


Research Article

On Solitary Wave Solutions for the Camassa-Holm and the Rosenau-RLW-Kawahara Equations with the Dual-Power Law Nonlinearities

Nattakorn Sukantamala^{1,2} and Supawan Nanta³ 

¹Chiang Mai University, Chiang Mai 50200, Thailand

²Advanced Research Center for Computational Simulation, Chiang Mai University, Chiang Mai 50200, Thailand

³Phetchabun Rajabhat University, Phetchabun 67000, Thailand

Correspondence should be addressed to Supawan Nanta; nanta.supawan@gmail.com

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The nonlinear wave equation is a significant concern to describe wave behavior and structures. Various mathematical models related to the wave phenomenon have been introduced and extensively being studied due to the complexity of wave behaviors. In the present work, a mathematical model to obtain the solution of the nonlinear wave by coupling the classical Camassa-Holm equation and the Rosenau-RLW-Kawahara equation with the dual term of nonlinearities is proposed. The solution properties are analytically derived. The new model still satisfies the fundamental energy conservative property as the original models. We then apply the energy method to prove the well-posedness of the model under the solitary wave hypothesis. Some categories of exact solitary wave solutions of the model are described by using the Ansatz method. In addition, we found that the dual term of nonlinearity is essential to obtain the class of analytic solution. Besides, we provide some graphical representations to illustrate the behavior of the traveling wave solutions.

1. Introduction

In the study of nonlinear wave phenomena, the nonlinear partial differential equations are one of the great mathematical models to investigate the problems. A variety of the mathematical theory for the wave equations has been achieved theoretically and numerically, arising in empirical applications on ion-acoustic and magnetohydrodynamics waves in plasma, longitudinal dispersive waves in elastic rods, pressure waves in liquid-gas bubble mixtures, and rotating flow down a tube. For instance, the various phenomena of shallow-water waves are led by nonlinear partial differential equations such as the Korteweg-de Vries (KdV) equation [1–7], the Benjamin-Bona-Mahony (BBM) equation [8–11], the Symmetric Regularized Long Wave (SRLW) equation [12–15], the Kawahara equation [16–19], and the Rosenau equation [20–23]. For further understanding of nonlinear behaviors

of shallow-water waves, the generalized Rosenau-RLW equation was introduced in the following:

$$u_t - u_{xxt} + u_{xxxxt} + u_x + \beta u^p u_x = 0, \quad (1)$$

where $p \geq 1$ and β are a constant. Equation (1) is an extension of the Rosenau equation by adding a viscous term $-u_{xxt}$ and replacing the nonlinear term with a general power of nonlinearity $u^p u_x$. If $p = 1$ and $\beta = 1$, then equation (1) is called usual Rosenau-RLW equation. When $p = 2$, then equation (1) is called the modified Rosenau-RLW equation. For numerical study for the Rosenau-RLW equation, we refer to [24–27]. Later, many models related to the Rosenau and the Rosenau-RLW equations have been studied and become an essential topic in the study of shallow-water wave behavior. In [28], the solitons and periodic solutions for the Rosenau-KdV and Rosenau-Kawahara equations were

obtained utilizing the sine-cosine and the tanh methods. Wongsajai and Poochinapan [29] numerically studied the Rosenau-RLW-KdV equation by coupling the Rosenau-RLW equation and the Rosenau-KdV equation. In [30], Labidi and Biswas obtained the analytical one-soliton solution for the Rosenau-Kawahara equation by using He's semi-inverse variational principle. Biswas et al. [31] obtained the solitary solution and two invariance of the generalized Rosenau-Kawahara equation with power law nonlinearity. Recently, by adding a viscous term $-u_{xxt}$ into the Rosenau-Kawahara equation, which is called the Rosenau-RLW-Kawahara equation,

$$u_t - u_{xxt} + u_{xxxxt} + u_x + u_{xxx} - u_{xxxxx} + \beta u^p u_x = 0. \quad (2)$$

It has been a growing interest in computation nonlinear wave equations. In [32], He and Pan initially studied a second-order three-level linearly implicit difference scheme which is energy-conserved and unconditionally stable. In [33], two conservative high-order accurate finite difference schemes for the periodic initial value generalized Rosenau-Kawahara-RLW equation were introduced and extensively studied. For more related nonlinear wave equations, readers can refer to [34–43].

