admissible

Then,  $T$  has a fixed point.

**Corollary 25.** *Let  $(X, d, \preceq)$  be a complete partially ordered b-metric space with parameter  $s \geq 1$ . Let  $T : X \rightarrow X$  be a map. Suppose there exists a simulation function  $\zeta$  such that*

$$\zeta(d(Tx, Ty), M(x, y)) \geq 0, \tag{74}$$

for all  $x, y \in X$  with  $x \preceq y$  where  $M(x, y)$  is defined in Theorem 20.

Also, assume that the following conditions hold;

- (i)  $T$  is nondecreasing and continuous
- (ii) There exists  $x_0 \in X$  such that  $x_0 \preceq Tx_0$

Then,  $T$  has a fixed point.

Now, we provide an example in support of Theorem 20.

**Example 3.** Let  $X = \{0, 1, 2, 3, 4\}$  and let  $d : X \times X \rightarrow \mathbb{R}^+$  be defined by the following:

$$d(x, y) = |x - y|^2, \tag{75}$$

for all  $x, y \in X$ .

Hence,  $d$  is a complete b-metric space with parameter  $s = 2$ .

Define a partial order on  $X$  as follows:

$$\preceq : \{(0, 0), (0, 3), (0, 4), (1, 1), (2, 2), (2, 4), (3, 3), (3, 4), (4, 4)\}. \tag{76}$$

Then,  $(X, \preceq)$  is a partially ordered set.

We define the maps  $T : X \rightarrow X$  and  $\alpha : X \times X \rightarrow \mathbb{R}^+$  as follows:

$$\begin{aligned} T0 &= T1 = 2, \\ T2 &= 3, \\ T3 &= T4 = 4, \end{aligned} \tag{77}$$

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in A, \\ 0, & \text{if } (x, y) \in X \times X/A, \end{cases}$$

where

$$A = \{(0, 3), (0, 4), (2, 4), (3, 4), (4, 4)\}. \tag{78}$$

Clearly,  $T$  is a continuous, nondecreasing, and triangular  $\alpha$ -orbital admissible map.

Moreover, we choose  $x_0 = 3 \in X$ ; then,  $3 = x_0 \preceq Tx_0 = T3 = 4$  and  $\alpha(x_0, Tx_0) = \alpha(3, 4) = 1 \geq 1$ .

Now, we verify Equation (9) by choosing  $L = 10$  and  $\zeta(t, s) = s - \phi(s) - t$  for all  $t, s \in [0, \infty)$  where  $\phi(s) = s/2$ . For simplicity, let  $t = \alpha(x, y)d(Tx, Ty)$  and  $s = M(x, y) + Lm(x, y)$ .

The cases where we have  $x = y$  are trivial. So, we consider only the following three cases.



Case (i): When  $x = 0$  and  $y = 3$ .

$$t = \alpha(x, y)d(Tx, Ty) = \alpha(0, 3)d(T0, T3) = \alpha(0, 3)d(2, 4) = 4,$$

$$s = M(x, y) + Lm(x, y) = M(0, 3) + L \bullet m(0, 3) = \frac{9}{2} + 10 \bullet \frac{4}{10} = \frac{17}{2}.$$

(79)

Hence,  $\zeta(t, s) = \zeta(4, 17/2) = 17/2 - \phi(17/2) - 4 = 17/2 - 17/4 - 4 = 1/4 \geq 0$ .

Case (ii): When  $x = 0$  and  $y = 4$ .

$$t = \alpha(x, y)d(Tx, Ty) = \alpha(0, 4)d(2, 4) = 4,$$

$$s = M(x, y) + Lm(x, y) = M(0, 4) + L \bullet m(0, 4) = 8 + 20 \bullet 0 = 8.$$

(80)

Hence,  $\zeta(t, s) = s - \phi(s) - t \Rightarrow \zeta(4, 8) = 8 - \phi(8) - 4 = 8 - 4 - 4 = 0 \geq 0$ .

Case (iii): When  $x = 2$  and  $y = 4$ .

$$t = \alpha(x, y)d(Tx, Ty) = \alpha(2, 4)d(3, 4) = 1,$$

$$s = M(x, y) + Lm(x, y) = M(2, 4) + L \bullet m(2, 4) = 2 + 20 \bullet 0 = 2.$$

(81)

Hence,  $\zeta(t, s) = \zeta(1, 2) = 2 - \phi(2) - 1 = 0 \geq 0$ .

