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Research Article

Fixed Point Theorems for Set-Valued $\mathscr{L}$-Contractions in Branciari Distance Spaces

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In this paper, the notion of set-valued $\mathscr{L}$-contractions is introduced, and a new fixed point theorem for such contractions is established. An example to illustrate main theorem is given.

1. Introduction and Preliminaries

Branciari [1] introduced a Branciari distance by replacing the triangle inequality in a metric with the rectangular inequality as follows.

A map $d : X \times X \longrightarrow [0,\infty)$, where X is a nonempty set, is said to be Branciari distance on X if and only if it satisfies the following conditions:

For all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y is as follows:

(d1) $d(x, y) = 0$ if and only if $x = y$

(d2) $d(x, y) = d(y, x)$

(d3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$

The pair (X, d) is called a Branciari distance space, whenever d is a Branciari distance on X.

In many papers, for example, [2–8], it is called generalized metric space, Branciari metric space, or rectangular metric space. However, these names do not reflect and indicate the meaning well of the notion of Branciari distance spaces because Branciari distance can not reduce to the standard metric. Further it is well known that a Branciari distance space (X, d) does not have a topology which is compatible with d (see [8]). For these reasons, we rename and use it as Branciari distance space.

Branciari [1] extended the Banach contraction principle to Branciari distance space.

After that, a lot of authors, for example, [2–15] and references therein, obtained fixed point results in such spaces. Jain et al. [16] obtained fixed point results in extended Branciari b-distance spaces [17] by defining the notion of certain contractive conditions, and they gave an application to nonlinear fractional differential equations.

Branciari [1] investigated the existence of fixed points with the following two conditions:

(i) The topology of a Branciari distance space is a Hausdorff topological space

(ii) Any Branciari distance is continuous in each coordinates

However, it is known that the above two conditions are not correct (see [14, 15]).

Sarma et al. [15] and Samet [14] (see also [3, 4, 8, 18]) show that Branciari distances have the following topological disadvantages.

(B1) A Branciari distance does not need to be continuous in each coordinates

(B2) A convergent sequence in Branciari distance spaces does not need to be Cauchy

(B3) The topology of a Branciari distance space does not need to be a Hausdorff topological space

(B4) An open ball does not need to be an open set

Note that it follows from (B3) that the uniqueness of limits can not be guaranteed.

In despite of the above toplogical feature, the existence of fixed points can be investigated without additional conditions such as continuity of Branciari distances or/and Hausdorffness of the topology of Branciari distance spaces. This is why researchers are interested in Branciari distance spaces.

Let $\theta : (0,\infty) \longrightarrow (1,\infty)$ be a function.

Consider the following conditions:

(θ1) θ is nondecreasing, i.e., $\theta(t_1) \leq \theta(t_2)$, whenever $t_1 \leq t_2$

(θ2) For any sequence $\{t_n\}$ of points in $(0, \infty)$

$$\lim_{n\longrightarrow\infty} \theta(t_n) = 1 \Longleftrightarrow \lim_{n\longrightarrow\infty} t_n = 0. \tag{1}$$

(θ3) There exist $r \in (0, 1)$ and $l \in (0,\infty)$ such that

$$\lim_{t\longrightarrow 0^+} \frac{\theta(t) - 1}{t^r} = l. \tag{2}$$

(θ4) θ is continuous

(θ5) θ is strictly increasing

(θ6) $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$

(θ7) For any finite sequence $\{t_k\}_{k=1}^{n} \subset (0,\infty)$

$$\theta\left(\sum_{k=1}^{n} t_k\right) \leq \prod_{k=1}^{n} \theta(t_k). \tag{3}$$

Jleli and Samet [19] introduced the notion of θ-contractions and generalized the Banach contraction principle in the setting of Branciari distance spaces, where $\theta : (0,\infty) \longrightarrow (1,\infty)$ is a function such that (θ1), (θ2), and (θ3) are satisfied.

Since then, Jleli et al. [7] obtained a generalization of result of [19] with conditions (θ1), (θ2), (θ3), and (θ4). Arshad et al. [20] extended the result of [19] by using the notion of α-orbital admissible mapping with conditions (θ 1), (θ2), and (θ3). Also, Ahmad et al. [21] extended the result of Jleli and Samet [19] to metric spaces by using conditions (θ1), (θ2), and (θ4). Durmaz and Altun [22] obtained a generalization of the result of Klim and Wardowski [23] by defining the concept of set-valued θ-contractions with control function θ satisfying (θ1), (θ2), (θ3), and (θ6). Abdeljawad et al. [17] introduced the concept of extended Branciari b-distance, and they extended the result of Jleli and Samet [19] to extended Branciari b-distance spaces with control function θ satisfying (θ2) and (θ3). Cho [24] introduced the notion of generalized set-valued weak θ-contractions in metric spaces and obtained fixed point results for such contractions with control function θ satisfying (θ1), (θ2), (θ4), and (θ7).

Recently, Cho [25] introduced the concept of $\mathscr{L}$-contractions in Branciari distance spaces and established a fixed point theorem for such contractions. He unified concepts of some contractions which exist in literature including θ-contractions.

Very recently, Saleh et al. [26] extended the result of Cho [25] by introducing the concepts of generalized $\mathscr{L}$-contractions in Branciari distance spaces. Aydi et al. [27] extended the result of Cho [25] to partial metric spaces.

In the paper, we introduce notions of set-valued $\mathscr{L}$-contractions and set-valued $\mathscr{L}_*$-contractions in Branciari distance spaces and prove the existence of fixed points for both type of contractions.

Khojasteh et al. [28] introduced the notion of $\mathscr{Z}$-contractions by using the concept of simulation functions and unified the some existing metric fixed point results. The authors of [29–32] gave generalizations of simulation functions and obtained generalizations of results of [28]. Moreover, Demma et al. [33] and Yamaod and Sintunavarat [34] extended the results of [28] to b-metric spaces by using the notion of b-simulation functions and s-simulation functions, respectively.