As furthermore consideration of the unidirectional shallow-water waves, one of the equations is a Camassa-Holm (CH) equation which can be founded:

$$u_t + \kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad (3)$$

where κ is a constant. The equation has been derived by Camassa and Holm [44] in 1993 and has a solitary peaked solution which discontinuity in the first derivative. For the significance of κ , it was shown in [45] that for all $\kappa > 0$, there are smooth solitary wave solutions, and for $\kappa = 0$, it has peaked soliton solution (peakon). A classification of weak traveling wave solution of Camassa-Holm equation was given in [46]. Furthermore, Kalisch and Lenells have investigated the kind of traveling wave solution, smooth traveling waves, cusped traveling waves, and composited traveling waves [47]. The orbital stability of the peakons and the solitons of the smooth solitary wave of CH equation were shown in [46, 48, 49]. In 2010, Lai [50] established the existence and uniqueness of a local solution of the CH equation in Sobolev space $H^s(\mathbb{R})$, and the well-posedness was established by Li and Olver [49]. Very recently, Nanta et al. [51] obtained the numerical study of the generalized Camassa-Holm equation involving dual-power law nonlinearities. Other studies of CH-related equation are also reported by various publications [52–57].

In this paper, our purpose is to investigate the coupling of the original CH equation and the Rosenau-RLW-Kawahara equation with the dual-power law nonlinearity:

$$u_t - \mu u_{xxt} + 2\kappa u_x + \eta u_{xxx} + u_{xxxxt} + \gamma u_{xxxxx} = f(u)u_x + s[2u_x u_{xx} + uu_{xxx}], \quad (4)$$

with the initial condition

$$u(x, 0) = u_0(x), x \in [x_L, x_R], \quad (5)$$

where $u_0(x)$ is a known smooth function, $\kappa, \eta \in \mathbb{R}$, $\mu > 0$. The function $f(u) = Au + Bu^m$ represents the dispersive nonlinear terms in both low and high-order nonlinearity, where $A, B \in \mathbb{R}$, and $m \in \mathbb{N}$ indicates the power law nonlinearity. Moreover, the solitary wave solution and its derivatives have the following asymptotic values:

$$u \longrightarrow 0 \text{ as } x \longrightarrow \pm\infty, \text{ and for } n \geq 1, \frac{\partial^n u}{\partial x^n} \longrightarrow 0 \text{ as } x \longrightarrow \pm\infty. \quad (6)$$

Note that equation (4) reduces to Rosenau-RLW-Kawahara equation when $s = A = 0$, and it reduces to CH equation (3) when $\mu = 1, \kappa = 1/2, \eta = 0, \gamma = 0, B = 0$, and $s = 1$ and removing viscous term u_{xxxxt} . Moreover, when $A = \gamma = \eta = s = 0$, equation (4) reduces to the Rosenau-RLW equation.

To study the nature of solutions, researchers have attempted to find the exact solution of the Rosenau-type equation. Many methods were introduced and developed to explore the analytical solution corresponding to nonlinear partial differential equations. By using the sech and trigonometric function method, Esfahani [58] (Esfahani and Pourgholi [59]) studied solitary wave solutions to the generalized Rosenau-KdV and Rosenau-RLW equation, respectively. The solitons and shock waves were discussed by Razborave et al. [60] by applying a semi-inverse variational method. In [29], Wongsajai and Poochinapan used the sine-cosine method to find the exact solution of the Rosenau-RLW-KdV equation. He and Pan [32] also used the sine-cosine method to obtain the solitary solution for the generalized Rosenau-Kawahara-RLW equation, and the solution for the Rosenau-Kawahara-RLW equation with, notably, the generalized Novikov type perturbation was solely derived by He [38]. The solution of (2+1) dimensional of nonlinear wave equation using modified exponential function method and Ansatz function technique with symbolic computation was proposed in [61]. In [62], solitary wave solution for Ablowitz-Kaup-Newell-Segur water wave equation was obtained by using the simple equation method and modified simple equation method. The generalized extended tanh method and the F-expansion method were used to derive exact solutions for the Kadomtsev-Petviashvili and the modified Kadomtsev-Petviashvili dynamical equations [63]. In addition, readers can refer to [64, 65] for more methods to find analytic wave solutions.

The paper has been organized as follows. In Section 2, the fundamental energy-preserving property of the initial boundary value problems is proved. By applying the energy method, the well-posedness of the new model is obtained in the solution space $H_0^2(\Omega)$. In addition, the traveling wave solutions of the equation were employed by the Ansatz method, which determines solitary solutions and periodic

solutions. Finally, concluding remarks are reported in the last section.

2. Solution Properties

We first state that the solution of equations (4)–(6) satisfies the following energy conservative property.