Case (iv): When  $x = 3$  and  $y = 4$ .

$$t = \alpha(x, y)d(Tx, Ty) = \alpha(3, 4)d(4, 4) = 0,$$

$$s = M(x, y) + Lm(x, y) = M(3, 4) + L \bullet m(3, 4) = \frac{1}{2} + 20 \bullet 0 = \frac{1}{2}.$$

(82)

Hence,  $\zeta(t, s) = \zeta(0, 1/2) = 1/2 - \phi(1/2) - 0 = 1/4 \geq 0$ .

From cases (i)-(iv) considered above,  $T$  satisfies Equation (9) and hence all the hypotheses of Theorem 20. Thus,  $T$  has a fixed point which is  $x = 4$ .

The following is an example in support of Corollary 24.

*Example 4.* Let  $X = [0, 4]$  and let  $d : X \times X \rightarrow \mathbb{R}^+$  is the same as in Example 3..

Define a partial order on  $X$  as follows:

$$\leq : \{(x, y) : x, y \in [0, 2), x = y\} \cup \{x, y \in [2, 4], x \leq y\}. \quad (83)$$

Then,  $(X, \leq)$  is a partially ordered set.

We define the following mappings:

$$T(x) = \begin{cases} x + 2, & \text{if } 0 \leq x < 1, \\ \frac{1}{2}(x + 5), & \text{if } 1 \leq x < 3, \\ 4, & \text{if } 3 \leq x \leq 4, \end{cases} \quad (84)$$

$$\alpha(x, y) = \begin{cases} 1, & \text{if } 2 \leq x \leq 4 \text{ and } y = 4, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly,  $T$  is a continuous, nondecreasing, and triangular  $\alpha$ -orbital admissible mapping.

We choose  $x_0 = 3 \in X$  and then  $3 = x_0 \leq Tx_0 = T3 = 4$  and  $\alpha(x_0, Tx_0) = \alpha(3, 4) = 1 \geq 1$ .

Now, we verify Equation (9) by choosing the simulation function:

$$\zeta(t, s) = s - \phi(s) - t, \quad (85)$$

for all  $t, s \in [0, \infty)$  where  $\phi(s) = (1/2)^* s$ .

*Case 1.*  $2 \leq x < 3$  and  $y = 4$ .

For simplicity, let  $t = \alpha(x, y)d(Tx, Ty)$  and  $s = M(x, y) + Lm(x, y)$ .

In this case,  $Tx \in [7/2, 4)$  and  $Ty = 4$ . Moreover,

$$t = \alpha(x, y)d(Tx, Ty) = \alpha(x, 4)d\left(\frac{1}{2}(x + 5), 4\right) = \frac{1}{4}(x - 3)^2,$$

$$M(x, y) = M(x, 4) = \left\{ \frac{1}{16}(x - 3)^2, \frac{1}{2}(x - 4)^2 \right\},$$

$$m(x, y) = m(x, 4) = 0.$$

(86)

Hence,

$$\zeta(t, s) = \zeta\left(\frac{1}{4}(x - 3)^2, \frac{1}{2}(x - 4)^2\right)$$

$$= \frac{1}{2}(x - 4)^2 - \frac{1}{4}(x - 4)^2 - \frac{1}{4}(x - 3)^2 \quad (87)$$

$$= \frac{7 - 2x}{4} \geq 0.$$

*Case 2.*  $3 \leq x \leq 4$  and  $y = 4$ .

In this case,  $Tx = 4$  and  $Ty = 4$ . Moreover,

$$t = \alpha(x, y)d(Tx, Ty) = \alpha(x, 4)d(4, 4) = 0,$$

$$M(x, y) = M(x, 4) = \frac{1}{2}(x - 4)^2, \quad (88)$$

$$m(x, y) = m(x, 4) = 0.$$

Hence,

$$\zeta(t, s) = \zeta\left(0, \frac{1}{2}(x - 4)^2\right) = \frac{1}{2}(x - 4)^2 - \frac{1}{4}(x - 4)^2 - 0$$

$$= \frac{1}{4}(x - 4)^2 \geq 0.$$

(89)

Therefore,  $T$  satisfies Equation (73) and hence all the assertions of Corollary 24. Thus,  $T$  has a unique fixed point, namely,  $x = 4$ .