Let $\zeta : [0,\infty) \times [0,\infty) \longrightarrow R$ be a function. Consider the following conditions:

(ζ1) $\xi(0, 0) = 0$

(ζ2) $\xi(t, s) < s - t \forall s, t > 0$

($\zeta 2'$) $\xi(t, s) < \psi(s) - \psi(t) \forall s, t > 0$, where $\psi : [0,\infty) \longrightarrow [0,\infty)$ is continuous and strictly increasing function with $\psi(0) = 0$

(ζ3) For any sequence $\{t_n\}, \{s_n\} \subset (0,\infty)$

$$\lim_{n\longrightarrow\infty} t_n = \lim_{n\longrightarrow\infty} s_n > 0 \Longrightarrow \limsup_{n\longrightarrow\infty} \zeta(t_n, s_n) < 0. \tag{4}$$

(ζ4) For any sequence $\{t_n\}, \{s_n\} \subset (0,\infty)$ with $t_n < s_n \forall n = 1, 2, 3, \cdots$

$$\lim_{n\longrightarrow\infty} t_n = \lim_{n\longrightarrow\infty} s_n > 0 \Longrightarrow \limsup_{n\longrightarrow\infty} \zeta(t_n, s_n) < 0. \tag{5}$$

(ζ5) If for any sequence $\{t_n\}, \{s_n\} \subset (0,\infty)$

$$0 < \lim_{n\longrightarrow\infty} t_n \leq \liminf_{n\longrightarrow\infty} s_n \leq \limsup_{n\longrightarrow\infty} s_n \leq b \lim_{n\longrightarrow\infty} t_n < +\infty, \tag{6}$$

where $b \geq 1$, then we have

$$\limsup_{n\longrightarrow\infty} \zeta(bt_n, s_n) < 0. \tag{7}$$

(ζ6) If for any sequence $\{t_n\}, \{s_n\} \subset (0,\infty)$

$$\begin{aligned} &0 < \liminf_{n\longrightarrow\infty} t_n \leq s \limsup_{n\longrightarrow\infty} s_n \leq s^2 \liminf_{n\longrightarrow\infty} t_n, \\ &0 < \liminf_{n\longrightarrow\infty} s_n \leq s \limsup_{n\longrightarrow\infty} t_n \leq s^2 \liminf_{n\longrightarrow\infty} s_n, \end{aligned} \tag{8}$$

where $s \geq 1$, then we have

$$\limsup_{n\longrightarrow\infty} \zeta(t_n, s_n) < 0. \tag{9}$$

Let $\zeta : [0,\infty) \times [0,\infty) \longrightarrow R$ be a function.
Then, we say that

(1) $\zeta : [0,\infty) \times [0,\infty) \longrightarrow R$ is called a simulation function in the sense of Khojasteh et al. [28] if and only if (ζ1), (ζ2), and (ζ3) hold

(2) $\zeta : [0,\infty) \times [0,\infty) \longrightarrow R$ is called a simulation function in the sense of Argoubi et al. [30] if and only if (ζ2) and (ζ3) hold

(3) $\zeta : [0,\infty) \times [0,\infty) \longrightarrow R$ is called a simulation function in the sense of Roldán-López-de-Hierro et al. [32] if and only if (ζ1), (ζ2), and (ζ4) hold

(4) $\zeta : [0,\infty) \times [0,\infty) \longrightarrow R$ is called a simulation function in the sense of Isik et al. [31] if and only if (ζ2) and (ζ4) hold

(5) $\zeta : [0,\infty) \times [0,\infty) \longrightarrow R$ is called a ψ-simulation function [29] if and only if ($\zeta 2'$) and (ζ3) hold

(6) $\zeta : [0,\infty) \times [0,\infty) \longrightarrow R$ is called a b-simulation function [33] if and only if (ζ2) and (ζ5) hold

(7) $\zeta : [0,\infty) \times [0,\infty) \longrightarrow R$ is calld a s -simulation function [34] if and only if (ζ2) and (ζ6) hold

Denote $\mathcal{Z}_K$(resp., $\mathcal{Z}_A$, $\mathcal{Z}_\psi$) by the family of all simulation functions in the sense of Khojasteh et al. (resp., all simulation functions in the sense of Argoubi et al., all ψ -simulation functions).

Note that every simulation function in the sense of Argoubi et al. is a ψ-simulation function. In fact, let $\zeta \in \mathcal{Z}_A$. If we take $\psi(t) = t, \forall t \geq 0$, then $\zeta \in \mathcal{Z}_\psi$.

Proposition 1. *The following are satisfied.*

(1) $\mathcal{Z}_A \subset \mathcal{Z}_\psi$ and $\mathcal{Z}_A \neq \mathcal{Z}_\psi$ ([29])

(2) $\mathcal{Z}_K \subset \mathcal{Z}_A$ and $\mathcal{Z}_K \neq \mathcal{Z}_A$ ([30])

Denote $\mathcal{Z}_R$ (resp., $\mathcal{Z}_I$) by the class of all simulations functions in the sense of Roldán-López-de-Hierro et al. (resp., in the sense of Isik et al.). Also, we denote by $\mathcal{Z}_b$ and $\mathcal{Z}_s$ the set of all b-simulation functions and s-simulation functions, respectively.

Example 1 (see [28, 32, 35, 36]). Let $\zeta_i : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}, i = 1, 2, 3, 4, 5, 6, 7$ be a function defined as follows:

(1) $\zeta_1(t, s) = ks - t, \forall s, t \geq 0$, where $0 \leq k < 1$

(2) $\zeta_2(t, s) = \phi(s) - t, \forall s, t \geq 0$, where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a function such that $\phi(s) < s, \forall s > 0$ and $\limsup_{t\longrightarrow s}\phi(t) < s$

(3) $\zeta_3(t, s) = s\phi(s) - t, \forall s, t \geq 0$, where $\phi : [0,\infty) \longrightarrow [0, 1)$ is a function such that $\limsup_{t\longrightarrow s^+} \phi(t) < 1, \forall s > 0$

(4) $\zeta_4(t, s) = s - \phi(s) - t, \forall s, t \geq 0$, where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a function such that $\forall s > 0, \liminf_{t\longrightarrow s} \phi(t) > 0$ or $\phi : [0,\infty) \longrightarrow [0,\infty)$ is continuous such that $\phi(t) = 0$ if and only if $t = 0$

(5) $\zeta_5(t, s) = \psi(s) - \varphi(t), \forall t, s \geq 0$ where $\psi, \varphi : [0,\infty) \longrightarrow [0,\infty)$ are continuous functions such that $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$, $\psi(t) < t \leq \varphi(t), \forall t > 0$ and φ is increasing

(6) $\zeta_6(t, s) = \eta(s) - t, \forall s, t \geq 0$, where $\eta : [0,\infty) \longrightarrow [0,\infty)$ is upper semicontinuous with $\eta(t) < t, \forall t > 0$ and $\eta(t) = 0$ if and only if $t = 0$

(7) $\zeta_7(t, s) = s - \int_0^t \phi(u)du, \forall s, t \geq 0$, where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a function such that for each $t > 0$, $\int_0^t \phi(u)du$ exists and $\int_0^t \phi(u)du > t$

Then, $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7 \in \mathcal{Z}_K$.

Example 2. Let $\zeta : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ be a function defined by

$$\zeta(t, s) = s - t - 1. \tag{10}$$

Then, (ζ2) is satisfied.