Theorem 1. *If the solution of equations (4)–(6) u and its derivatives $\partial_x u$, $\partial_x^2 u$ go to zero when $|x| \rightarrow \infty$, then equations (4)–(6) have the following global conservation law:*

$$E(t) = \int_{-\infty}^{\infty} u^2(x, t) + \mu u_x^2(x, t) + u_{xx}^2(x, t) dx = \int_{-\infty}^{\infty} u^2(x, 0) + \mu u_x^2(x, 0) + u_{xx}^2(x, 0) dx = E(0), \tag{7}$$

for all $t \in [0, T]$.

Proof. Let $u_t - \mu u_{xxt} + u_{xxxxt} = -2\kappa u_x - \eta u_{xxx} + \gamma u_{xxxxx} + f(u)u_x + s(2u_x u_{xx} + uu_{xxx})$. Then,

$$\begin{aligned} \frac{dE(t)}{dt} &= 2 \int_{-\infty}^{\infty} uu_t dx + 2\mu \int_{-\infty}^{\infty} u_x u_{xt} dx + 2 \int_{-\infty}^{\infty} u_{xx} u_{xxt} dx \\ &= 2 \int_{-\infty}^{\infty} uu_t dx - 2\mu \int_{-\infty}^{\infty} uu_{xxt} dx + 2 \int_{-\infty}^{\infty} uu_{xxxxt} dx \\ &= 2 \int_{-\infty}^{\infty} u(-2\kappa u_x - \eta u_{xxx} - \gamma u_{xxxxx} + f(u)u_x + s(2u_x u_{xx} + uu_{xxx})) dx. \end{aligned} \tag{8}$$

Using the integration by parts and the assumption u and its derivatives $\partial_x u$, $\partial_x^2 u \rightarrow 0$ as $|x| \rightarrow \infty$, we obtain

$$\int_{-\infty}^{\infty} uu_x dx = 0,$$

$$\int_{-\infty}^{\infty} uu_{xxx} dx = 0,$$

$$\int_{-\infty}^{\infty} uu_{xxxxx} dx = 0,$$

$$\begin{aligned} \int_{-\infty}^{\infty} uf(u)u_x dx &= \int_{-\infty}^{\infty} (Au^2 + Bu^{m+1}) du \\ &= \lim_{x_L \rightarrow -\infty} \lim_{x_R \rightarrow \infty} \left[\frac{A}{3} u^3 \Big|_{x_L}^{x_R} + \frac{B}{m+2} u^{m+2} \Big|_{x_L}^{x_R} \right] = 0, \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} u(2u_x u_{xx} + uu_{xxx}) dx &= \int_{-\infty}^{\infty} u(u_x u_{xx} + (uu_{xx})_x) dx \\ &= \int_{-\infty}^{\infty} uu_x u_{xx} dx - \int_{-\infty}^{\infty} uu_x u_{xx} dx = 0. \end{aligned} \tag{9}$$

Therefore, $E(t)$ is a constant function, that is,

$$\frac{d}{dt} \int_{-\infty}^{\infty} u^2(x, t) + \mu u_x^2(x, t) + u_{xx}^2(x, t) dx = 0, \tag{10}$$

which yields $E(t) = E(0)$ for all $t \in [0, T]$, as desired.

By assumption (6), problem (4) can be set up in a compact subset of \mathbb{R} , namely, $\Omega = [x_L, x_R]$. Thereby, we consider the following initial-boundary value problem (4) with the initial condition (5) and the boundary conditions

$$\begin{aligned} u(x_L, t) = u(x_R, t) = 0 \quad u_x(x_L, t) = u_x(x_R, t) = 0 \quad \text{and} \quad u_{xx}(x_L, t) \\ = u_{xx}(x_R, t) = 0 \quad t \in [0, T]. \end{aligned} \tag{11}$$

For a nonnegative integer k , let $H^k(\Omega)$ denote the usual Sobolev space of real valued functions defined on the interval Ω . We define the following Sobolev space:

$$H_0^k(\Omega) = \left\{ u \in H^k(\Omega) \mid \frac{\partial^i u}{\partial x^i} = 0 \text{ on } \partial\Omega, i = 1, 2, \dots, k-1 \right\}. \tag{12}$$

The solutions of equations (4) and (5) with the boundary condition (11) satisfy the following energy conservative property. \square

Theorem 2. *Suppose $u_0 \in H_0^2(\Omega)$; then, the solution of equations (4), (5), and (11) satisfies the following:*

$$\begin{aligned} E(t) &= \int_{x_L}^{x_R} u^2(x, t) + \mu u_x^2(x, t) + u_{xx}^2(x, t) dx \\ &= \int_{x_L}^{x_R} u^2(x, 0) + \mu u_x^2(x, 0) + u_{xx}^2(x, 0) dx = E(0), \end{aligned} \tag{13}$$

for all $t \in [0, T]$.