#### 4. An Application to an Integral Equation

In this section, we give an application of our result to an integral equation.

Consider the following integral equation:

$$u(r) = v(r) + \theta \int_a^b H(r, z) f(z, u(z)) dz, \quad r \in I = [a, b], \quad (90)$$

where  $v : I \rightarrow \mathbb{R}$ ,  $H : I \times I \rightarrow \mathbb{R}$ , and  $f : I \times \mathbb{R} \rightarrow \mathbb{R}$  are given continuous functions and  $\theta$  is a constant such that  $\theta \geq 0$ .

Let  $X$  be the set of continuous real functions defined on  $[a, b]$ . Define the b-metric by the following:

$$d(u, v) = \sup_{r \in I} |u(r) - v(r)|^s, \quad (91)$$

for all  $x, y \in X$ . Consider  $s > 1$ . Then,  $(X, d, \leq)$  is a complete partially ordered b-metric space with the usual order.

Now, define a mapping  $T : X \rightarrow X$  by the following:

$$Tu(r) := v(r) + \theta \int_a^b H(r, z) f(z, u(z)) dz, \quad r \in I. \quad (92)$$

We will use the following assumptions to prove the existence of a solution of Equation (90):

- (a)  $\theta \leq 1/s$
- (b)  $\sup_{r \in I} \int_a^b H(r, z) dz \leq 1/b - a$
- (c) For all  $u, v \in \mathbb{R}$ ,  $|f(z, u) - f(z, v)| \leq |u - v|$
- (d) There exists a function  $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$  such that for  $r \in I$  and for all  $a, b \in X$  with  $\zeta(b, a) \geq 0$

The existence of a solution to Equation (90) is equivalent to the existence of a fixed point of  $T$ . Now, we prove the following result.

**Theorem 26.** *Under the assumptions (a)-(d), Equation (90) has a unique solution in  $X$ .*

*Proof.*

$$\begin{aligned} d(Tu_1, Tu_2) &= \sup_{r \in I} |Tu_1(r) - Tu_2(r)|^s \\ &= \sup_{r \in I} \left| v(r) + \theta \int_a^b H(r, z) f(z, u_1(z)) dz \right. \\ &\quad \left. - \left( v(r) + \theta \int_a^b H(r, z) f(z, u_2(z)) dz \right) \right|^s \\ &= \sup_{r \in I} \left| \theta \int_a^b H(r, z) [f(z, u_1(z)) - f(z, u_2(z))] dz \right|^s \\ &= |\theta|^s \sup_{r \in I} \left| \int_a^b H(r, z) [f(z, u_1(z)) - f(z, u_2(z))] dz \right|^s \\ &\leq |\theta|^s \sup_{r \in I} \left\{ \int_a^b H(r, z) dz \right\}^s \left( \int_a^b |f(z, u_1(z)) - f(z, u_2(z))| dz \right)^s \\ &\leq |\theta|^s \left( \frac{1}{b-a} \right)^s \sup_{r \in I} \left( \int_a^b (d(u_1, u_2))^{1/s} dz \right)^s \\ &= |\theta|^s d(u_1, u_2) \leq \frac{1}{s} d(u_1, u_2) \leq M(u_1, u_2) + L \bullet N(u_1, u_2), \end{aligned} \quad (93)$$

where  $M$  and  $L$  are defined as in Theorem 20. Hence,  $T$  satisfies all the conditions of Corollary 23. Therefore,  $T$  has a fixed point; that is, Equation (90) has a solution in  $X$ .  $\square$

## 5. Conclusion

In this paper, we introduce a new class of maps, namely, an almost generalized  $\alpha$ -admissible  $Z$ -contraction, and establish fixed point theorems. Moreover, we prove the existence and uniqueness of fixed points in the setting of partially ordered b-metric spaces with the help of simulation function. Our results unify several related results in the existing literature. The given results not only unify several existing results but also extend and improve them. Finally, we verify the established theorems by some examples and provide an application of our result to an integral equation.

## Data Availability

The authors confirm that the data supporting the findings of this study are available within this article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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