Let $\{t_n\}, \{s_n\}$ be sequnces of points in $(0, \infty)$ such that

$$t_n < s_n \forall n = 1, 2, 3, \cdots \text{and} \lim_{n\longrightarrow\infty} t_n = \lim_{n\longrightarrow\infty} s_n > 0. \tag{11}$$

Then, we have

$$\begin{aligned} \limsup_{n\longrightarrow\infty} \zeta(t_n, s_n) &= \limsup_{n\longrightarrow\infty} (s_n - t_n - 1) \\ &= \limsup_{n\longrightarrow\infty} s_n - \liminf_{n\longrightarrow\infty} t_n - 1 < 0. \end{aligned} \tag{12}$$

Thus, $\zeta \in \mathcal{Z}_I$, but $\zeta \notin \mathcal{Z}_R$, because $\zeta(0, 0) \neq 0$. Hence, $\mathcal{Z}_I \neq \mathcal{Z}_R$.

Example 3 (see [31]). Let $\zeta_I : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ be a function defined as

$$\zeta_I(t, s) = \begin{cases} 1 & \text{if}\,(s, t) = (0, 0)\,\text{or}\, s = t, \\ 2(s - t) & \text{if}\, s < t, \\ \lambda s - t & \text{otherwise}, \end{cases} \tag{13}$$

where $0 < \lambda < 0$.

Then, $\zeta_I \in \mathcal{Z}_I$, but $\zeta_I \notin \mathcal{Z}_A$. Hence, $\mathcal{Z}_A \neq \mathcal{Z}_I$. Also, we know that $\zeta_I \notin \mathcal{Z}_K$ and $\zeta_I \notin \mathcal{Z}_R$.

Note that $\mathcal{Z}_K \subset \mathcal{Z}_R$. The following examples show that $\mathcal{Z}_K \neq \mathcal{Z}_R$.

Example 4. Let $\zeta : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ be a function defined as

$$\zeta(t, s) = \begin{cases} 0 & \text{if}\,(s, t) = (0, 0), \\ -1 & \text{if}\, s = t, \\ 2(s - t) & \text{if}\, s < t, \\ \lambda s - t & \text{otherwise}, \end{cases} \tag{14}$$

where $0 < \lambda < 0$. Then, (ζ1) and (ζ2) are satisfied.

We show that (ζ5) is not satisfied.

To show this, let $\{t_n\}, \{s_n\} \subset (0,\infty)$ be two sequences such that

$$\lim_{n\longrightarrow\infty} t_n = \lim_{n\longrightarrow\infty} s_n > 0. \quad (15)$$

We may assume that $s_n < t_n, \forall n = 1, 2, 3, \cdots$.
Then

$$\lim_{n\longrightarrow\infty} \sup \zeta(t_n, s_n) = 0. \quad (16)$$

Hence, (ζ5) is not satisfied. Thus, $\zeta \notin \mathscr{Z}_K$.
We now show that (ζ6) is satisfied.
Let $\{a_n\}, \{b_n\} \subset (0,\infty)$ be two sequences such that

$$a_n < b_n \forall n = 1, 2, 3, \cdots \text{and} \lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} b_n = c > 0. \quad (17)$$

Then

$$\lim_{n\longrightarrow\infty} \sup \zeta(a_n, b_n) = \lambda \lim_{n\longrightarrow\infty} \sup b_n - \lim_{n\longrightarrow\infty} \inf a_n = \lambda c - c < 0. \quad (18)$$

Hence, (ζ6) holds. Thus, $\zeta \in \mathscr{Z}_R$. Therefore, $\mathscr{Z}_R \neq \mathscr{Z}_K$.

Proposition 2. *The following inclusion relations are satisfied.*

(1) $\mathscr{Z}_K \subsetneq \mathscr{Z}_A \subsetneq \mathscr{Z}_I$

(2) $\mathscr{Z}_K \subsetneq \mathscr{Z}_A \subsetneq \mathscr{Z}_\psi$

(3) $\mathscr{Z}_K \subsetneq \mathscr{Z}_R \subsetneq \mathscr{Z}_I$

Proposition 3. *If $\zeta \in \mathscr{Z}_s$ is decreasing in the first coordinate, then $\zeta \in \mathscr{Z}_b$.*

Proof. Let $\zeta \in \mathscr{Z}_s$. Then, (ζ2) and (ζ6) hold.
Assume that $\zeta \in \mathscr{Z}_s$ is decreasing in the first coordinate.
We show that (ζ5) holds.
Let $\{t_n\}, \{s_n\} \subset (0,\infty)$ be two sequences such that

$$0 < \lim_{n\longrightarrow\infty} t_n \leq \lim_{n\longrightarrow\infty} \inf s_n \leq \lim_{n\longrightarrow\infty} \sup s_n \leq b \lim_{n\longrightarrow\infty} t_n < +\infty, \quad (19)$$

where $b > 1$.
From (19), we infer that

$$\begin{aligned} 0 < \lim_{n\longrightarrow\infty} \inf t_n = \lim_{n\longrightarrow\infty} t_n \leq b \lim_{n\longrightarrow\infty} \inf s_n \leq b \lim_{n\longrightarrow\infty} \sup s_n \\ \leq b^2 \lim t_n = b^2 \lim_{n\longrightarrow\infty} \inf t_n, \end{aligned} \quad (20)$$

which implies

$$0 < \lim_{n\longrightarrow\infty} \inf t_n \leq b \lim_{n\longrightarrow\infty} \sup s_n \leq b^2 \lim_{n\longrightarrow\infty} \inf t_n. \quad (21)$$

It follows from (19) that

$$\begin{aligned} 0 < \lim t_n \leq \lim_{n\longrightarrow\infty} \inf s_n \leq b \lim_{n\longrightarrow\infty} \sup t_n, \\ b \lim t_n = b \lim_{n\longrightarrow\infty} \inf t_n \leq b^2 \lim_{n\longrightarrow\infty} \inf s_n. \end{aligned} \quad (22)$$

Thus, we obtain

$$0 < \lim_{n\longrightarrow\infty} \inf s_n \leq b \lim_{n\longrightarrow\infty} \sup t_n \leq b^2 \lim_{n\longrightarrow\infty} \inf s_n. \quad (23)$$

By applying (21) and (23) to (ζ6), we have

$$\lim_{n\longrightarrow\infty} \sup \zeta(t_n, s_n) < 0. \quad (24)$$

Since ζ is decreasing in the first coordinate,

$$\lim_{n\longrightarrow\infty} \sup \zeta(bt_n, s_n) \leq \lim_{n\longrightarrow\infty} \sup \zeta(t_n, s_n) < 0. \quad (25)$$

Hence, it follows from (19) and (25) that (ζ5) is satisfied. Thus, $\zeta \in \mathscr{Z}_b$. □

Note that simulation functions $\zeta_i \in \mathscr{Z}_K, i = 1, 2, 3, 4, 5, 6, 7$, given in Example 1 are s-simulation and decreasing functions in the first coordinate. Hence, $\zeta_i \in \mathscr{Z}_b, i = 1, 2, 3, 4, 5, 6, 7$ (see also [33]).

For more details and examples of simulation functions, we refer to [28, 33, 34, 36–40], and for ψ-simulation functions, we refer to [29, 41, 42].