It should be pointed out that the invariant function $E(t)$ indicates the energy conservation for equations (4) and (5). Next, we provide the well-posedness of problems (4) and (5) with the boundary condition (11) on the solution space $H_0^2[x_L, x_R]$. Before providing the well-posedness, we first state the existence, which can be proved by the standard energy method. By combining the local existence and uniqueness with Theorem 2, we obtain the global existence. Therefore, we leave the proof.

Lemma 3 (existence). *Suppose $u_0 \in H_0^2[x_L, x_R]$; then, there exists a positive constant δ such that $\|u\|_{H^2} \leq \delta$, and then, the initial value of problems (4) and (5) has a unique global solution $u(x, t)$ with $u(x, t) \in C(0, \infty; H^2[x_L, x_R])$.*

Theorem 4. *Suppose $u_0 \in H_0^2[x_L, x_R]$; then, problems (4) and (5) with the boundary condition (11) are well-posed.*

Proof. First, we let u_1 and u_2 are two solutions of (4) and (5) with the boundary condition (11) satisfying the initial conditions $u_{0,1}$ and $u_{0,2}$, respectively. Let $\varepsilon = u_1 - u_2$; then, by substituting, δ corresponds to the following equation:

$$\begin{aligned} \varepsilon_t - \mu\varepsilon_{xxt} + 2\kappa\varepsilon_x + \eta\varepsilon_{xxx} + \varepsilon_{xxxxt} + \gamma\varepsilon_{xxxxx} \\ = f(u_1)(u_1)_x - f(u_2)(u_2)_x + s[2(u_1)_x(u_1)_{xx} + (u_1)(u_1)_{xxx}] \\ - s[2(u_2)_x(u_2)_{xx} + (u_2)(u_2)_{xxx}]. \end{aligned} \tag{14}$$

with the initial conditions

$$\varepsilon(x, 0) = u_{0,1} - u_{0,2} \tag{15}$$

and boundary conditions

$$\begin{aligned} \varepsilon(x_L, t) = \varepsilon(x_R, t) = 0 \quad \varepsilon_x(x_L, t) = \varepsilon_x(x_R, t) \\ = 0 \text{ and } \varepsilon_{xx}(x_L, t) = \varepsilon_{xx}(x_R, t) = 0 \quad t \in [0, T], \end{aligned} \tag{16}$$

where $t \in [0, T]$ and $x \in [x_L, x_R]$. By the standard energy method, we introduce the following energy function

$$E^*(t) = \int_{x_L}^{x_R} \varepsilon^2 + \mu\varepsilon_x^2 + \varepsilon_{xx}^2 dx. \tag{17}$$

By similar arguments as that in the proof of Theorem 1, we have

$$\begin{aligned} \frac{dE^*(t)}{dt} &= 2 \int_{x_L}^{x_R} \varepsilon \varepsilon_t dx + 2\mu \int_{x_L}^{x_R} \varepsilon_x \varepsilon_{xt} dx + 2 \int_{x_L}^{x_R} \varepsilon_{xx} \varepsilon_{xxt} dx \\ &= 2 \int_{x_L}^{x_R} \varepsilon \varepsilon_t dx - 2\mu \int_{x_L}^{x_R} \varepsilon \varepsilon_{xxt} dx + 2 \int_{x_L}^{x_R} \varepsilon \varepsilon_{xxxxt} dx \\ &= 2 \int_{x_L}^{x_R} \varepsilon (-2\kappa\varepsilon_x - \eta\varepsilon_{xxx} - \gamma\varepsilon_{xxxxx}) dx \\ &\quad + 2 \int_{x_L}^{x_R} \varepsilon [f(u_1)(u_1)_x - f(u_2)(u_2)_x] dx \\ &\quad + 2s \int_{x_L}^{x_R} \varepsilon [2(u_1)_x(u_1)_{xx} + (u_1)(u_1)_{xxx} \\ &\quad - (2(u_2)_x(u_2)_{xx} + (u_2)(u_2)_{xxx})] dx \\ &= 2 \int_{x_L}^{x_R} \varepsilon [f(u_1)(u_1)_x - f(u_2)(u_2)_x] dx \\ &\quad + 2s \int_{x_L}^{x_R} \varepsilon [2((u_1)_x(u_1)_{xx} - (u_2)_x(u_2)_{xx}) \\ &\quad + ((u_1)(u_1)_{xxx} - (u_2)(u_2)_{xxx})] dx. \end{aligned} \tag{18}$$

Noting that the first nonlinear term can be estimated as

$$\begin{aligned} \int_{x_L}^{x_R} \varepsilon [f(u_1)(u_1)_x - f(u_2)(u_2)_x] \\ = A \int_{x_L}^{x_R} \varepsilon (\varepsilon(u_1)_x + u_2 \varepsilon_x) dx + 2sB \int_{x_L}^{x_R} \varepsilon \\ \times \left(\varepsilon_x (u_2)^m + \varepsilon (u_1)_x \sum_{k=1}^{m-1} (u_1)^{m-k-1} (u_2)^k \right) \\ \leq C \int_{x_L}^{x_R} |\varepsilon^2 + \varepsilon \varepsilon_x| dx \leq C \left(\int_{x_L}^{x_R} \varepsilon^2 dx + \int_{x_L}^{x_R} \varepsilon_x^2 dx \right), \end{aligned} \tag{19}$$

where Theorem 1 and the Cauchy-Schwarz inequality are used. For the second term, we see that

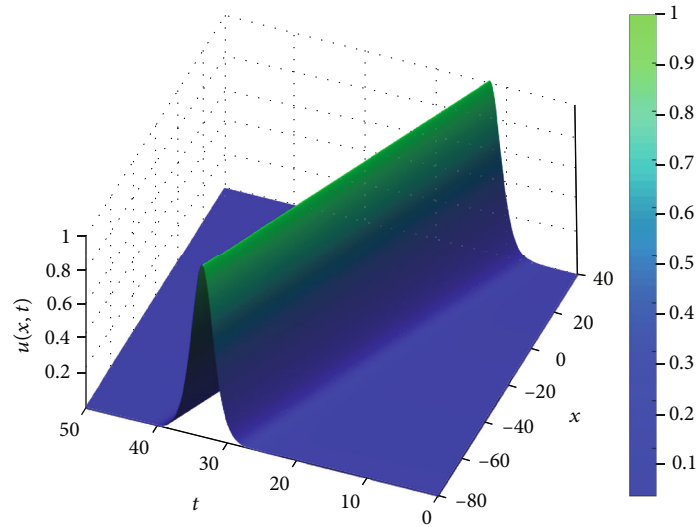
$$\begin{aligned} 2s \int_{x_L}^{x_R} \varepsilon [2((u_1)_x(u_1)_{xx} - (u_2)_x(u_2)_{xx}) \\ + ((u_1)(u_1)_{xxx} - (u_2)(u_2)_{xxx})] dx \\ = 4s \int_{x_L}^{x_R} \varepsilon ((u_1)_x(u_1)_{xx} - (u_2)_x(u_2)_{xx}) dx \\ + 2s \int_{x_L}^{x_R} \varepsilon ((u_1)(u_1)_{xxx} - (u_2)(u_2)_{xxx}) dx := M_1 + M_2. \end{aligned} \tag{20}$$

For the term M_1 , by Theorem 1 and the Cauchy-Schwarz inequality, we have

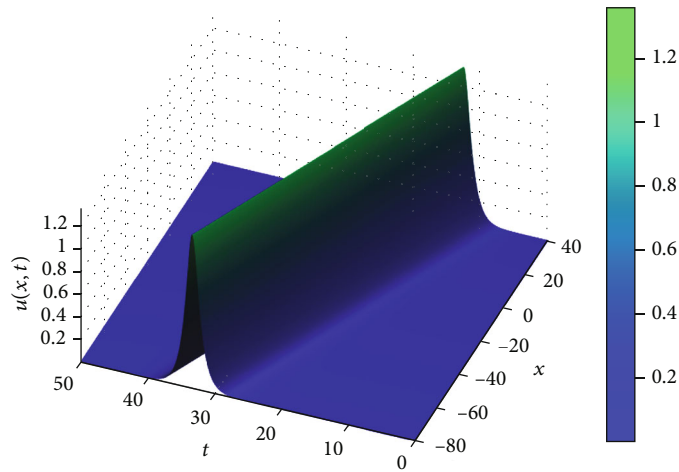
$$M_1 = 4s \int_{x_L}^{x_R} \varepsilon (\varepsilon_x (u_1)_{xx} + (u_2)_x \varepsilon_{xx}) dx \leq C \int_{x_L}^{x_R} \varepsilon^2 dx + \varepsilon_x^2 + \varepsilon_{xx}^2 dx. \tag{21}$$