Now, we recall the concept of $\mathscr{L}$-simulation function and give the definition of $\mathscr{L}_*$-simulation function.

Let $\xi : [1,\infty) \times [1,\infty) \longrightarrow R$ be a function. Consider the following conditions:

(ξ1) $\xi(\cdot, \cdot)$ is a decreasing function on the first coordinate

(ξ2) $\xi(1, 1) = 1$

(ξ3) $\xi(rt, s) = (1/r)\xi(t, s), \forall r > 1$

(ξ4) $\xi(t, s) < (s/t) \forall s, t > 1$

(ξ5) For any sequence $\{t_n\}, \{s_n\} \subset (1,\infty)$ with $t_n \leq s_n \forall n = 1, 2, 3, \cdots$

$$\lim_{n\longrightarrow\infty} t_n = \lim_{n\longrightarrow\infty} s_n > 1 \Longrightarrow \lim_{n\longrightarrow\infty} \sup \xi(t_n, s_n) < 1. \quad (26)$$

Let $\xi : [1,\infty) \times [1,\infty) \longrightarrow R$ be a function.
Then, we say that

(1) ξ is $\mathscr{L}$ -simulation function [25] if and only if it satisfies conditions (ξ2), (ξ4), and (ξ5)

(2) ξ is $\mathscr{L}_*$ -simulation function if and only if it satisfies conditions (ξ1), (ξ2), (ξ3), (ξ4), and (ξ5)

Denote $\mathscr{L}_*$ by the family of all $\mathscr{L}_*$-simulation functions, and $\mathscr{L}$ by the class of all $\mathscr{L}$-simulation functions.

Note that $\xi(t, t) < 1 \forall t > 1$, and $\mathscr{L}_* \subset \mathscr{L}$.

Example 5 (see [25]). Let $\xi_b, \xi_w, \xi : [1,\infty) \times [1,\infty) \longrightarrow \mathbb{R}$ be functions defined as follows, respectively:

(1) $\xi_b(t,s) = (s^k/t) \forall t, s \geq 1$ where $k \in (0,1)$

(2) $\xi_w(t,s) = (s/t\phi(s)) \forall t, s \geq 1$ where $\phi : [1,\infty) \longrightarrow [1,\infty)$ is nondecreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$

$$\xi(t,s) \begin{cases} 1 & \text{if } (s,t) = (1,1), \\ \dfrac{s}{2t} & \text{if} s < t, \\ \dfrac{s^\lambda}{t} & \text{otherwise,} \end{cases} \quad (27)$$

$\forall s, t \geq 1$, where $\lambda \in (0,1)$.

Then, $\xi_b, \xi_w, \xi \in \mathscr{L}_*$.

Example 6. Let $\xi_i : [1,\infty) \times [1,\infty) \longrightarrow \mathbb{R}, i = 1, 2, 3,$ be a function defined as follows:

(1) $\xi_1(t,s) = (\psi(s)/\varphi(t)), \forall t, s \geq 1$ where $\psi, \varphi : [1,\infty) \longrightarrow [1,\infty)$ are continuous functions such that $\psi(t) = \varphi(t) = 1$ if and only if $t = 1$, $\psi(t) < t \leq \varphi(t), \forall t > 1$ and φ is an increasing function

(2) $\xi_2(t,s) = (\eta(s)/t), \forall s, t \geq 1,$ where $\eta : [1,\infty) \longrightarrow [1,\infty)$ is upper semicontinuous with $\eta(t) < t, \forall t > 1$ and $\eta(t) = 1$ if and only if $t = 1$

(3) $\xi_3(t,s) = (s/\int_0^t \phi(u)du), \forall s, t \geq 1,$ where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a function such that for each $t > 0$, $\int_0^t \phi(u)du$ exists and $\int_0^t \phi(u)du > t$, and $\int_0^1 \phi(u)du = 1$

Then, $\xi_1, \xi_3 \in \mathscr{L}$, and $\xi_2 \in \mathscr{L}_*$.

Note that if $\varphi : [1,\infty) \longrightarrow [1,\infty)$ is satisfied condition

$$\varphi(rt) = r\varphi(t) \forall r > 1, t \geq 1, \quad (28)$$

then $\xi_1 \in \mathscr{L}_*$.

We recall the following definitions which are in [1].

Let (X,d) be a Branciari distance space, $\{x_n\} \subset X$ be a sequence, and $x \in X$.

Then, we say that

(1) $\{x_n\}$ is convergent to x (denoted by $\lim_{n\longrightarrow\infty} x_n = x$) if and only if $\lim_{n\longrightarrow\infty} d(x, x_n) = 0$

(2) $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\longrightarrow\infty} d(x_n, x_m) = 0$

(3) (X,d) is complete if and only if every Cauchy sequence in X is convergent to some point in X

Let (X,d) be a Branciari distance space.

We denote by $CL(X)$ the class of nonempty closed subsets of X. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $CL(X)$, i.e., for all $A, B \in CL(X)$,

$$H(A,B) = \begin{cases} \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}, & \text{if the maximum exists,} \\ \infty, & \text{otherwise,} \end{cases} \quad (29)$$

where $d(a,B) = \inf\{d(a,b) : b \in B\}$ is the distance from the point a to the subset B.

For $A, B \in CL(X)$, let $D(A,B) = \sup_{x\in A} d(x,B)$.

Then, we have $D(A,B) \leq H(A,B)$ for all $A, B \in CL(X)$.

Lemma 4 (see [43]). *Let (X,d) be a Branciari distance space, $\{x_n\} \subset X$ be a Cauchy sequence, and $x, y \in X$. If there exists a positive integer N such that*

(1) $x_n \neq x_m \forall n, m > N$

(2) $x_n \neq x \forall n > N$

(3) $x_n \neq y \forall n > N$

(4) $\lim_{n\longrightarrow\infty} d(x_n, x) = 0 = \lim_{n\longrightarrow\infty} d(x_n, y)$

then $x = y$.

Lemma 5. *If (X,d) is a Branciari distance space, then $C(X) \subset CL(X)$, where $C(X)$ is the class of nonempty compact subsets of X.*

Proof. Let $A \in C(X)$, and let $\{x_n\} \subset A$ be a sequence such that

$$\lim_{n\longrightarrow\infty} d(x_n, x) = 0, \text{where } x \in X. \quad (30)$$

It follows from compactness of A that there exists a convergent subsequence $\{x_{n(k)}\}$ of $\{x_n\}$.

Let

$$\lim_{n\longrightarrow\infty} d\left(x_{n(k)}, a\right) = 0, \text{for } a \in A. \quad (31)$$

Since

$$\lim_{n\longrightarrow\infty} d\left(x_{n(k)}, x\right) = 0, \quad (32)$$

from Lemma 4, $x = a \in A$. Hence, $A \in CL(X)$. □

Lemma 6. *Let (X,d) be a Branciari distance space, and let $A, B \in CL(X)$.*

If $a \in A$ and $d(a,B) < c$, then there exists $b \in B$ such that $d(a,b) < c$.