Next, by simple calculations, we can estimate the term M_2 as

$$\begin{aligned} M_2 &= 2s \int_{x_L}^{x_R} \varepsilon ((u_1)(u_1)_{xxx} - (u_2)(u_2)_{xxx}) dx \\ &= -2s \int_{x_L}^{x_R} ((\varepsilon u_1)_x (u_1)_{xx} - (\varepsilon u_2)_x (u_2)_{xx}) dx \\ &= -2s \int_{x_L}^{x_R} \varepsilon ((u_1)_x (u_1)_{xx} - (u_2)_x (u_2)_{xx}) \\ &\quad + \varepsilon_x (u_1 (u_1)_{xx} - u_2 (u_2)_{xx}) dx \\ &= -2s \int_{x_L}^{x_R} \varepsilon (\varepsilon_x (u_1)_{xx} + (u_2)_x \varepsilon_{xx}) + \varepsilon_x (\varepsilon (u_1)_{xx} + u_2 \varepsilon_{xx}) dx \\ &\leq 2s \int_{x_L}^{x_R} \|\varepsilon\|_\infty |\varepsilon_x| |(u_1)_{xx}| + C(|\varepsilon| |\varepsilon_x| + |\varepsilon_x| |\varepsilon_{xx}|) dx \\ &\leq C \int_{x_L}^{x_R} \varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2 dx + s \|\varepsilon\|_\infty^2 \int_{x_L}^{x_R} (u_1)_{xx}^2 dx \\ &\leq C \int_{x_L}^{x_R} \varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2 dx, \end{aligned} \tag{22}$$



(a) $m = 2$



(b) $m = 4$

FIGURE 1: Plot solitary wave solutions (32) when $A = -1$ and $s = 2$.

where Theorem 1, the Cauchy-Schwarz inequality, and the Sobolev's inequality are used. Substituting equations (19)–(22) into equation (18) gives

$$\frac{dE^*(t)}{dt} \leq CE^*(t), \tag{23}$$

which yields $E^*(t) \leq e^{CT}E^*(0)$ for all $t \in [0, T]$. Obviously, the uniqueness is consequently obtained when the initial conditions for u_1 and u_2 are the same. Moreover, if $\varepsilon(x, 0) < \delta$, $\varepsilon_x(x, 0) < \delta$, and $\varepsilon_{xx}(x, 0) < \delta$, then we have

$$\frac{dE^*(t)}{dt} \leq e^{CT}E^*(0) \leq \delta e^{CT}, \tag{24}$$

for all $t \in [0, T]$. That is, the solution is continuously dependent on the initial condition. Since the existence and uniqueness are obtained by Lemma 3, therefore equations (4) and (5) with the boundary condition (11) are well-posed as required. \square

3. Solitary Wave Solutions

Next, we focus on problems (4) and (5). By introducing $\xi = x - ct$, we see that equation (4) reduces to

$$\begin{aligned} & -cu_\xi + \mu cu_{\xi\xi\xi} + 2\kappa u_\xi + \eta u_{\xi\xi\xi} - cu_{\xi\xi\xi\xi} + \gamma u_{\xi\xi\xi\xi\xi} \\ & = f(u)u_\xi + s[2u_\xi u_{\xi\xi} + uu_{\xi\xi\xi}], \end{aligned} \tag{25}$$

that is,

$$\begin{aligned} & [2\kappa - c]u_\xi + [\mu c + \eta]u_{\xi\xi\xi} + [\gamma - c]u_{\xi\xi\xi\xi\xi} \\ & = f(u)u_\xi + s[2u_\xi u_{\xi\xi} + uu_{\xi\xi\xi}], \end{aligned} \tag{26}$$

where $f(u) = Au + Bu^m$. The solitary wave Ansatz method admits the used assumption