Proof. Let $\varepsilon = c - d(a,B)$.

It follows from definition of infimum that there exists $b \in B$ such that $d(a,b) < d(a,B) + \varepsilon$. Hence, $d(a,b) < c$. □

2. Fixed Point Theorems

We denote by Θ the class of all functions $\theta : (0,\infty) \longrightarrow (1,\infty)$ such that conditions $(\theta2)$, $(\theta4)$, and $(\theta5)$ hold.

Let (X, d) be a Branciari distance space.

A set-valued mapping $T : X \longrightarrow 2^X$, where 2^X is the family of all nonempty subsets of X, is called set-valued $\mathscr{L}$-contraction with respect to $\xi \in \mathscr{L}$ if and only if for all $x, y \in X$ with $d(x, y) > 0$, and for all $u \in Tx$, there exists $v \in Ty$ with $d(u, v) > 0$ such that

$$\xi(\theta(d(u, v)), \theta(d(x, y))) \geq 1, \tag{33}$$

where $\theta \in \Theta$.

Now, we prove our main result.

Theorem 7. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set-valued $\mathscr{L}$-contraction with respect to $\xi \in \mathscr{L}$.*

Then, T has a fixed point.

Proof. Let $x_0 \in X$ be a point, and let $x_1 \in Tx_0$ be such that $d(x_0, x_1) > 0$.

From (33), there exists $x_2 \in Tx_1$ with $d(x_1, x_2) > 0$ such that

$$1 \leq \xi(\theta(d(x_1, x_2)), \theta(d(x_0, x_1))) < \frac{\theta(d(x_0, x_1))}{\theta(d(x_1, x_2))}, \tag{34}$$

which implies

$$\theta(d(x_1, x_2)) < \theta(d(x_0, x_1)), \tag{35}$$

and so

$$d(x_1, x_2) < d(x_0, x_1). \tag{36}$$

Again, from (33), there exists $x_3 \in Tx_2$ with $d(x_2, x_3) > 0$ such that

$$1 \leq \xi(\theta(d(x_2, x_3)), \theta(d(x_1, x_2))) < \frac{\theta(d(x_1, x_2))}{\theta(d(x_2, x_3))}, \tag{37}$$

which implies

$$\theta(d(x_2, x_3)) < \theta(d(x_1, x_2)), \tag{38}$$

and so

$$d(x_2, x_3) < d(x_1, x_2). \tag{39}$$

Inductively, we can find a sequence $\{x_n\} \subset X$ such that, $\forall n = 1, 2, 3, \cdots$,

$$x_{n-1} \neq x_n, x_n \in Tx_{n-1} \text{ and } d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{40}$$

Since $\{d(x_{n-1}, x_n)\}$ is a decreasing sequence, there exists $r \geq 0$ such that

$$\lim_{n \longrightarrow \infty} d(x_{n-1}, x_n) = r. \tag{41}$$

We now show that $r = 0$.

Assume that $r \neq 0$.

Then, it follows from ($\theta 2$) that

$$\lim_{n \longrightarrow \infty} \theta(d(x_{n-1}, x_n)) \neq 1, \text{and so} \lim_{n \longrightarrow \infty} \theta(d(x_{n-1}, x_n)) > 1. \tag{42}$$

Let $t_n = \theta(d(x_n, x_{n+1}))$ and $s_n = \theta(d(x_{n-1}, x_n)) \forall n = 1, 2, 3, \cdots$.

Then, $t_n \leq s_n \forall n = 1, 2, 3, \cdots$ and

$$\lim_{n \longrightarrow \infty} t_n = \lim_{n \longrightarrow \infty} s_n > 1. \tag{43}$$

It follows from ($\xi 5$) that

$$1 \leq \limsup_{n \longrightarrow \infty} \xi(t_n, s_n) < 1, \tag{44}$$

which is a contradiction.

Thus, we have

$$\lim_{n \longrightarrow \infty} d(x_{n-1}, x_n) = 0, \tag{45}$$

and so

$$\lim_{n \longrightarrow \infty} \theta(d(x_{n-1}, x_n)) = 1. \tag{46}$$

We now show that $\{x_n\}$ is a Cauchy sequence.

On the contrary, assume that $\{x_n\}$ is not a Cauchy sequence.

Then, there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is the smallest index for which $m(k) > n(k) > k \forall k = 1, 2, 3, \cdots$

$$d\left(x_{m(k)}, x_{n(k)}\right) \geq \varepsilon \text{ and } d\left(x_{m(k)-1}, x_{n(k)}\right) < \varepsilon. \tag{47}$$

It follows from (47) and condition (d3) that

$$\begin{aligned} \varepsilon \leq d\left(x_{m(k)}, x_{n(k)}\right) &\leq d\left(x_{n(k)}, x_{m(k)-2}\right) \\ &+ d\left(x_{m(k)-2}, x_{m(k)-1}\right) + d\left(x_{m(k)-1}, x_{m(k)}\right) \\ &< \varepsilon + d\left(x_{m(k)-2}, x_{m(k)-1}\right) + d\left(x_{m(k)-1}, x_{m(k)}\right). \end{aligned} \tag{48}$$

Letting $k \longrightarrow \infty$ in above inequality, we have

$$\lim_{k \longrightarrow \infty} d\left(x_{m(k)}, x_{n(k)}\right) = \varepsilon = \lim_{k \longrightarrow \infty} d\left(x_{m(k)-2}, x_{n(k)}\right). \tag{49}$$

From (33), there exists $z_{n(k)+1} \in Tx_{n(k)}$ with $d(x_{m(k)+1}, z_{n(k)+1}) > 0 \forall k = 1, 2, 3, \cdots$ such that

$$1 \leq \xi\left(\theta\left(d\left(x_{m(k)+1}, z_{n(k)+1}\right)\right), \theta\left(d\left(x_{m(k)}, x_{n(k)}\right)\right)\right) < \frac{\theta\left(d\left(x_{m(k)}, x_{n(k)}\right)\right)}{\theta\left(d\left(x_{m(k)+1}, z_{n(k)+1}\right)\right)}, \tag{50}$$

which implies

$$\theta\left(d\left(x_{m(k)+1}, z_{n(k)+1}\right)\right) < \theta\left(d\left(x_{m(k)}, x_{n(k)}\right)\right), \forall k = 1, 2, 3, \cdots. \tag{51}$$

So

$$d\left(x_{m(k)+1}, z_{n(k)+1}\right) < d\left(x_{m(k)}, x_{n(k)}\right), \forall k = 1, 2, 3, \cdots. \tag{52}$$

Taking limit supremum in above inequality and using (49), we have

$$\limsup_{n\longrightarrow\infty} d\left(x_{m(k)+1}, z_{n(k)+1}\right) \leq \varepsilon. \tag{53}$$

We deduce that

$$\begin{aligned} d\left(x_{m(k)}, x_{n(k)}\right) &\leq d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{m(k)+1}, Tx_{n(k)}\right) + d\left(Tx_{n(k)}, x_{n(k)}\right) \\ &\leq d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{m(k)+1}, z_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right). \end{aligned} \tag{54}$$

Taking limit infimum in above inequality and using (53), we have

$$\varepsilon \leq \liminf_{n\longrightarrow\infty} d\left(x_{m(k)+1}, z_{n(k)+1}\right). \tag{55}$$

It follows from (53) and (55) that

$$\lim_{n\longrightarrow\infty} d\left(x_{m(k)+1}, z_{n(k)+1}\right) = \varepsilon. \tag{56}$$

Let

$$t_k = \theta\left(d\left(x_{m(k)+1}, z_{n(k)+1}\right)\right) \text{ and } s_k = \theta\left(d\left(x_{m(k)}, x_{n(k)}\right)\right). \tag{57}$$

Then, $t_k \leq s_k \forall k = 1, 2, 3, \cdots$.

It follows from (49), (56), (θ2), and (θ4) that

$$\lim_{k\longrightarrow\infty} s_k = \lim_{k\longrightarrow\infty} t_k > 1. \tag{58}$$

It follows from (ξ5) that

$$1 \leq \lim_{k\longrightarrow\infty} \sup \xi(t_k, s_k) < 1, \tag{59}$$

which is a contradiction.

Thus, $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists a point $x_* \in X$ such that

$$\lim_{n\longrightarrow\infty} d(x_*, x_n) = 0. \tag{60}$$

It follows from (33) that there exists $y_n \in Tx_*$ with $d(x_{n+1}, y_n) > 0$ such that

$$1 \leq \xi(\theta(d(x_{n+1}, y_n)), \theta(d(x_n, x_*))) < \frac{\theta(d(x_n, x_*))}{\theta(d(x_{n+1}, y_n))}, \tag{61}$$

which implies

$$\theta(d(x_{n+1}, y_n)) < \theta(d(x_n, x_*)), \tag{62}$$

and hence

$$d(x_{n+1}, y_n) < d(x_n, x_*). \tag{63}$$

Thus, we have

$$\lim_{n\longrightarrow\infty} d(x_{n+1}, y_n) = 0. \tag{64}$$

Since

$$\begin{gathered} d(x_*, y_n) \leq d(x_*, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, y_n), \\ \lim_{n\longrightarrow\infty} d(x_*, y_n) = 0. \end{gathered} \tag{65}$$

Because $Tx_* \in CL(X)$ and $\{y_n\} \subset Tx_*$, $x_* \in Tx_*$. □

We give an example to illustrate Theorem 7.

Example 1. Let $X = \{1, 2, 3, 4\}$ and define $d : X \times X \longrightarrow [0, \infty)$ as follows:

$$\begin{gathered} d(1, 2) = d(2, 1) = 3, \\ d(2, 3) = d(3, 2) = d(1, 3) = d(3, 1) = 1, \\ d(1, 4) = d(4, 1) = d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) = 4, \\ d(x, x) = 0 \forall x \in X. \end{gathered} \tag{66}$$

Then, (X, d) is a complete Branciari distance space but not a metric space (see [9]).

Define a map $T : X \longrightarrow CL(X)$ by

$$Tx = \begin{cases} \{2, 3\} & (x = 1, 2, 3), \\ \{1, 4\} & (x = 4), \end{cases} \tag{67}$$

and a function $\theta : (0,\infty) \longrightarrow (1,\infty)$ by

$$\theta(t) = e^t. \tag{68}$$

We now show that T is a set valued $\mathscr{L}$-contraction with respect to ξ_w, where $\xi_w(t, s) = (s/t\phi(s)) \forall t, s \geq 1, \phi(s) = (1/2)s + (1/2) \forall s \geq 1$.

We consider the following cases.

Case 1. $x = 1$ and $y = 2$.

For $2 \in T1$, there exists $3 \in T2$ with $d(2,3) > 0$ such that

$$\xi_w(\theta(d(2,3)), \theta(d(1,2))) = \xi_w(\theta(1), \theta(3)) = \frac{\theta(3)}{\theta(1)\phi(3)} = \frac{e^2}{2} > 1, \tag{69}$$

and for $3 \in T1$, there exists $2 \in T2$ with $d(3,2) > 0$ such that

$$\xi_w(\theta(d(3,2)), \theta(d(1,2))) = \xi_w(\theta(1), \theta(3)) = \frac{\theta(3)}{\theta(1)\phi(3)} = \frac{e^2}{2} > 1. \tag{70}$$

Case 2. $x = 1$ and $y = 3$.

For $2 \in T1$, there exists $3 \in T3$ with $d(2,3) > 0$ such that

$$\xi_w(\theta(d(2,3)), \theta(d(1,3))) = \xi_w(\theta(1), \theta(1)) = \frac{\theta(1)}{\theta(1)\phi(1)} = 1, \tag{71}$$

and for $3 \in T1$, there exists $2 \in T3$ with $d(3,2) > 0$ such that

$$\xi_w(\theta(d(3,2)), \theta(d(1,3))) = \xi_w(\theta(1), \theta(1)) = \frac{\theta(1)}{\theta(1)\phi(1)} = 1. \tag{72}$$

Case 3. $x = 1$ and $y = 4$.

For $2 \in T1$, there exists $1 \in T4$ with $d(2,1) > 0$ such that

$$\xi_w(\theta(d(2,1)), \theta(d(1,4))) = \xi_w(\theta(3), \theta(4)) = \frac{\theta(4)}{\theta(3)\phi(4)} = \frac{e}{2.5} > 1, \tag{73}$$

and for $3 \in T1$, there exists $1 \in T4$ with $d(3,1) > 0$ such that

$$\xi_w(\theta(d(3,1)), \theta(d(1,4))) = \xi_w(\theta(1), \theta(4)) = \frac{\theta(4)}{\theta(1)\phi(4)} = \frac{e^3}{2.5} > 1. \tag{74}$$

Case 4. $x = 2$ and $y = 3$.

For $2 \in T2$, there exists $3 \in T3$ with $d(2,3) > 0$ such that

$$\xi(\theta(d(2,3)), \theta(d(2,3))) = \xi(\theta(1), \theta(1)) = \frac{\theta(1)}{\theta(1)\phi(1)} = 1, \tag{75}$$

and for $3 \in T2$, there exists $2 \in T3$ with $d(3,2) > 0$ such that

$$\xi(\theta(d(3,2)), \theta(d(2,3))) = \xi(\theta(1), \theta(1)) = \frac{\theta(1)}{\theta(1)\phi(1)} = 1. \tag{76}$$

Case 5. $x = 2$ and $y = 4$.