$$u(\xi) = \lambda \operatorname{sech}^\beta(\alpha\xi). \tag{27}$$

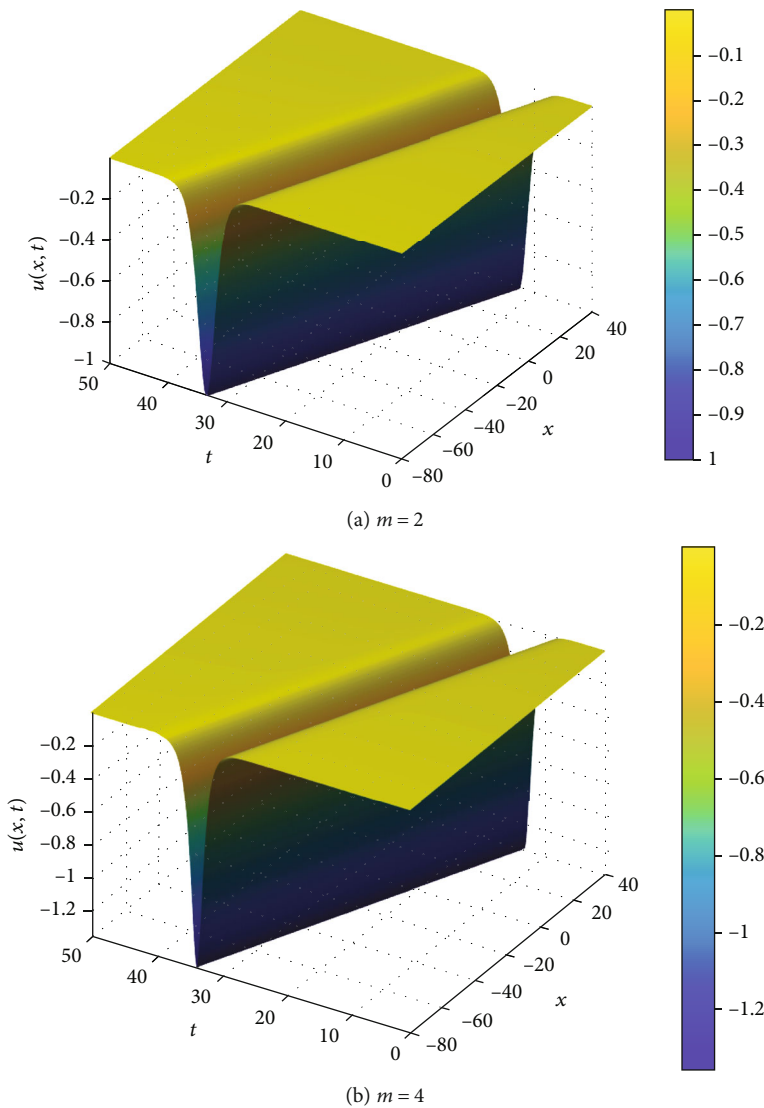


FIGURE 2: Plot solitary wave solutions (32) when $A = 1$ and $s = -2$.

Simple calculations give

$$\begin{aligned}
 u_\xi &= -\lambda\alpha\beta \sec h^\beta(\alpha\xi) \tanh(\alpha\xi), \\
 u_{\xi\xi} &= \lambda\alpha^2\beta^2 \sec h^\beta(\alpha\xi) - \lambda\alpha^2\beta(\beta+1) \sec h^{\beta+2}(\alpha\xi), \\
 u_{\xi\xi\xi} &= \lambda\alpha^3\beta(\beta+1)(\beta+2) \sec h^{\beta+2}(\alpha\xi) \tanh(\alpha\xi) \\
 &\quad - \lambda\alpha^3\beta^3 \sec h^\beta(\alpha\xi) \tanh(\alpha\xi), \\
 u_{\xi\xi\xi\xi} &= \lambda\alpha^4\beta^4 \sec h^\beta(\alpha\xi) - \lambda\alpha^4\beta(\beta+1) \\
 &\quad \cdot (2\beta^2 + 4\beta + 4) \sec h^{\beta+2}(\alpha\xi) + \lambda\alpha^4\beta(\beta+1)(\beta+2) \\
 &\quad \cdot (\beta+3) \sec h^{\beta+4}(\alpha\xi), \\
 u_{\xi\xi\xi\xi\xi} &= -\lambda\alpha^5\beta^5 \sec h^\beta(\alpha\xi) \tanh(\alpha\xi) + \lambda\alpha^5\beta(\beta+1)(\beta+2) \\
 &\quad \cdot (2\beta^2 + 4\beta + 4) \sec h^{\beta+2}(\alpha\xi) \tanh(\alpha\xi) \\
 &\quad - \lambda\alpha^5\beta(\beta+1)(\beta+2)(\beta+3)(\beta+4) \sec h^{\beta+4} \\
 &\quad \cdot (\alpha\xi) \tanh(\alpha\xi).
 \end{aligned}
 \tag{28}$$

Therefore, equation (26) turns into

$$\begin{aligned}
 &[-(2\kappa + c) - (\mu c + \eta)\alpha^2\beta^2 - (\gamma - c)\alpha^4\beta^4] \lambda\alpha\beta \sec h^\beta(\alpha\xi) \\
 &\quad + [(\mu c + \eta) + (\gamma - c)(2\beta^2 + 4\beta + 4)\alpha^3] \lambda\alpha^3\beta(\beta+1) \\
 &\quad \times (\beta+2) \sec h^{\beta+2}(\alpha\xi) - (\gamma - c)\lambda\alpha^5\beta(\beta+1)(\beta+2) \\
 &\quad \times (\beta+3)(\beta+4) \sec h^{\beta+4}(\alpha\xi) \\
 &= [-A\lambda^2\beta - 3s\lambda^2\alpha^3\beta^3] \sec h^{2\beta}(\alpha\xi) + s\lambda^2\alpha^3\beta(\beta+1) \\
 &\quad \times (3\beta+2) \sec h^{2\beta+2}(\alpha\xi) - B\lambda^{m+1}\alpha\beta \sec h^{(m+1)\beta}(\alpha\xi).
 \end{aligned}
 \tag{29}$$

Balancing $\sec h^{2\beta+2}(\alpha\xi)$ and $\sec h^{(m+1)\beta}(\alpha\xi)$, we obtain that $2\beta + 2 = (m + 1)\beta$; so, $\beta = 1/m - 1$.

Setting the coefficients of each term of $\sec h^j(\mu\xi)$ to zero, we have the following system:

$$-A\alpha\lambda^2\beta - 3s\lambda^2\alpha^3\beta^3 = 0,$$

$$\begin{aligned}
 & s\lambda^2\alpha^3\beta(\beta+1)(3\beta+2) - B\lambda^{m+1}\alpha\beta = 0, \\
 & (2\kappa+c) + (\mu c + \eta)\alpha^2\beta^2 + (\gamma-c)\alpha^4\beta^4 = 0, \\
 & (\mu c + \eta) + (\gamma-c)(2\beta^2 + 4\beta + 4)\alpha^3 = 0, \\
 & (\gamma-c)\lambda\alpha^5\beta(\beta+1)(\beta+2)(\beta+3)(\beta+4) = 0.
 \end{aligned}
 \tag{30}$$

Solving system (30), we obtain the set of parameters.

$$\begin{aligned}
 \alpha &= \left[\frac{-A(m-1)^2}{12s} \right]^{1/2}, \\
 \lambda &= \left[\frac{-A(m^2+3m+2)}{6B} \right]^{1/(m-1)}, \\
 \gamma = c &= -\frac{\eta}{\mu}, \\
 \kappa &= -\frac{\eta}{2\mu}.
 \end{aligned}
 \tag{31}$$

For $sA < 0$, we can obtain the following solitary wave solutions for equation (4):

$$\begin{aligned}
 u(x, t) &= \left[\frac{-A(m^2+3m+2)}{6B} \right]^{1/m-1} \operatorname{sech}^{2/m-1} \\
 &\cdot \left((m-1)\sqrt{\frac{-A}{12s}} \left(x + \frac{\eta}{\mu}t \right) \right).
 \end{aligned}
 \tag{32}$$

Additionally, the following periodic wave solutions for equation (4) can be obtained when $sA > 0$

$$\begin{aligned}
 u(x, t) &= \left[\frac{-A(m^2+3m+2)}{6B} \right]^{1/m-1} \operatorname{sec}^{2/m-1} \\
 &\cdot \left((m-1)\sqrt{\frac{A}{12s}} \left(x + \frac{\eta}{\mu}t \right) \right).
 \end{aligned}
 \tag{33}$$

Figures 1 and 2 plot the analytical solutions in the case of $A = -1$ and $s = 2$ and $A = 1$ and $s = -2$, respectively, when $m = 2$ and $m = 4$.

4. Concluding Remarks

In this paper, we successfully studied the nonlinear wave equation by coupling the classical Camassa-Holm equation and the Rosenau-RLW-Kawahara equation in the case of asymptotic boundary conditions. Based on the boundary conditions, we obtained that the equation possesses the conservative energy, which was used to derive the well-posedness in $H_0^2(\Omega)$. Moreover, to seek the analytic solution in $H_0^2(\Omega)$, we applied the Ansatz method to derive the solitary wave solution class by balancing linear and nonlinear terms. One can see that the dual term of nonlinearity $f(u) = Au + Bu^m$ is essential to derive the class of analytic solutions.

In view of Theorem 4, the order of the highest-order derivative appearing in equation (4) is five, but there are six boundary conditions as defined in equation (11), which seems that it is overdetermined for the problem on a bounded interval. It should be pointed out that the boundary condition (11) is logical to study under the solitary wave conditions, that is, u and its derivative approach to zero when $|x| \rightarrow \infty$ (see equation (6)). However, there are many qualitative differences in the behavior of solutions depending on the number of boundary conditions used. Therefore, this question should be of interest in the future.

Data Availability

No data were available in the manuscript.

Conflicts of Interest

No conflict of interest exists. We wish to confirm that there are no known conflicts of interest associated with this publication, and there has been no significant financial support for this work that could have influenced its outcome.

Authors' Contributions

All authors developed the theoretical formalism and performed the analytic calculations to the writing final version of the manuscript.

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