For $2 \in T3$, there exists $1 \in T4$ with $d(2,1) > 0$ such that

$$\xi(\theta(d(2,1)), \theta(d(3,4))) = \xi(\theta(3), \theta(4)) = \frac{\theta(4)}{\theta(3)\phi(4)} = \frac{e}{2.5} > 1, \tag{77}$$

and for $3 \in T3$, there exists $1 \in T4$ with $d(3,1) > 0$ such that

$$\xi(\theta(d(3,1)), \theta(d(3,4))) = \xi(\theta(1), \theta(4)) = \frac{\theta(4)}{\theta(1)\phi(4)} = \frac{e^3}{2.5} > 1. \tag{78}$$

Case 6. $x = 3$ and $y = 4$.

For $2 \in T3$, there exists $1 \in T4$ with $d(2,1) > 0$ such that

$$\xi(\theta(d(2,1)), \theta(d(3,4))) = \xi(\theta(3), \theta(4)) = \frac{\theta(4)}{\theta(3)\phi(4)} = \frac{e}{2.5} > 1, \tag{79}$$

and for $3 \in T3$, there exists $1 \in T4$ with $d(3,1) > 0$ such that

$$\xi(\theta(d(3,1)), \theta(d(3,4))) = \xi(\theta(1), \theta(4)) = \frac{\theta(4)}{\theta(1)\phi(4)} = \frac{e^3}{2.5} > 1. \tag{80}$$

Hence, T is a set valued $\mathscr{L}$-contraction with respect to ξ_w

Thus, all hypotheses of Theorem 7 are satisfied, and T has fixed points 2, 3, and 4.

Note that the Nadler's fixed point theorem does not hold. If $x = 2$ and $y = 4$, then

$$H(T2, T4) \leq kd(2,4), k \in (0,1), \tag{81}$$

so

$$4k \geq 4, \tag{82}$$

which is a contradiction.

From Theorem 7, we have the following corollary.

Corollary 8. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set valued map such that for all $x, y \in X$ with $d(x, y) > 0$ and $\inf_{z \in Tx} d(z, Ty) > 0$*

$$\xi(\theta(D(Tx, Ty)), \theta(d(x, y))) \geq r, \tag{83}$$

where $\xi \in \mathscr{L}_*, \theta \in \Theta$ and $r > 1$.

Then, T has a fixed point.

Proof. Since

$$\inf_{z \in Tx} d(z, Ty) > 0, \tag{84}$$

for each $u \in Tx, d(u, Ty) > 0$. Hence, we have

$$\theta(d(u, Ty)) \leq \theta(D(Tx, Ty)) < r\theta(D(Tx, Ty)). \tag{85}$$

It follows from ($\theta 5$) that there exists the inverse function θ^{-1} of θ, and so from (85), we have

$$d(u, Ty) < \theta^{-1}(r\theta(D(Tx, Ty))). \tag{86}$$

From Lemma 6, there exists $v \in Ty$ such that

$$d(u, v) < \theta^{-1}(r\theta(D(Tx, Ty))). \tag{87}$$

Hence

$$\theta(d(u, v)) < r\theta(D(Tx, Ty)). \tag{88}$$

Thus, we have

$$\begin{aligned}\xi(\theta(d(u, v)), \theta(d(x, y))) &> \xi(r\theta(D(Tx, Ty)), \theta(d(x, y))) \\ &= \frac{1}{r}\xi(\theta(D(Tx, Ty)), \theta(d(x, y))) \geq \frac{1}{r}r = 1.\end{aligned} \tag{89}$$

□

Thus, condition (33) holds and from Theorem 7, T has a fixed point.

Corollary 9. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set valued map such that for all $x, y \in X$ with $d(x, y) > 0$ and $\inf_{z\in Tx} d(z, Ty) > 0$*

$$\xi(\theta(H(Tx, Ty)), \theta(d(x, y))) \geq r, \tag{90}$$

where $\xi \in \mathscr{L}_*, \theta \in \Theta$ and $r > 1$.

Then, T has a fixed point.

Proof. It follows from ($\xi 1$) and (90) that

$$\xi(\theta(D(Tx, Ty)), \theta(d(x, y))) \geq \xi(\theta(H(Tx, Ty)), \theta(d(x, y))) \geq r, \tag{91}$$

which implies

$$\xi(\theta(D(Tx, Ty)), \theta(d(x, y))) \geq r. \tag{92}$$

□

Thus, from Corollary 8, T has a fixed point.

3. Consequence

We have fixed point results by applying simulation functions given in Example 5 and Example 6 to Theorem 7.

By taking $\xi = \xi_b$ in Theorem 7, we obtain Corollary 10.

Corollary 10. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set valued map.*

Suppose that for all $x, y \in X$ with $d(x, y) > 0$ and $u \in Tx$, there exists $v \in Ty$ with $d(u, v) > 0$ such that

$$\theta(d(u, v)) \leq [\theta(d(x, y))]^k, \tag{93}$$

where $\theta \in \Theta$ and $k \in (0, 1)$.

Then, T has a fixed point.

Remark 11. Corollary 10 is a generalization of Theorem 7 of [19] and Theorem 2.2 of [21] without condition ($\theta 3$) to set-valued maps.

Corollary 12. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set valued map such that for all $x, y \in X$ with $d(x, y) > 0$ and $\inf_{z\in Tx} d(z, Ty) > 0$*

$$\theta(D(Tx, Ty)) < [\theta(d(x, y))]^k, \tag{94}$$

where $\theta \in \Theta, k \in (0, 1)$.

Then, T has a fixed point.

Proof. Since $\inf_{z\in Tx} d(z, Ty) > 0$, we have that, for each $u \in Tx$,

$$\theta(d(u, Ty)) \leq \theta(D(Tx, Ty)) < [\theta(d(x, y))]^k, \tag{95}$$

and so

$$d(u, Ty) < \theta^{-1}\left([\theta(d(x, y))]^k\right). \tag{96}$$

Hence, there exists $v \in Ty$ such that

$$d(u, v) < \theta^{-1}\left([\theta(d(x, y))]^k\right), \text{and so } \theta(d(u, v)) < [\theta(d(x, y))]^k. \tag{97}$$

By Corollary 10, T has a fixed point. □

Corollary 13. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set valued map such that for all $x, y \in X$ with $d(x, y) > 0$ and $\inf_{z\in Tx} d(z, Ty) > 0$*

$$\theta(H(Tx, Ty)) < [\theta(d(x, y))]^k, \tag{98}$$

where $\theta \in \Theta, k \in (0, 1)$.

Then, T has a fixed point.

Remark 14. Corollary 13 is a generalization of Theorem 2 of [44] to Branciari distance spaces with conditions:

$$(\theta 5) \text{ and } \inf_{z\in Tx} d(z, Ty) > 0, \forall x, y \in X. \tag{99}$$

By taking $\xi = \xi_w$ in Theorem 7, we obtain Corollary 15.

Corollary 15. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set valued map.*

Suppose that for all $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ with $d(u, v) > 0$ such that

$$\theta(d(u,v)) \leq \frac{\theta(d(x,y))}{\phi(\theta(d(x,y)))}, \tag{100}$$

where $\theta \in \Theta$ and $\phi : [1,\infty) \longrightarrow [1,\infty)$ is nondecreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$.

Then, T has a fixed point.

Remark 16. Corollary 15 is a generalization of Corollary 2.7 [24] to Branciari distance spaces without condition (θ7).

Corollary 17. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set valued map such that for all $x, y \in X$ with $d(x, y) > 0$ and $\inf_{z\in Tx} d(z, Ty) > 0$*

$$\theta(D(Tx,Ty)) < \frac{\theta(d(x,y))}{\phi(\theta(d(x,y)))}, \tag{101}$$

where $\theta \in \Theta$ and $\phi : [1,\infty) \longrightarrow [1,\infty)$ is nondecreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$.

Then, T has a fixed point.

Proof. Suppose that condition (101) satisfied.

Since $\inf_{z\in Tx} d(z, Ty) > 0$, we have that, for each $u \in Tx$,

$$\theta(d(u,Ty)) \leq \theta(D(Tx,Ty)) < \frac{\theta(d(x,y))}{\phi(\theta(d(x,y)))}. \tag{102}$$

Thus

$$d(u,Ty) < \theta^{-1}\left(\frac{\theta(d(x,y))}{\phi(\theta(d(x,y)))}\right), \tag{103}$$

and so there exists $v \in Ty$ such that

$$\theta(d(u,v)) < \frac{\theta(d(x,y))}{\phi(\theta(d(x,y)))}. \tag{104}$$

By Corollary 15, T has a fixed point. □

Corollary 18. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set valued map such that for all $x, y \in X$ with $d(x, y) > 0$ and $\inf_{z\in Tx} d(z, Ty) > 0$*

$$\theta(H(Tx,Ty)) < \frac{\theta(d(x,y))}{\phi(\theta(d(x,y)))}, \tag{105}$$

where $\theta \in \Theta$ and $\phi : [1,\infty) \longrightarrow [1,\infty)$ is nondecreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$.

Then, T has a fixed point.

Corollary 19. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set valued map.*

Suppose that for all $x, y \in X$ with $d(x, y) > 0$ and for all $u \in Tx$, there exists $v \in Ty$ with $d(u, v) > 0$ such that

$$d(u,v) \leq d(x,y) - \varphi(d(x,y)), \tag{106}$$

where $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is nondecreasing and lower semicontinuous such that $\varphi^{-1}(\{0\}) = 0$.

Then, T has a fixed point.

By the same way as proof of Corollary 15 in [24], we can prove Corollary 19.

Corollary 20. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set valued map such that for all $x, y \in X$ with $d(x, y) > 0$ and $\inf_{z\in Tx} d(z, Ty) > 0$*

$$D(Tx,Ty) < d(x,y) - \varphi(d(x,y)), \tag{107}$$

where $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is nondecreasing and lower semicontinuous such that $\varphi^{-1}(\{0\}) = 0$.

Then, T has a fixed point.

Corollary 21. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set valued map such that for all $x, y \in X$ with $d(x, y) > 0$ and $\inf_{z\in Tx} d(z, Ty) > 0$*

$$H(Tx,Ty) < d(x,y) - \varphi(d(x,y)), \tag{108}$$

where $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is nondecreasing and lower semicontinuous such that $\varphi^{-1}(\{0\}) = 0$.

Then, T has a fixed point.

By taking $\theta(t) = 2 - (2/\pi)\arctan(1/t^{\alpha})$, where $\alpha \in (0, 1)$, $t > 0$ in Corollary 15, we obtain the following result.

Corollary 22. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set valued map.*

Suppose that for all $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ with $d(u, v) > 0$ such that

$$2 - \frac{2}{\pi}\arctan\left(\frac{1}{[d(u,v)]^{\alpha}}\right) \leq \frac{2-(2/\pi)\arctan(1/[d(x,y)]^{\alpha})}{\phi(2-(2/\pi)\arctan(1/[d(x,y)]^{\alpha}))}, \tag{109}$$

where $\alpha \in (0, 1)$ and $\phi : [1,\infty) \longrightarrow [1,\infty)$ is nondecreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$.

Then, T has a fixed point.

Corollary 23. *Let (X, d) be a complete Branciari distance space, and let $T : X \longrightarrow CL(X)$ be a set valued map such that for all $x, y \in X$ with $d(x, y) > 0$ and $\inf_{z\in Tx} d(z, Ty) > 0$*

$$2-\frac{2}{\pi}\arctan\left(\frac{1}{[D(Tx,Ty)]^{\alpha}}\right)<\frac{2-(2/\pi)\arctan\left(1/[d(x,y)]^{\alpha}\right)}{\phi\left(2-(2/\pi)\arctan\left(1/[d(x,y)]^{\alpha}\right)\right)}, \tag{110}$$

where $\alpha\in(0,1)$ *and* $\phi:[1,\infty)\longrightarrow[1,\infty)$ *is nondecreasing and lower semicontinuous such that* $\phi^{-1}(\{1\})=1$.

Then, T has a fixed point.

Corollary 24. *Let* (X,d) *be a complete Branciari distance space, and let* $T:X\longrightarrow CL(X)$ *be a set valued map such that for all* $x,y\in X$ *with* $d(x,y)>0$ *and* $\inf_{z\in Tx}d(z,Ty)>0$

$$2-\frac{2}{\pi}\arctan\left(\frac{1}{[H(Tx,Ty)]^{\alpha}}\right)<\frac{2-(2/\pi)\arctan\left(1/[d(x,y)]^{\alpha}\right)}{\phi\left(2-(2/\pi)\arctan\left(1/[d(x,y)]^{\alpha}\right)\right)}, \tag{111}$$

where $\alpha\in(0,1)$ *and* $\phi:[1,\infty)\longrightarrow[1,\infty)$ *is nondecreasing and lower semicontinuous such that* $\phi^{-1}(\{1\})=1$.

Then, T has a fixed point.

4. Conclusion

One can unify and merge some existing fixed point theorems by using $\mathscr{L}$-simulation functions and $\mathscr{L}_*$-simulation functions in Branciari distance spaces. One can obtain some concequence of the main theorem by applying $\mathscr{L}$-simulation functions and $\mathscr{L}_*$-simulation functions given in Example 1 and Example 2. Further, one can derive all the results of the paper in the setting of metric spaces.

5. Suggestion

We suggest that the b-simulation function can be extended in a similar way to the one in which the simulation function is extended to the $\mathscr{L}$-simulation function. The main theorem can be extended and generalized to b-metric space, Branciari b-distance space, and extended Branciari b-distance space using certain extended simulation functions, and the existing fixed point theorem can be interpreted.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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